ARITHMETIC PROBLEMS CONCERNING CAUCHY'S FUNCTIONAL EQUATION II[1]

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I. Introduction

1. Statement of problem and main result. In a previous paper [4] of the same title the authors have studied the real-valued *monotone* solutions $f(x)$ of the functional equation

$$(1.1)\quad f\left(\textstyle\sum_1^m u_i\,\alpha_i\right) = \textstyle\sum_1^m f(u_i\,\alpha_i) \qquad (u_i \text{ arbitrary non-negative integers}),$$

under various assumptions on m and the real constants α_i. In the present sequel to [4], which does not assume a knowledge of [4], we propose to study the *uniformly continuous* solutions of (1.1). Although some of the features of [4] will again appear in the present situation, the methods now required are different and they also permit a setting of the problem in higher dimensions.

Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be elements of the real n-dimensional space R^n $(n < m)$ satisfying the following conditions:

1. Every set of n among the α_i are linearly independent over the real field.
2. The elements $\alpha_1, \cdots, \alpha_m$ are rationally independent, i.e., linearly independent over the rational field.

Let $f(x)$ denote a solution of (1.1) having values in the Banach space B. Such a solution needs to be defined only on the set

$$(1.2)\qquad S = \left\{x = \textstyle\sum_1^m u_i\,\alpha_i \mid u_i \text{ integers } \geqq 0\right\}.$$

Without further conditions on $f(x)$ the problem is of little interest for we clearly obtain the most general solution of (1.1) by assigning at will the values of $f(u_i\,\alpha_i)$ for $u_i = 1, 2, \cdots$ and $i = 1, \cdots, m$. We propose, however, to determine those solutions $f(x)$ of (1.1) which are *uniformly continuous* (abbreviated below to UC), i.e. are such that to every ε there corresponds a δ such that

$$\| f(x) - f(y) \| < \varepsilon \quad \text{if} \quad | x - y | < \delta \qquad (x, y \in S).$$

Here we denote by $| \cdots |$ and $\| \cdots \|$ the norms of the spaces R^n and B, respectively.

If $\lambda(x)$ is a linear function from R^n into B then it is clear that $f(x) = \lambda(x)$ is a UC solution of (1.1). Other such solutions are obtained as follows: For every $i = 1, \cdots, m$ we consider the set

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$$(1.3) \quad S_i = \{x = u_i \alpha_i + \textstyle\sum_{j \neq i} k_j \alpha_j \mid u_i \text{ integer} \geqq 0, \ k_j \text{ integers}\}.$$

Observe that S_i has the periods α_j $(j \neq i)$ since $x \in S_i$ implies that $x + \alpha_j \in S_i$. Let the function $\phi_i(x)$ be defined in S_i, with values in B, such that

1°. $\phi_i(0) = 0$,
2°. $\phi_i(x + \alpha_j) = \phi_i(x)$ $(j \neq i; x \in S_i)$,
3°. $\phi_i(x)$ is UC on S_i.

We claim that $\phi_i(x)$ is a solution of (1.1). Indeed, observe that $S \subset S_i$ and that by 1° and 2° we may write

$$\phi_i\left(\textstyle\sum_1^m u_j \alpha_j\right) = \phi_i(u_i \alpha_i) = \phi_i(u_i \alpha_i) + \textstyle\sum_{j \neq i} \phi_i(u_j \alpha_j) = \textstyle\sum_{j=1}^m \phi_i(u_j \alpha_j).$$

Adding together all solutions so far obtained we see that

$$(1.4) \qquad f(x) = \lambda(x) + \textstyle\sum_1^m \phi_i(x) \qquad (x \in S),$$

represents a UC solution of (1.1). Indeed, observe that $S \subset \bigcap_i S_i$ and that (1.1) is a linear relation.

Our aim is to establish the converse

THEOREM 1. *If $f(x)$ is a solution of* (1.1) *which is UC on S then $f(x)$ admits a unique representation of the form* (1.4) *in which $\lambda(x)$ is a linear function from R^n into B, while the $\phi_i(x)$ satisfy the conditions* 1°, 2° *and* 3° *stated above.*

2. Consequences of Theorem 1. Given n, the value of m is crucial in this problem. First of all we required that $m > n$ and for a good reason. Indeed, if $m \leqq n$ and we still assume the $\alpha_1, \cdots, \alpha_m$ to be linearly independent, then the distances between two distinct points of S have a positive lower bound. But then our requirement of uniform continuity becomes meaningless.

Let us now assume that $m = n + 2$. Now $\phi_i(x)$ is to have $n + 1$ periods $\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_{n+2}$ which are rationally independent. From $\phi_i(0) = 0$ we conclude that

$$(2.1) \qquad \phi_i\left(\textstyle\sum_{j \neq i} k_j \alpha_j\right) = 0.$$

However, the arguments of ϕ_i appearing here are dense in R^n; as first observed by Jacobi, the relations (2.1) in conjunction with the uniform continuity of ϕ_i imply that $\phi_i(x) = 0$ if $x \in S_i$ and thus (1.4) reduces to $f(x) = \lambda(x)$. This reasoning is valid a fortiori if $m > n + 2$. This proves

THEOREM 2. *If $m \geqq n + 2$ and if $f(x)$ is a solution of* (1.1) *which is UC on S, then $f(x)$ is the restriction to S of a linear function $\lambda(x)$ from R^n to B.*

We now deal with the only remaining case when $m = n + 1$. The main result for this case will readily appear as soon as we settle the following question: Let $f(x)$ be a solution of (1.1) UC on S. Is it possible to extend $f(x)$ to a UC solution $F(x)$ of the unrestricted functional equation

$$F\left(\textstyle\sum_1^{n+1} k_i \alpha_i\right) = \textstyle\sum_1^{n+1} F(k_i \alpha_i) \qquad (k_i \text{ arbitrary integers})? \tag{2.2}$$

The answer is affirmative and very simply settled as follows: Let (1.4) be the representation of our solution according to Theorem 1. The function $\phi_i(x)$ is UC on S_i having the n periods α_j $(j \neq i)$. Since S_i is dense in R^n we may extend $\phi_i(x)$ uniquely to a function $\Phi_i(x)$ defined throughout R^n by means of

$$\Phi_i(x) = \lim_{y \to x, y \in S_i} \phi_i(y).$$

The function $\Phi_i(x)$ is likewise UC in R^n and has the same periods as $\phi_i(x)$. But then the relation

$$F(x) = \lambda(x) + \textstyle\sum_{i=1}^{n+1} \Phi_i(x) \qquad (x \in R^n) \tag{2.3}$$

defines a function $F(x)$ which is UC on R^n and evidently satisfies the unrestricted equation (2.2). Moreover $F(x) = f(x)$ if $x \in S$. This extension and representation (2.3) is unique because (1.4) was unique. This establishes

THEOREM 3. *Let $m = n + 1$. We obtain the most general uniformly continuous solution $f(x)$ of* (1.1) *as the restriction to the set S, defined by* (1.2), *of a function $F(x)$, defined by* (2.3), *where $\lambda(x)$ is a linear function from R^n to B, while $\Phi_i(x)$ $(i = 1, \cdots, n + 1)$ is a continuous function from R^n to B having the n periods $\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_{n+1}$, while $\Phi_i(0) = 0$. This construction is unique in the sense that two distinct sets $\{\lambda(x), \Phi_i(x)\}$ as above, furnish distinct solutions of* (1.1).

In particular, every UC solution $f(x)$ of (1.1) *has a unique extension $F(x)$ UC on all of R^n which is a solution of the unrestricted functional equation* (2.2).

In Part II we establish Theorem 1. In the brief Part III we give some examples and also mention a theorem of Erdös which suggested the present investigation.

II. Proof of Theorem 1

3. A fundamental inequality. Let $f(x)$ be a UC solution of (1.1), and let $x = \sum u_\nu \alpha_\nu$, $y = \sum v_\nu \alpha_\nu$ be two elements of S. Finally, ε being given let δ be such that

$$\| f(x) - f(y) \| < \varepsilon \quad \text{if} \quad | x - y | < \delta. \tag{3.1}$$

We set $q_\nu = u_\nu - v_\nu$ and divide the numbers $1, \cdots, m$ into two disjoint classes $I = \{i\}$ and $J = \{j\}$. For each $j \in J$ let w_j be a given non-negative integer. We now define for $k = 1, 2, \cdots$

$$\begin{aligned} u_j^{(k)} &= w_j + kq_j, & v_j^{(k)} &= w_j + (k-1)q_j \quad &&\text{if } q_j \geqq 0, \\ u_j^{(k)} &= w_j + (k-1)|q_j|, & v_j^{(k)} &= w_j + k|q_j| &&\text{if } q_j < 0. \end{aligned}$$

Observe that in either case $u_j^{(k)} - v_j^{(k)} = q_j$. For each k we have

$$\textstyle\sum_{i \in I} u_i \alpha_i + \sum_{j \in J} u_j^{(k)} \alpha_j - \sum_{i \in I} v_i \alpha_i - \sum_{j \in J} v_j^{(k)} \alpha_j = \sum_1^m q_\nu \alpha_\nu = x - y$$

so that if $|x - y| < \delta$ then (3.1) and (1.1) imply that

$$\| \textstyle\sum_{i \in I} (f(u_i \alpha_i) - f(v_i \alpha_i)) + \sum_{j \in J} (f(u_j^{(k)} \alpha_j) - f(v_j^{(k)} \alpha_j)) \| < \varepsilon.$$

Letting $k = 1, \cdots, M$ and forming the arithmetic mean of the M quantities within the norm bars we obtain the inequality

$$\left\| \sum_{i \in I} (f(u_i \alpha_i) - f(v_i \alpha_i)) + \frac{1}{M} \sum_{j \in J} \eta_j \{ f((w_j + M \mid q_j \mid) \alpha_j) - f(w_j \alpha_j) \} \right\| < \varepsilon, \tag{3.2}$$

where $\eta_j = +1$ if $q_j \geqq 0$ and $\eta_j = -1$ if $q_j < 0$. The inequality (3.2) will be applied below on two occasions.

4. The asymptotic behavior of solutions. As a first application of the inequality (3.2) let us show that the limits

$$\lim_{N \to +\infty} f(N\alpha_j)/N = \lambda_j \qquad (j = 1, \cdots, m) \tag{4.1}$$

exist. To see this let us choose integers q_ν so that $|\sum q_\nu \alpha_\nu| < \delta$ with $q_j > 0$, and set $u_\nu = \max(q_\nu, 0)$, $v_\nu = \max(-q_\nu, 0)$. Defining $x = \sum u_\nu \alpha_\nu$, $y = \sum v_\nu \alpha_\nu$, we have $|x - y| = |\sum q_\nu \alpha_\nu| < \delta$. To these points x and y we now apply the inequality (3.2), where J consists of the single subscript j, I denoting the set of $\nu \neq j$, and obtain

$$\left\| \sum_{i \neq j} (f(u_i \alpha_i) - f(v_i \alpha_i)) + \frac{1}{M} f((w_j + Mq_j)\alpha_j) - \frac{1}{M} f(w_j \alpha_j) \right\| < \varepsilon. \tag{4.2}$$

Let now N be an arbitrary natural number. Dividing N by q_j let $N = w_j + q_j M$, where $0 \leqq w_j < q_j$. The numbers M and w_j so determined (as functions of N) we select for M and w_j appearing in (4.2). If $N \to \infty$ then also $M \to \infty$ while w_j remains bounded. Thus in (4.2) the term $(1/M)f(w_j \alpha_j) \to 0$. Let E denote the sum appearing in (4.2). If λ denotes one of the limits of the sequence $\Sigma_j = \{f(N\alpha_j)/N\}$ and if we observe that $N/M \to q_j$ we see that on letting $N \to \infty$ through appropriate values the inequality (4.2) becomes

$$\| E + q_j \lambda \| \leqq \varepsilon.$$

Thus if λ' and λ'' are any two of the limits of the sequence Σ_j, then

$$\| q_j \lambda' - q_j \lambda'' \| \leqq 2\varepsilon$$

hence $\| \lambda' - \lambda'' \| \leqq 2\varepsilon q_j^{-1} \leqq 2\varepsilon$. Since ε is arbitrary we conclude that $\lambda' = \lambda''$ and (4.1) is established.

5. The linear component $\lambda(x)$. We shall now use the relations (4.1) to isolate the linear component of a solution $f(x)$ of (1.1). We define $\lambda(x)$ as a linear mapping of R^n into B as follows:

$$\text{If} \quad x = \textstyle\sum_1^m x_i \alpha_i (x_i \text{ real}) \quad \text{then} \quad \lambda(x) = \sum x_i \lambda_i. \tag{5.1}$$

The linearity of $\lambda(x)$ is apparent from this definition, but its being a *function* from R^n into B is still in doubt. To establish this we have to show that a relation

(5.2) $$\sum_1^m x_i\,\alpha_i = 0 \qquad (x_i \text{ real, } x_l \neq 0 \text{ for some } l)$$

implies the relation

(5.3) $$\sum_1^m x_i\,\lambda_i = 0.$$

This may be shown as follows: In the space R^m of the m-tuples $(x_1, \cdots, x_m)$ the vector relation (5.2) defines an $(m - n)$-dimensional subspace V_{m-n}. As the α_i are rationally independent, we conclude that V_{m-n} contains none of the points of the lattice L of points of R^m having integral coordinates with the exception of the origin. However, the sequence of points

$$\{(tx_1, tx_2, \cdots, tx_m)\} \qquad (t = 1, 2, \cdots)$$

comes arbitrarily close to such lattice points. Indeed, by a theorem of Dirichlet (see [3, page 170]) we know that for each natural number ν we can find integers $t^{(\nu)}, k_1^{(\nu)}, \cdots, k_m^{(\nu)}$ $(t^{(\nu)} > 0)$ such that

(5.4) $$|\, t^{(\nu)}x_i - k_i^{(\nu)} \,| < 1/\nu \qquad (i = 1, \cdots, m);$$

in fact $k_i^{(\nu)} = 0$ for all ν if $x_i = 0$. But then, in view of (5.2) and (5.4)

$$\begin{aligned} |\, \textstyle\sum_i k_i^{(\nu)}\alpha_i \,| &= |\, \textstyle\sum_i k_i^{(\nu)}\alpha_i - \sum_i t^{(\nu)}x_i\,\alpha_i \,| \\ &= |\, \textstyle\sum_i (k_i^{(\nu)} - t^{(\nu)}x_i)\alpha_i \,| < (1/\nu)\textstyle\sum_i |\,\alpha_i\,| \end{aligned}$$

and hence

(5.5) $$\lim_{\nu\to\infty} |\, \textstyle\sum_i k_i^{(\nu)}\alpha_i \,| = 0.$$

On the other hand (5.4) implies the following: If $x_l \neq 0$ then

(5.6) $$\lim_{\nu\to\infty} k_i^{(\nu)}/k_l^{(\nu)} = x_i/x_l\,.$$

Let $U = \{i \mid x_i > 0\}$, $V = \{i \mid x_i < 0\}$, $W = \{i \mid x_i = 0\}$. Moreover, it is clear that $\operatorname{sgn} k_i^{(\nu)} = \operatorname{sgn} x_i$ $(i = 1, \cdots, m)$ provided that ν is sufficiently large. But then we can rewrite (5.5) as

$$\lim_{\nu\to\infty} |\, \textstyle\sum_{i\in U} k_i^{(\nu)}\alpha_i - \sum_{i\in V} |\,k_i^{(\nu)}\,|\alpha_i \,| = 0$$

and now the uniform continuity of $f(x)$ and (1.1) imply that

$$\lim_{\nu\to\infty} \|\, \textstyle\sum_{i\in U} f(k_i^{(\nu)}\alpha_i) - \sum_{i\in V} f(|\,k_i^{(\nu)}\,|\alpha_i) \,\| = 0.$$

Choosing a fixed $l \in U$ and dividing the last relation by $k_l^{(\nu)}$ we obtain a fortiori (because $\lim k_l^{(\nu)} = +\infty$ as $\nu \to \infty$)

$$\lim_{\nu\to\infty} \left\| \sum_{i\in U} \frac{k_i^{(\nu)}}{k_l^{(\nu)}} \frac{f(k_i^{(\nu)}\alpha_i)}{k_i^{(\nu)}} - \sum_{i\in V} \frac{|\,k_i^{(\nu)}\,|}{k_l^{(\nu)}} \frac{f(|\,k_i^{(\nu)}\,|\,\alpha_i)}{|\,k_i^{(\nu)}\,|} \right\| = 0.$$

If we now perform the passage to the limit within the norm bars we obtain by (4.1) and (5.6) the relation

$$\left\| \sum_U \frac{x_i}{x_l}\lambda_i + \sum_V \frac{x_i}{x_l}\lambda_i \right\| = 0$$

which is equivalent to the relation (5.3) to be established.

6. The periodic components. The linear function $\lambda(x)$ constructed in §5 is now used as follows: We define a new function $\omega(x)$ by

$$\omega(x) = f(x) - \lambda(x). \tag{6.1}$$

Evidently also $\omega(x)$ is a solution of (1.1) UC on S. Moreover

$$\lim_{N\to\infty} \omega(N\alpha_i)/N = 0 \qquad (i = 1, \cdots, m) \tag{6.2}$$

because of (4.1), (6.1) and the relation $\lambda(N\alpha_i)/N = \lambda_i$ implied by (5.1).

For each $i = 1, \cdots, m$ we now define a function $\phi_i(x)$ throughout the set S_i, described by (1.3), by the following requirements:

1. $\phi_i(0) = 0$,
2. $\phi_i(x + \alpha_j) = \phi_i(x)$ $(j \neq i; x \in S_i)$,
3. $\phi_i(u_i\,\alpha_i) = \omega(u_i\,\alpha_i)$ $(u_i \geqq 0)$.

Evidently $x = \sum u_i\,\alpha_i$ implies

$$\begin{aligned} f(x) &= \lambda(x) + \omega(x) = \lambda(x) + \textstyle\sum_i \omega(u_i\,\alpha_i) \\ &= \lambda(x) + \textstyle\sum_i \phi_i(u_i\,\alpha_i) = \lambda(x) + \textstyle\sum_i \phi_i(x) \end{aligned}$$

and the desired representation (1.4) is seen to hold.

We are still *to show that* $\phi_i(x)$ *is* UC *on* S_i. Given ε, let δ_1 be such that

$$x \in S,\ y \in S \quad \text{and} \quad |\, x - y \,| < \delta_1 \quad \text{imply} \quad \|\, \omega(x) - \omega(y) \,\| < \varepsilon.$$

Let

$$\xi = u_i\,\alpha_i + \textstyle\sum_{j\neq i} k_j\,\alpha_j\,, \qquad \eta = v_i\,\alpha_i + \textstyle\sum_{j\neq i} l_j\,\alpha_j$$

be two points of S_i such that $|\, \xi - \eta \,| < \delta_1$ and let us show that

$$|\, \phi_i(\xi) - \phi_i(\eta) \,| \leqq \varepsilon. \tag{6.3}$$

For this purpose we write $k_j - l_j = q_j$ and select non-negative u_j and v_j such that $q_j = u_j - v_j$ $(j \neq i)$. Finally let

$$x = u_i\,\alpha_i + \textstyle\sum_{j\neq i} u_j\,\alpha_j\,, \qquad y = v_i\,\alpha_i + \textstyle\sum_{j\neq i} v_j\,\alpha_j \tag{6.4}$$

observing that x *and* y *are elements of* S. Moreover

$$\begin{aligned} x - y &= u_i\,\alpha_i - v_i\,\alpha_i + \textstyle\sum_{j\neq i} q_j\,\alpha_j \\ &= u_i\,\alpha_i - v_i\,\alpha_i + \textstyle\sum_{j\neq i} (k_j - l_j)\alpha_j = \xi - \eta \end{aligned}$$

so that $|\, x - y \,| = |\, \xi - \eta \,| < \delta_1$. We may therefore apply the fundamental inequality of §3 to the solution $\omega(x)$, rather than $f(x)$, and the points (6.4) with $I = \{i\}$, $J = \{j \mid j \neq i\}$, $q_j = u_j - v_j$, and $w_j = 0$, obtaining

$$\left\| \omega(u_i\,\alpha_i) - \omega(v_i\,\alpha_i) + \frac{1}{M} \sum_{j\neq i} \eta_j\, \omega(M \,|\, q_j \,|\, \alpha_j) \right\| < \varepsilon.$$

Letting $M \to \infty$ we know by (6.2) that the terms of the sum converge to zero, so that we obtain in the limit

$$\|\, \omega(u_i\,\alpha_i) - \omega(v_i\,\alpha_i) \,\| \leqq \varepsilon.$$

On the other hand, from the periodicities of ϕ_i and its defining property 3, we know that

$$\phi_i(\xi) = \phi_i(u_i\,\alpha_i) = \omega(u_i\,\alpha_i), \qquad \phi_i(\eta) = \phi_i(v_i\,\alpha_i) = \omega(v_i\,\alpha_i)$$

so that our last inequality furnishes the desired inequality (6.3). This completes a proof of Theorem 1.

III. Concluding remarks

7. Examples and applications. We discuss some applications of Theorems 2 and 3 for the simplest case when $n = 1$ and $B = R^1$.

a. Let $n = 1$, $m = n + 2 = 3$, hence α_1, α_2, α_3 real, all $\neq 0$ and all three rationally independent. By Theorem 2 we conclude that the UC solutions of

$$f(u_1\,\alpha_1 + u_2\,\alpha_2 + u_3\,\alpha_3) = f(u_1\,\alpha_1) + f(u_2\,\alpha_2) + f(u_3\,\alpha_3) \quad (u_\nu \geqq 0), \tag{7.1}$$

are of the form $f(x) = Cx$ (C real constant).

All conditions are met if $\alpha_i = \log p_i$, where p_1, p_2, p_3 are three distinct rational primes. Setting $f(\log y) = F(y)$, we see that $F(y)$ is defined on the set of integers

$$A = \{p_1^{u_1} p_2^{u_2} p_3^{u_3} \mid u_\nu \geqq 0\} \tag{7.2}$$

on which it is *additive* in the sense that

$$F(p_1^{u_1} p_2^{u_2} p_3^{u_3}) = F(p_1^{u_1}) + F(p_2^{u_2}) + F(p_3^{u_3}). \tag{7.3}$$

We now observe that the uniform continuity of $f(x)$ on the set

$$S = \{x = u_1\,\alpha_1 + u_2\,\alpha_2 + u_3\,\alpha_3 \mid u_\nu \geqq 0\}$$

amounts to the condition that

$$x_\nu \in S,\ y_\nu \in S,\ x_\nu \neq y_\nu \text{ and } x_\nu - y_\nu \to 0 \quad \text{imply} \quad f(x_\nu) - f(y_\nu) \to 0.$$

Thus by the change of variable $x = \log y$, Theorem 1 furnishes the

COROLLARY 1. *If the real-valued $F(y)$ is additive on the set* (7.2) *in the sense that* (7.3) *holds and if*

$$r_\nu \in A,\ s_\nu \in A,\ r_\nu \neq s_\nu \text{ and } r_\nu/s_\nu \to 1 \quad \text{imply} \quad F(r_\nu) - F(s_\nu) \to 0$$

then $F(y) = C \log y$.

This corollary (and the paper [4]) suggested the present investigation. The Corollary 1 in turn owes its origin to the following theorem of Erdös:

Let $F(y)$ $(y = 1, 2, \cdots)$ be an arithmetic function which is additive in the sense that $F(rs) = F(r) + F(s)$ whenever $(r, s) = 1$. If we also assume that $F(r + 1) - F(r) \to 0$ as $r \to \infty$, then $F(y) = C \log y$ (see [2, Theorem XIII on p. 18] and [5], [1] for more recent and elementary proofs).

Corollary 1 and Erdös' theorem now suggest the following open problem: *Let* $\alpha_i = \log p_i$ $(i = 1, 2, 3)$, *where* p_i *are three distinct primes. Let*

$$S = \{\log (p_1^{u_1} p_2^{u_2} p_3^{u_3})\} = \{\xi_1, \xi_2, \xi_3, \cdots\}$$

be our familiar set with its elements arranged in increasing order $(\xi_1 < \xi_2 < \cdots)$. *If* $f(x)$ *is a solution of* (7.1) *such that*

$$f(\xi_{\nu+1}) - f(\xi_\nu) \to 0 \quad as \quad \nu \to \infty,$$

is it still true that $f(x) = Cx$ *on* S?

An affirmative answer to this problem would certainly contain Corollary 1 (since $\xi_{\nu+1} - \xi_\nu \to 0$), but would say much more.

b. We return to the assumptions of Corollary 1 with the difference that we now have only *two* primes, hence the relation

$$F(p_1^{u_1} p_2^{u_2}) = F(p_1^{u_1}) + F(p_2^{u_2}) \tag{7.4}$$

with solutions $F(y)$ defined on the set $A' = \{p_1^{u_1} p_2^{u_2}\}$. Here we may apply Theorem 3 with $n = 1$, $m = n + 1 = 2$ and obtain the following curious

COROLLARY 2. *The most general solution* $F(y)$ *of the functional equation* (7.4) *having the property that*

$$r_\nu \in A', \ s_\nu \in A', \ r_\nu \neq s_\nu \ \text{and} \ r_\nu/s_\nu \to 1 \quad \text{imply} \quad F(r_\nu) - F(s_\nu) \to 0 \tag{7.5}$$

is given by the formula

$$F(y) = C \log y + \phi_1(\log y) + \phi_2(\log y), \tag{7.6}$$

where $\phi_1(x)$ *and* $\phi_2(x)$ *are everywhere continuous functions having the periods* $\log p_2$ *and* $\log p_1$, *respectively, while* $\phi_1(0) = \phi_2(0) = 0$. *The representation* (7.6) *is unique.*

REFERENCES

1. A. S. BESICOVITCH, *On additive functions of a positive integer*, Studies in Mathematical Analysis and Related Topics, Stanford University Press, 1962, pp. 38–41.
2. P. ERDÖS, *On the distribution function of additive functions*, Ann. of Math. (2), vol. 47 (1946), pp. 1–20.
3. G. H. HARDY AND E. M. WRIGHT, *An introduction to the theory of numbers*, Oxford, 1954.
4. CH. PISOT AND I. J. SCHOENBERG, *Arithmetic problems concerning Cauchy's functional equation*, Illinois J. Math., vol. 8 (1964), pp. 40–56.
5. A. RÉNYI, *On a theorem of P. Erdös and its application in information theory*, Mathematica (Cluj), Vol. 1 (24) (1959), pp. 341–344.

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