A CHARACTERIZATION OF CERTAIN REGULAR d-CLASSES IN SEMIGROUPS

BY

R. J. WARNE

The concept of a d-class in a semigroup was introduced and investigated by Green [3]. The importance of this concept in the study of semigroups is indicated in [2]. Regular d-classes have been studied by Miller and Clifford [2], [4]. A semigroup consisting of a single d-class is called bisimple. Clifford [1] has determined the structure of all bisimple inverse semigroups with identity.

Let S be a semigroup and A a non-empty subset of S. Let E_A denote the collection of idempotents of A. E_A may be partially ordered as follows: $e \leq f$ if and only if $ef = fe = e$. We characterize regular d-classes D for which E_D is linearly ordered and we determine the structure of bisimple inverse semigroups G for which E_G is linearly ordered. The connection between certain regularity conditions [2] and the linear ordering of idempotents is considered.

Two elements of S are said to be R-(L-) equivalent if they generate the same principal right (left) ideal. Two elements a, b of S are d-equivalent if there exists x in S such that $a\ R\ x$ and $x\ L\ b$ (or equivalently there exists y in S such that $a\ L\ y$ and $y\ R\ b$). An element a in S is called right (left) regular if $a\ R\ a^2$ ($a\ L\ a^2$). a is called biregular if it is either right regular or left regular. S is said to be biregular if all its elements are biregular. An element a in S is regular if a in aSa. A subset of S is regular if all its elements are regular. A regular semigroup in which the idempotents commute is called an inverse semigroup [2], [5].

Let e be an idempotent element of S. $P_e(Q_e)$ will denote the right (left) unit subsemigroup of eSe (the set of elements of eSe having a right (left) inverse with respect to e the identity of eSe). H_e will denote the group of units of eSe.

By a decomposition of S we mean a partition of S into a union of disjoint subsemigroups.

S^1 will denote S with an appended identity [2, p. 4].

LEMMA. *Let S be a bisimple inverse semigroup. Then E_S is linearly ordered if and only if S is biregular.*

Proof. Suppose that E_S is linearly ordered. Then, if a in S there exist e, f in E_D such that $a\ R\ e$ and $a\ L\ f$ [2], [4]. If $ef = fe = e$, then $aea\ L\ fea$ or $a^2\ L\ a$ [2], [4]. If $ef = fe = f$, $a^2\ R\ a$. Conversely, suppose that S is biregular. If e, f in E_D, there exists a in D such that $e\ R\ a$ and $a\ L\ f$. Hence, if $a\ R\ a^2$ then $a\ R\ ae$. Since $a\ L\ f$, there exists x in S such that $xa = f$. Thus, $xa\ R\ xae$

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implies $f\ R\ fe$. Hence, $f = fe = ef$ [2], [4]. In a similar fashion, $a\ L\ a^2$ implies $ef = fe = e$.

THEOREM. *The following four conditions are equivalent on any d-class D of a semigroup*:

(A) *D is regular and E_D is linearly ordered.*

(B) *D is a biregular, bisimple, inverse semigroup.*

(C) *D is the union of a chain of bisimple, biregular inverse semigroups with identity.*

(D) *D is a group or the idempotents of D commute and D has the following decomposition into groups, right cancellative semigroups without idempotent and left cancellative semigroups without idempotent*:

$$(*) \qquad D = \bigcup\{H_e \cup (P_e - H_e) \cup (Q_e - H_e) : e \in E_D\}$$

Proof. (A) $\Rightarrow$ (B). If a, b in D, there exists f, h in E_D such that $a\ L\ f$ and $b\ R\ h$ [2], [4]. Thus, $ab\ R\ ah$ and $ah\ L\ fh$. Hence, $ab\ d\ fh$ and D is a semigroup. D is therefore a bisimple inverse semigroup [2, p. 62]. Thus, D is biregular by the lemma.

(B) $\Rightarrow$ (C). E_D is linearly ordered by the lemma. If a in eDe there exists x in D such that $e\ R_D\ x$ (R_D denotes R-equivalence on D) and $x\ L_D\ a$. There exists f in E_D such that $x\ L_D\ f$ [4], [2] and hence one may find u, v in D such that $f = ux$ and $x = vf$. If $fe = f$, $xe = vfe = vf = x$ and x in eDe. Thus, there exist r, p, q in D such that $e = x(ere)$, $a = (epe)x$, $x = (eqe)a$ and eDe is thus bisimple. If $ef = e$, $e = eux = (euqe)a$ and eDe is again bisimple. Since eDe is clearly an inverse semigroup, eDe is biregular by the lemma. If a in D, there exist e, f in E_D such that a in eDf. Clearly, $eDf \subseteq eDe$ or fDf and hence $D = \bigcup(eDe : e \text{ in } E_D)$. If e, f in E_D, $eDe \subseteq fDf$ or $fDf \subseteq eDe$.

(C) $\Rightarrow$ (D). $P_e - H_e(Q_e - H_e)$ is a right (left) cancellative semigroup without idempotent. It follows from the lemma that E_D is linearly ordered. Clearly, D is regular. If a in D, there exists f, g in E_D such that $a\ R\ f$ and $a\ L\ g$ [4], [2]. If $gf = g$, $af = agf = ag = a$ and a in fSf [4],[6]. There exists y in S^1 such that $a(fyf) = f$ and a in P_f [4], [2]. If $gf = f$, a in Q_g. Thus, the equality $(*)$ is satisfied. If the second or third factor of $(*)$ is empty, D is a group. If, for example, the third factor is empty, $Q_e \subseteq P_e$ and $P_e = H_e = Q_e$ for all e in E_D. Thus, $D_e = R_e = H_e = L_e$. Otherwise, a in $(P_e - H_e) \cap (Q_f - H_f)$ implies a in $eSe \cap fSf$ and there exist z_1, z_2 in S such that $az_1 = e$ and $z_2\, a = f$. If $fe = e$, $z_2\, ae = fe = e$ and $z_2\, a = e = f$. If $fe = f$, $az_1 = f = e$ and we have a contradiction in both cases. If a in $P_e \cap P_f$, $a\ R\ e$ and $a\ R\ f$; i.e. $e\ R\ f$. Since the idempotents of D commute $e = f$. Similarly, a in $Q_e \cap Q_f$ implies $e = f$.

(D) $\Rightarrow$ (A). If a in P_e, a in eSe and there exists y in Q_e such that $ay = e$. Thus, $aya = ea = a$ and $a^2y = ae = a$. Thus, a is regular and right regular. If a in Q_e, a is regular and left regular. If a in D, a^2 in D. If x, y in D, there exists a in D such that $a\ R\ y$ and $a\ L\ x$. Thus, $a^2\ R\ ay$ and $ay\ L\ xy$ [4], [2].

Hence, $xy\ d\ a^2da$ and D is a semigroup and therefore a bisimple inverse semigroup [2, p. 62]. Thus, E_D is linearly ordered by the lemma, Q.E.D.

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Virginia Polytechnic Institute
Blacksburg, Virginia
West Virginia University
Morgantown, West Virginia