Illinois Journal of Mathematics
Volume 50, Number 1, Spring 2006, Pages 93–105
S 0019-2082

DIFFUSING POLYGONS AND SLE(κ, ρ)

ROBERT O. BAUER AND ROLAND M. FRIEDRICH

ABSTRACT. We give a geometric derivation of SLE(κ, ρ) in terms of conformally invariant random growing compact subsets of polygons. The parameters ρ_j are related to the exterior angles of the polygons. We also show that SLE(κ, ρ) can be generated by a metric Brownian motion, where metric and Brownian motion are coupled and the metric is a pull-back metric of the Euclidean metric of an evolving polygon.

1. Introduction

Stochastic Loewner evolution (or SLE) as introduced by Schramm in [13] describes random growing compacts in a simply connected planar domain D. Schramm discovered SLE by considering discrete random simple curves which satisfy (1) a Markovian-type property and whose scaling limit was conjectured to be (2) conformally invariant. These two properties (plus a reflection symmetry) render SLE canonical in the sense that there exists only a one-parameter family of random non-self-crossing curves γ with these properties. They are all obtained by solving Loewner's equation [9] with a driving function given in terms of Brownian motion. If D is the upper half-plane $\mathbb{H}$, and $\kappa \geq 0$, consider for each $z \in \overline{\mathbb{H}}$ the ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z, \tag{1.1}$$

where $W_t = \sqrt{\kappa} B_t$, and B_t is a one-dimensional standard Brownian motion. Let T_z be the duration for which this equation is well defined, i.e.,

$$T_z = \sup\{t : \inf_{s \in [0,t]} |g_t(z) - W_t| > 0\},$$

and set $K_t = \{z : T_z \leq t\}$. Then it is easy to show that g_t is a conformal map from $\mathbb{H} \backslash K_t$ onto $\mathbb{H}$ with $\lim_{z \to \infty} g(z) - z = 0$. It can also be shown

Received June 3, 2005; received in final form January 18, 2006.
2000 *Mathematics Subject Classification.* 60K35.
The research of the first author was supported by NSA grant H98230-04-1-0039. The research of the second author was supported by a grant from the Max-Planck-Gesellschaft.

[11] that with probability one the random growing compact set K_t is generated by a random non-self-crossing curve $t \mapsto \gamma_t$ in the sense that $\mathbb{H}\backslash K_t$ is the unbounded component of $\mathbb{H}\backslash\gamma[0,t]$. γ is a random curve connecting the boundary points 0 and ∞ and is called *chordal* SLE_κ in $\mathbb{H}$ from 0 to ∞. For an arbitrary domain D and prime ends z and w chordal SLE_κ in D from z to w is defined via conformal invariance up to a time-change. If $f(\mathbb{H}) = D$, $f(0) = z$, $f(\infty) = w$, then $f \circ \gamma_t$ is a chordal SLE_κ in D from z to w. If g is another conformal map from $\mathbb{H}$ onto D with $g(0) = z$, $g(\infty) = w$, then $\{g \circ \gamma_t, t \geq 0\}$ has the same law as a time-change of $\{f \circ \gamma_t, t \geq 0\}$.

For calculations involving SLE conformal invariance is a powerful tool as it is always permissible to choose the geometrically most convenient configuration to do a given calculation. The solution depends only on the conformal equivalence class, or the moduli, of the configuration. The determination of certain hitting probabilities is reduced to solving appropriate hypergeometric equations. In fact, what one does is to track the evolution of these hitting probabilities as the curve γ grows, which comes down to tracking the evolution of the image under the uniformizing map g_t of the set γ is supposed to hit. For example, if γ is chordal SLE_κ in the upper half-plane from 0 to ∞, $\kappa > 4$ and $x, y > 0$, then the probability that γ hits $(-\infty, -y)$ before (x, ∞) depends only on the cross ratio $-y/x$ and is given by

$$p = \frac{\Gamma(2-4a)}{\Gamma(2-2a)\Gamma(1-2a)} \left(\frac{y/x}{y/x+1}\right)^{1-2a} F(2a, 1-2a, 2-2a; \frac{y/x}{y/x+1}),$$

for $a = 2/\kappa$; see [7]. The calculation of p uses the movement x and $-y$ undergo under the uniformizing map g_t, i.e., $t \mapsto x_t \equiv g_t(x)$, $-y_t \equiv g_t(-y)$. We note that although x_t and y_t are coupled to W_t via

$$\partial_t x_t = \frac{2}{x_t - W_t}, \quad \partial_t(-y_t) = \frac{2}{-y_t - W_t},$$

there is no coupling of W_t to x_t or $-y_t$. If we do couple W_t to $x_t, -y_t$ via

$$dW_t = \sqrt{\kappa} dB_t + b(W_t, x_t, -y_t)dt,$$

then the requirement that the random curve γ that results from solving Löwner's equation for this W_t be both conformally invariant and satisfy a Markovian-type property, forces the function b to be homogenous of degree -1; see [2]. A particularly simple such function is

$$b(w, x, y) = \frac{\rho_1}{w - x} + \frac{\rho_2}{w - y}.$$

Coupling with this particular choice of drift b leads to an example of $\mathrm{SLE}(\kappa, \rho)$, which we now define.

Let $z_1 < z_2 < \cdots < z_n$ be real numbers all distinct from 0. Consider the system of stochastic differential equations

$$\begin{aligned} dW_t &= \sqrt{\kappa}\; dB_t + \sum_{k=1}^{n} \frac{\rho_k}{W_t - Z_t^k}\; dt, \\ dZ_t^k &= \frac{2}{Z_t^k - W_t}\; dt, \quad k = 1, \ldots, n, \end{aligned} \tag{1.2}$$

with $W_0 = 0, Z_0^1 = z_1, \ldots, Z_0^n = z_n$, and where B_t is a one-dimensional standard Brownian motion. The solution exists at least up to some small t. As above, let $g_t(z)$ be the solution to (1.1). Then the family of conformal maps g_t is called SLE(κ, ρ) in the upper half-plane from $(0, z_1, \ldots, z_n)$ to ∞. In this paper we will show that SLE(κ, ρ) arises naturally when one considers random growing compacts in polygons, i.e., that the particular drift of W_t in (1.2) can be derived from purely geometric considerations. SLE(κ, ρ), its properties and relation to SLE have been studied in several papers; see [8], [15], [4]. The recent paper [14] extends SLE(κ, ρ) to interaction with interior 'force points.'

2. Schwarz-Christoffel formula

Let D be a bounded simply connected domain whose boundary is a closed polygonal line without self-intersections. Let the consecutive vertices be $p_1, \ldots, p_n$ in positive cyclic order. The angle at p_k is given by the value of $\arg(p_{k-1} - p_k)/(p_{k+1} - p_k)$ between 0 and 2π (we set $p_{n+1} = p_1$). Denote this angle by $\alpha_k \pi$, $0 < \alpha_k < 2$. We also introduce the outer angles $\beta_k \pi = (1 - \alpha_k)\pi$, $-1 < \beta_k < 1$, and note that

$$\beta_1 + \cdots + \beta_n = 2. \tag{2.1}$$

The polygon is convex if and only if all $\beta_k > 0$. We will call the pairs (p_k, β_k) the *corners* of the polygon.

Let f be a conformal map from D onto the upper half-plane $\mathbb{H}$ and let $z_k = f(p_k)$. Assume that none of the z_k equals ∞. For $z \in \mathbb{H}$ define the Schwarz-Christoffel mapping

$$SC(z) = SC\left[\begin{array}{c|c} z_1, \ldots, z_n & z \\ \beta_1, \ldots, \beta_n & z^* \end{array}\right] = \int_{z^*}^{z} \prod_{k=1}^{n} (z - z_k)^{-\beta_k}\; dz, \tag{2.2}$$

where the powers $(z - z_k)^{-\beta_k}$ denote analytic branches in $\mathbb{H}$. Note that

$$SC'(z) = \prod_{k=1}^{n} (z - z_k)^{-\beta_k}, \tag{2.3}$$

and

$$\frac{SC''(z)}{SC'(z)} = -\sum_{k=1}^{n} \frac{\beta_k}{z - z_k}. \tag{2.4}$$

Then it is well known that for some constants $a \neq 0, b \in \mathbb{C}$,

$$f^{-1} = aSC + b;$$

see [1]. We also note for future reference that for any $\lambda > 0$

$$SC\left[\begin{array}{c|c} \lambda z_1, \ldots, \lambda z_n & \lambda z \\ \beta_1, \ldots, \beta_n & \lambda z^* \end{array}\right] = \frac{1}{\lambda} SC\left[\begin{array}{c|c} z_1, \ldots, z_n & z \\ \beta_1, \ldots, \beta_n & z^* \end{array}\right]. \tag{2.5}$$

This is a consequence of (2.1).

3. Loewner evolution in a polygon

Let D be a polygon with corners $(p_1, \beta_1), \ldots, (p_n, \beta_n)$ as above. Let u and χ be two points on the boundary of D which are not vertices. Then there is a conformal map f from D onto $\mathbb{H}$ such that $f(u) = 0$ and $f(\chi) = \infty$. Any other such map is of the form λf for some $\lambda > 0$. Via a translation and rotation we can move D into position so that $u = 0$ and $\{z \in \mathbb{H} : |z| < \epsilon\} \subset D$ for some $\epsilon > 0$. In particular, the edge containing 0 is real. Assume now that D is in such a position. We choose f so that f^{-1} is given by the Schwarz-Christoffel map

$$z \in \mathbb{H} \mapsto SC\left[\begin{array}{c|c} z_1, \ldots, z_n & z \\ \beta_1, \ldots, \beta_n & 0 \end{array}\right] \in D, \tag{3.1}$$

where $z_k = f(p_k)$.

If γ is a simple curve in D from u to χ and γ' is a subarc of γ from u to $u' \in D$, then there is a unique conformal map g from $\mathbb{H}\backslash f \circ \gamma'$ onto $\mathbb{H}$ such that $\lim_{z\to\infty} g(z) - z = 0$. The expansion of g at infinity is $g(z) = z + 2t/z + o(1/|z|)$, where t is a positive real number. If γ'' is a subarc from u to u'' strictly contained in γ', and $\tilde{g}$ the conformal map from $\mathbb{H}\backslash f \circ \gamma''$ onto $\mathbb{H}$ with expansion $\tilde{g}(z) = z + 2s/z + o(1/|z|)$ at infinity, then $s < t$ and in fact we may parametrize $t \in [0, \infty) \mapsto \gamma(t)$ so that $g_t : \mathbb{H}\backslash f \circ \gamma[0, t] \to \mathbb{H}$ has expansion

$$g_t(z) = z + 2t/z + o(1/|z|), \quad z \to \infty.$$

This is known as parametrization by half-plane capacity; see [7]. Let

$$z_t^k = g_t(z_k)$$

and set

$$f_t = SC\left[\begin{array}{c|c} z_t^1, \ldots, z_t^n & \cdot \\ \beta_1, \ldots, \beta_n & 0 \end{array}\right] \circ g_t \circ SC\left[\begin{array}{c|c} z_1, \ldots, z_n & \cdot \\ \beta_1, \ldots, \beta_n & 0 \end{array}\right]^{-1}.$$

Then f_t maps $D\backslash\gamma[0,t]$ conformally onto the polygon D_t with vertices

$$p_t^k = SC\left[\begin{array}{c} z_t^1,\ldots,z_t^n \\ \beta_1,\ldots,\beta_n \end{array}\middle|\begin{array}{c} z_t^k \\ 0 \end{array}\right].$$

This parametrization is natural in the sense that $s \in [0,\infty) \mapsto f_t \circ \gamma(t+s)$ is parametrized by half-plane capacity in D_t, if $t \in [0,\infty) \mapsto \gamma(t)$ is parametrized by half-plane capacity in D. Indeed, this follows readily from the fact that it is true for parametrization by half-plane capacity in the upper half-plane itself and the commutative diagram

$$\begin{array}{ccccc} D\backslash\gamma[0,t+s] & \xrightarrow{f_t} & D_t\backslash f_t\circ\gamma[t,t+s] & \xrightarrow{f_{t+s}\circ f_t^{-1}} & D_{t+s} \\ {\scriptstyle SC}\uparrow & & {\scriptstyle SC_t}\uparrow & & {\scriptstyle SC_{t+s}}\uparrow \\ \mathbb{H}\backslash SC^{-1}\circ\gamma[0,t+s] & \xrightarrow{g_t} & \mathbb{H}\backslash g_t\circ SC^{-1}\circ\gamma[t,t+s] & \xrightarrow{g_{t+s}\circ g_t^{-1}} & \mathbb{H} \end{array}$$

Here

$$SC = SC\left[\begin{array}{c} z_1,\ldots,z_n \\ \beta_1,\ldots,\beta_n \end{array}\middle|\begin{array}{c} \cdot \\ 0 \end{array}\right], \text{ and } SC_t = SC\left[\begin{array}{c} z_t^1,\ldots,z_t^n \\ \beta_1,\ldots,\beta_n \end{array}\middle|\begin{array}{c} \cdot \\ 0 \end{array}\right].$$

From Loewner's equation in the upper half-plane we get

$$\partial_t z_t^k = \partial_t g_t(z_k) = \frac{2}{z_t^k - w_t},$$

where $w_t = g_t(f \circ \gamma(t))$. Set $u_t = SC_t(w_t)$. Then

$$\begin{aligned} \partial_t f_t(\zeta) &= (\partial_t SC_t)\left[\left(SC_t^{-1}\circ f_t\right)(\zeta)\right] \\ &\quad + \frac{2}{\left(SC_t^{-1}\right)'(f_t(\zeta))\left[SC_t^{-1}(f_t(\zeta)) - SC_t^{-1}(\eta_t)\right]} \\ &\equiv \Xi(f_t(\zeta),\eta_t;\zeta_t^1,\ldots,\zeta_t^n). \end{aligned} \tag{3.2}$$

Explicitly,

$$\begin{aligned} \Xi(\zeta,u;p_1,\ldots,p_n) = &\int_0^z \prod_{k=1}^n (v-z_k)^{-\beta_k} \sum_{l=1}^n \frac{2\beta_l}{(v-z_l)(z_l-w)}\,dv \\ &+ \frac{2}{z-w}\prod_{k=1}^n (z-z_k)^{-\beta_k}, \end{aligned} \tag{3.3}$$

The vectorfield

$$\zeta \in D \mapsto \Xi(\zeta,u;p_1,\ldots,p_n) \in \mathbb{C}$$

defined in (3.2) has residue

$$\frac{2}{\left[\left(SC^{-1}\right)'(u)\right]^2} = 2\prod_{k=1}^n (w-z_k)^{-2\beta_k}$$

at $\zeta = u$. Also, for $\lambda > 0$,

$$\Xi(\lambda\zeta, \lambda u; \lambda p_1, \dots, \lambda p_n) = \lambda^3 \Xi(\zeta, u; p_1, \dots, p_n).$$

If we change time $s = s(t)$ so that

$$\frac{\partial s}{\partial t} = \left[\left(SC_t^{-1}\right)' (u_t) \right]^{-2}$$

and let $\tilde{\gamma}(s) = \gamma(t)$, $\tilde{f}_s = f_t$, $\tilde{g}_s = g_t$, $\tilde{u}_s = u_t$, and $\widetilde{SC}_s = SC_t$, then

$$\partial_s \tilde{f}_s(\zeta) = \left(\partial_s \widetilde{SC}_s\right) \left[\left(\widetilde{SC}_s^{-1} \circ \tilde{f}_s \right)(\zeta) \right] + \frac{2\left[\left(\widetilde{SC}_s^{-1}\right)' (\tilde{u}_s) \right]^2}{\left(\widetilde{SC}_s^{-1}\right)' (\tilde{f}_s(\zeta)) \left[\widetilde{SC}_s^{-1}(\tilde{f}_s(\zeta)) - \widetilde{SC}_s^{-1}(\tilde{u}_s) \right]}. \tag{3.4}$$

The vectorfield on the right is given by

$$\zeta \in D \mapsto \Xi(\zeta, u; p_1, \dots, p_n) \prod_{k=1}^{n} (w - z_k)^{2\beta_k}.$$

Finally, if

$$\tilde{\tilde{f}}_s = \tilde{f}_s - \int_0^s \left(\partial_r \widetilde{SC}_r\right) \left[\widetilde{SC}_r^{-1} \circ \tilde{f}_r(\tilde{u}_r) \right] dr, \tag{3.5}$$

then the vectorfield $\zeta \mapsto \Xi(\zeta, u; p_1, \dots, p_n)$ defined by

$$\partial_s \tilde{\tilde{f}}_s(\zeta) = \Xi\left(\tilde{\tilde{f}}_s(\zeta), \tilde{\tilde{\eta}}_s; \tilde{\tilde{\zeta}}_s^1, \cdots, \tilde{\tilde{\zeta}}_s^n \right), \tag{3.6}$$

has residue 2 at $\zeta = u$ and

$$\lim_{\zeta \to \eta} \Xi(\zeta, \eta; \zeta_1, \dots, \zeta_n) - \frac{2}{\zeta - \eta} = 0. \tag{3.7}$$

Explicitly,

$$\begin{aligned} \Xi(\zeta, u; p_1, \dots, p_n) &= \prod_{k=1}^{n} (w - z_k)^{2\beta_k} \int_w^z \prod_{l=1}^{n} (v - z_k)^{-\beta_k} \sum_{m=1}^{n} \frac{2\beta_m}{(v - z_m)(z_m - w)} \, dv \\ &\quad + \frac{2}{z - w} \prod_{k=1}^{n} (z - z_k)^{-\beta_k} (w - z_k)^{2\beta_k}. \end{aligned} \tag{3.8}$$

In particular

$$\Xi(\lambda\zeta, \lambda u; \lambda p_1, \dots, \lambda p_n) = \frac{1}{\lambda} \Xi(\zeta, u; p_1, \dots, p_n).$$

Note that all scaling relations follow from the basic relation (2.5).

To make the notation less cumbersome, we now drop the symbols $\sim$ while still referring to the quantities defined in (3.5), (3.6). We then have

THEOREM 3.1 (Loewner equation in a polygon). *If γ is a simple curve in a polygon D (positioned as above) connecting boundary points 0 and χ which are not vertices of D, then there is a unique parametrization*

$$t \in (0,\infty) \mapsto \gamma(t) \in D$$

such that there is (1) *a family f_t of conformal maps from $D \backslash \gamma(0,t]$ onto a polygon D_t with the same angles as D and mapping vertices to vertices and* (2) *a vectorfield Ξ satisfying* (3.7)*, so that for every $\zeta \in D$*

$$\partial_t f_t(\zeta) = \Xi(f_t(\zeta), u_t; p_t^1, \dots, p_t^n), \quad f_0(\zeta) = \zeta.$$

Here $u_t = f_t(\gamma(t))$, $p_t^k = f_t(p_k)$, $k = 1, \dots, n$.

Property (3.7) of the vectorfield Ξ in the theorem says that near its singularity the vectorfield looks—to first order—like the vectorfield for the chordal Loewner equation in the upper half-pane. The term

$$\frac{2\left[\left(SC^{-1}\right)'(\eta)\right]^2}{\left(SC^{-1}\right)'(\zeta)\left[SC^{-1}(\zeta) - SC^{-1}(\eta)\right]}$$

in the definition of Ξ (see (3.6), (3.4)) is nothing but the variation kernel of Schiffer and Spencer for the sphere when viewed in polygonal coordinates; see [12]. The variation kernel transforms under a change of coordinates like a reciprocal differential (i.e., holomorphic vectorfield) in the ζ coordinate—which explains the factor before the square bracket in the denominator—and transforms like a quadratic differential in the η-coordinate—which explains the numerator. It is thus natural to consider Ξ as the *variation kernel* for a polygon with corners $(p_1, \beta_1), \dots, (p_n, \beta_n)$.

THEOREM 3.2 (Loewner evolution in a polygon). *If $t \in [0,\infty) \mapsto u_t \in \mathbb{R}$ is smooth with $u_0 = 0$ and $\chi \neq 0$ is a boundary point of a polygon D positioned as above, then there exists* (1) *a simple curve $t \in (0,\infty) \mapsto \gamma(t) \in D$ and* (2) *a family f_t of conformal maps from $D \backslash \gamma(0,t]$ onto a polygon D_t with the same angles as D and mapping vertices to vertices, such that for every $\zeta \in D$*

$$\partial_t f_t(\zeta) = \Xi(f_t(\zeta), u_t; p_t^1, \dots, p_t^n), \quad f_0(\zeta) = \zeta.$$

Here Ξ is the vectorfield defined in (3.6) *and $\gamma(0) = 0$, $\lim_{t\to\infty} \gamma(t) = \chi$.*

Note that the endpoint χ enters in the definition of Ξ via SC. The property (3.7) of the vectorfield Ξ means here that to first order the slit $\gamma \in D$ obtained by solving the Loewner equation for η_t in a polygon grows the same way as the slit $\tilde{\gamma} \in \mathbb{H}$ obtained by solving the chordal Loewner equation in $\mathbb{H}$ with the same driving function η_t.

Proof. The theorem follows from the corresponding result for the chordal Loewner evolution in the upper half-plane and the fact that the *conformal parameters* p_t^k, $k = 1, \dots, n$, can be obtained by solving the system

$$\partial_t p_t^k = \Xi(p_t^k, u_t; p_t^1, \dots, p_t^n), \quad k = 1, \dots, n, \tag{3.9}$$

with initial condition $p_0^k = p_k$, $k = 1, \dots, n$. □

4. Stochastic Loewner evolution in a polygon and SLE(κ, ρ)

4.1. Conformally invariant measures. Let D be a polygon and u, χ points on the boundary of D minus its vertices. Suppose for each such triple (D, u, χ), we are given a random simple curve $\gamma = \gamma_{D,u\to\chi}$ in D from u to χ which is parametrized as in Theorem 3.1. Suppose further that the laws of the random simple curves γ are related as follows:

(1) If $a \neq 0, b \in \mathbb{C}$, $D' = aD + b$, $u' = au + b$, and $\chi' = a\chi + b$, then $a\gamma_{D,u\to\chi} + b$ has the same law as a timechange of $\gamma_{D',u'\to\chi'}$.
(2) If f_t is as in Theorem 3.1, and $D_t = f_t(D\backslash\gamma(0, t])$, $U_t = f_t(\gamma_{D,u\to\chi}(t))$, $\chi_t = f_t(\chi)$, then, conditional on $\gamma[0, t]$, $\{f_t \circ \gamma_{D,u\to\chi}(t + s) : s \geq 0\}$ has the same law as $\{\gamma_{D_t,U_t\to\chi_t}(s) : s \geq 0\}$.

Statement (1) is the invariance under certain conformal maps, namely linear transformations, and (2) is a combination of conformal invariance and the domain-Markovian property familiar from SLE.

By Theorem 3.1 and Theorem 3.2, knowing $\{\gamma_{D,u\to\chi}(t) : t \geq 0\}$ is equivalent to knowing $\{U_t, \chi_t, P_t^1, \dots, P_t^n)$, i.e., the random curve

$$t \in (0, \infty) \mapsto \gamma_{D,u\to\chi}(t) \in D$$

gives rise to a random process

$$t \in [0, \infty) \mapsto (U_t, \chi_t, P_t^1, \dots, P_t^n),$$

with $(U_0, \chi_0; P_0^1, \dots, P_0^n) = (u, \chi; p_1, \dots, p_n)$, and, conversely, we can recover $\gamma_{D,u\to\chi}$ from the process $(U_t, \chi_t, P_t^1, \dots, P_t^n)$. Note that the image of the endpoint of the curve γ, that is, χ_t, carries no additional information. It is determined as the image of ∞ under SC_t (only its initial value, χ, is required). In terms of the process $(U_t, \chi_t, P_t^1, \dots, P_t^n)$ the statement (2) is equivalent to the following statement:

(2') Conditioned on $\{(U_r, \chi_r, P_r^1, \dots, P_r^n) : r \leq t\}$, the law of

$$\{(U_{t+s}, \chi_{t+s}, P_{t+s}^1, \dots, P_{t+s}^n) : s \geq 0\},$$

where $(U_0, \chi_0, P_0^1, \dots, P_0^n) = (u, \chi, p_1, \dots, p_n)$, is equal to the law of

$$\{(\tilde{U}_s, \tilde{\chi}_s, \tilde{P}_s^1, \dots, \tilde{P}_s^n) : s \geq 0\},$$

an independent process with $(\tilde{U}_0, \tilde{\chi}_0, \tilde{P}_0^1, \dots, \tilde{P}_0^n) = (U_t, \chi_t, P_t^1, \dots, P_t^n)$.

But this is simply saying that the process $(U_t, \chi_t, p_t^1, \ldots, p_t^n)$ is a Markov process. Note that although it is possible to recover γ from $u_t, t \geq 0$, and $\chi, p_1, \ldots, p_n$, we cannot conclude that u_t is a Markov process, because knowledge of D_t requires knowledge of u_s for all $s \leq t$; see (3.9). The same situation arises in the case of multiply connected domains; see [2].

We now study what (1) says about the Markov process $(U_t, \chi_t, p_t^1, \ldots, p_t^n)$. If γ is a simple curve in D connecting two points on the boundary and $t \in (0, \infty) \mapsto \gamma(t) \in D$ is its natural parametrization as defined in Theorem 3.1, then, for any $\lambda > 0$, $\lambda\gamma$ is a simple curve in λD and

$$t \in (0, \infty) \mapsto \lambda\gamma(t/\lambda^2) \in \lambda D$$

is its natural parametrization. Indeed, the curve in its natural parametrization is created by the variation kernel, which transforms as a quadratic differential in the variable giving the singularity, see above.

For the Markov process $(U_t, \chi_t, p_t^1, \ldots, p_t^n)$ statement (1) thus implies that it has Brownian scaling. Since

$$dU_t = a(U_t, \chi_t; P_t^1, \ldots, P_t^n)\, dB_t + b(U_t, \chi_t; P_t^1, \ldots, P_t^n)\, dt,$$

for some coefficients a, b, it follows that for any $\lambda > 0$,

$$\begin{aligned} a(\lambda u, \lambda\chi; \lambda p_1, \ldots, \lambda p_n) &= a(u, \chi; p_1, \ldots, p_n), \\ b(\lambda u, \lambda\chi; \lambda p_1, \ldots, \lambda p_n) &= \frac{1}{\lambda} b(u, \chi; p_1, \ldots, p_n). \end{aligned} \tag{4.1}$$

The simplest nontrivial case is

$$a \equiv \sqrt{\kappa} \text{ and } b \equiv 0,$$

for some positive constant $\kappa \leq 4$, and we will show that this corresponds to a timechange of SLE(κ, ρ) when viewed in the upper half-plane.

4.2. SLE(κ, ρ) as polygon motion. Let $SC : \mathbb{H} \to D$ and $z_1, \ldots, z_n$ be defined as in (3.1), and set

$$\rho_k = \frac{\kappa}{2}\beta_k, \quad k = 1, \ldots, n. \tag{4.2}$$

Then $-\kappa/2 \leq \rho_k \leq \kappa/2$. Suppose that $(W_t, Z_t^1, \ldots, Z_t^n)$ is a solution to (1.2). For z in the upper half-plane, set

$$SC_t(z) = SC\left[\begin{array}{c|c} Z_t^1, \ldots, Z_t^n & z \\ \beta_1, \ldots, \beta_n & 0 \end{array}\right].$$

Then $z \mapsto SC_t(z)$ extends continuously to the real axis with the points Z_t^k removed and is differentiable there as a function of t. In particular, if $W_s \neq Z_s^1, \ldots, Z_s^n$ for $s \in [0, t]$, then we may define

$$h_t(z) = SC_t(z) - \int_0^t (\partial_s SC_s)(W_s)\, ds. \tag{4.3}$$

Define the stopping time σ by

$$\sigma = \sup\{t : W_s, Z_s^1, \dots, Z_s^n \text{ are all distinct for } 0 \le s \le t\}.$$

LEMMA 4.1. *The process $U_t \equiv h_t(W_t)$ is a martingale for $t < \sigma$. Furthermore, if*

$$A_t \equiv \kappa \int_0^t \left(SC_s'(W_s)\right)^2 \; ds$$

and $\tau(t)$ is defined by $A_{\tau(t)} = t$, then $t \mapsto U_{\tau(t)}$ is a standard Brownian motion.

Proof. By an appropriate Itô formula [10],

$$dU_t = (\partial_t h_t)(W_t)\; dt + h_t'(W_t)\; dW_t + \frac{\kappa}{2} h_t''(W_t)\; dt.$$

Thus (4.3), (1.2), and (2.4) imply

$$dU_t = \sqrt{\kappa} SC_t'(W_t)\; dB_t.$$

By (2.3), $0 < |SC_t'(W_t)| < \infty$ for $t < \sigma$, and the lemma follows. □

THEOREM 4.2. *Let $\gamma_{D,0\to\chi}$ be the random simple curve obtained by solving Loewner's equation in the polygon D for the process $(\sqrt{\kappa} B_t, \chi_t; P_t^1, \dots, P_t^n)$. Then $SC^{-1} \circ \gamma_{D,0\to\chi}$ is a timechange of SLE(κ,ρ) with*

$$\rho_k = \frac{\kappa}{2}\beta_k \text{ and } z_k = SC^{-1}(p_k), \quad k = 1, \dots, n.$$

Proof. This follows by noting that $h_t \circ g_t \circ SC^{-1}$ is a timechange of $\tilde{\tilde{f}}_t$, defined in (3.5). □

REMARK 4.3. Note that the integral term in the definition of h_t in (4.3) is precisely what is required to make both $\gamma_{D,0\to\chi}$ and $SC^{-1} \circ \gamma_{D,0\to\chi} \in \mathbb{H}$ grow according to a vector field with expansion $const./z + O(|z|)$ at its singularity, and a simple timechange then gives the singularity $2/z + O(|z|)$; see (3.7). If instead of starting with SLE(κ, ρ) we had begun with another diffusion $(W_t, Z_t^1, \dots, Z_t^n)$ we could always choose an integral drift term for a map $\tilde{h}_t$ from $\mathbb{H}$ onto a polygon $\tilde{D}$ sending Z_t^k to vertices, so that $\tilde{h}_t(W_t)$ is a martingale. However, in that case, not both curves $\gamma_{\tilde{D}}$ and $\gamma_{\mathbb{H}}$ would grow according to a vectorfield with expansion $const./z + O(|z|)$ at its singularity.

REMARK 4.4. If we begin with an arbitrary SLE(κ, ρ), i.e., we begin with a choice of $z_1, \dots, z_n$ and $\rho_1, \dots, \rho_n$, then the results of this section continue to hold. In this case the Schwarz-Christoffel mapping SC is no longer guaranteed to be one-to-one. However, it still maps the intervals $[z_k, z_{k+1}]$ onto straight line segments. By considering the Riemann surface of the analytic function SC we can still interpret the image $SC(\mathbb{H})$ as a polygon, albeit not a planar one. For example, SLE$(2,(-1,-1))$, up to a normalization, leads to the map $z^3 - 3z$ which is easily understood in terms of a 3-fold cover; see [1].

We close this section with an expression in terms of domain functionals for the evolution equation of the logarithmic derivative of the Loewner mappings in a polygon D. Denote by

$$(q, u) \in D \times \partial D \mapsto k_D(q, u)$$

the Poisson kernel of D. If $p \in \partial D$, denote by $\partial_2 H_{D,p}(q, u)$ the analytic function in q whose real part is $\partial_2 k_D(q, u)$ and which satisfies

$$\lim_{q \to p} \partial_2 H_{D,p}(q, u) = 0.$$

THEOREM 4.5. *Denote by K_t the hull of an SLE(κ, ρ) in the upper half-plane and $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ the normalized uniformizing map. Then*

$$f_t \equiv h_t \circ g_t \circ SC^{-1} : D\backslash SC(K_t) \to D_t$$

satisfies

$$\partial_t \ln f_t'(z) = h_t(W_t)^2 \partial_2 H_{D_t, h_t(\infty)}(f_t(z), h_t(W_t)). \tag{4.4}$$

Proof. Let $f_t = h_t \circ g_t \circ SC^{-1}$. Then

$$\begin{aligned} f_t'(z) &= h_t'(g_t(SC^{-1}(z)))g_t'(SC^{-1}(z))(SC^{-1})'(z) \\ &= \frac{\prod_{k=1}^n (g_t(SC^{-1}(z)) - Z_t^k)^{-\beta_k} g_t'(SC^{-1}(z))}{\prod_{k=1}^n (SC^{-1}(z) - z_k)^{-\beta_k}}. \end{aligned}$$

Set $w = SC^{-1}(z)$. As $\partial_t g_t'(z) = -2g_t'(z)/(g_t(z) - W_t)^2$, straightforward computation gives

(4.5)

$$\begin{aligned} \partial_t f_t'(z) &= \prod \left(\frac{g_t(w) - Z_t^k}{w - z_k} \right)^{-\beta_k} g_t'(w) \\ &\quad \times \left[\sum_{l=1}^n \left(\frac{2}{g_t(w) - W_t} - \frac{2}{Z_t^l - W_t} \right) \frac{-\beta_l}{g_t(w) - Z_t^l} - \frac{2}{(g_t(w) - W_t)^2} \right] \\ &= f_t'(z) \left[\frac{-2}{(g_t(w) - W_t)^2} + \frac{2}{(g_t(w) - W_t} \sum_{l=1}^n \frac{\beta_l}{Z_t^l - W_t} \right]. \end{aligned}$$

Now, we note that

$$H_{\mathrm{Pol}_t}(q, u) = h_t'(h_t^{-1}(u))^{-1} H_{\mathbb{H}}(h_t^{-1}(q), h_t^{-1}(u)),$$

whence, if $v_t = h_t^{-1}(u)$,

(4.6)

$$\partial_u H_{\mathrm{Pol}_t}(q, u) = h_t'(v_t)^{-2} \left[-\frac{h_t''(v_t)}{h_t'(v_t)} H_{\mathbb{H}}(h_t^{-1}(q), v_t) + \partial_2 H_{\mathbb{H}}(h_t^{-1}(q), v_t) \right].$$

Since $H_{\mathbb{H}}(z,w) = 2/(z-w)$, the theorem follows. □

5. SLE in variable background metric

Instead of mapping SLE(κ,ρ) into polygons we can also stay in the upper half-plane and change the metric. Indeed, $h_t : \mathbb{H} \to D_t$ is an immersion. If we endow D_t with the Euclidean metric, then the metric induced by h_t on $\mathbb{H}$ is

$$g_{ij} = \delta_{ij}|h_t'(z)|^2, \quad i,j = 1,2,$$

where the indices 1 and 2 refer to the real and imaginary coordinate, respectively. If $\Gamma = (\Gamma^i_{jk})$ denotes the Levi-Civita connection for this metric, then the (2-dimensional) Brownian motion $\tilde{W}$ for the metric (g_{ij}) solves the stochastic differential equation

$$d\tilde{W}^i_s = \sigma^i_j(\tilde{W}_s)\ dB^j_s - \frac{1}{2} g^{kl}(\tilde{W}_s)\Gamma^i_{kl}(\tilde{W}_s)\ ds;$$

see [6]. Here $g^{-1} = (g^{kl})$ is the inverse coefficient matrix of g and σ is a square root of g^{-1} (i.e., $\sigma\sigma^T = g^{-1}$), and we observe the Einstein summation convention according to which indices occurring once "upstairs" and once "downstairs" are to be summed over. For our particular metric g we find

$$\Gamma^1_{11} = \Gamma^1_{22} = -\Re\left(\frac{h_t''}{h_t'}\right);$$

see [3]. The boundary $\mathbb{R} = \partial\mathbb{H}$ is a one-dimensional sub-manifold of $\overline{\mathbb{H}}$. The metric g on $\mathbb{H}$ thus induces the metric $(h_t'(x))^2\ dx^2$ on $\mathbb{R}$. A (one-dimensional) Brownian motion W relative to this metric solves the stochastic differential equation

$$dW_s = \frac{dB_s}{h_t'(W_s)} - \frac{1}{2(h_t'(W_s))^2} \sum_{k=1}^{n} \frac{\beta_k}{W_s - Z^k_t}\ ds. \tag{5.1}$$

We now couple the metric to the Brownian motion W via

$$dZ^k_t = \frac{2}{\kappa(h_t'(W_t))^2(Z^k_t - W_t)}\ dt, \quad k = 1,\ldots,n. \tag{5.2}$$

Then, after a time-change, (5.1) and (5.2) become the SLE(κ,ρ)-system (1.2) with the convention $\rho_k = \kappa\beta_k/2$.

References

[1] L. V. Ahlfors, *Complex analysis*, McGraw-Hill Book Co., New York, 1978. MR 510197 (80c:30001)

[2] R. Bauer and R. Friedrich, *On chordal and bilateral SLE in multiply connected domains*, preprint, `arXiv:math.PR/0503178`.

[3] M. P. do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992. MR 1138207 (92i:53001)

[4] J. Dubédat, SLE(κ,ρ) *martingales and duality*, Ann. Probab. **33** (2005), 223–243. MR 2118865 (2005j:60180)

[5] P. Henrici, *Applied and computational complex analysis*, Wiley-Interscience, New York, 1974. MR 0372162 (51 #8378)
[6] E. P. Hsu, *Stochastic analysis on manifolds*, Graduate Studies in Mathematics, vol. 38, American Mathematical Society, Providence, RI, 2002. MR 1882015 (2003c:58026)
[7] G. F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, vol. 114, American Mathematical Society, Providence, RI, 2005. MR 2129588
[8] G. Lawler, O. Schramm, and W. Werner, *Conformal restriction: the chordal case*, J. Amer. Math. Soc. **16** (2003), 917–955 (electronic). MR 1992830 (2004g:60130)
[9] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*, Math. Ann. **89** (1923), 103–121. MR 1512136
[10] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Grundlehren der Mathematischen Wissenschaften, vol. 293, Springer-Verlag, Berlin, 1999. MR 1725357 (2000h:60050)
[11] S. Rohde and O. Schramm, *Basic properties of SLE*, Ann. of Math. (2) **161** (2005), 883–924. MR 2153402 (2006f:60093)
[12] M. Schiffer and D. C. Spencer, *Functionals of finite Riemann surfaces*, Princeton University Press, Princeton, N. J., 1954. MR 0065652 (16,461g)
[13] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288. MR 1776084 (2001m:60227)
[14] O. Schramm and D. Wilson, *SLE coordinate changes*, preprint, `arXiv:math.PR/0505368`.
[15] W. Werner, *Girsanov's transformation for* SLE(κ, ρ) *processes, intersection exponents and hiding exponents*, Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), 121–147. MR 2060031 (2005b:60262)

Robert O. Bauer, Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: `rbauer@math.uiuc.edu`

Roland M. Friedrich, Max-Planck Institut für Mathematik, D-53111 Bonn, Germany
E-mail address: `rolandf@mpim-bonn.mpg.de`