# THE MULTIPLIERS OF THE SPACE OF ALMOST CONVERGENT SEQUENCES

BY

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## 1. Introduction

Let $N$ be the set of all positive integers, $m(N)$ the space of bounded real-valued functions on $N$ with the sup norm. A continuous linear functional $\varphi$ on $m(N)$ is called a *Banach limit*, cf. [6], if for $f \in m(N)$,

$$\inf_n f(n) \leq \varphi(f) \leq \sup_n f(n) \quad \text{and} \quad \varphi(f) = \varphi(\tau f),$$

where $\tau f \in m(N)$ is defined by $(\tau f)(n) = f(n+1)$. Let $M$ be the set of all Banach limits. It is well-known that $M$ is non-empty, $w^*$-compact and convex.

Let $F$ be the set of all $f \in m(N)$ such that $\varphi(f)$ equals a fixed constant as $\varphi$ runs through $M$. If $f \in F$ then we say $f$ is *almost convergent*, cf. [6]. It is easy to see that $F$ is a closed subspace of $m(N)$ and it contains constant functions. $f \in m(N)$ is a *multiplier* of $F$ if $fF \subset F$. Since $F$ is not an algebra, $\mathfrak{M}_F$, the set of all multipliers of $F$, is properly contained in $F$. Lloyd [5] gave an example to show that $\mathfrak{M}_F$ is not even the largest subalgebra of $F$. The purpose of this paper is to provide a characterization of the set $\mathfrak{M}_F$. We show that $fF \subset F$ if and only if $f$ converges to a constant $\alpha$ in the following weak sense: given $\varepsilon > 0$ there is a set $A \subset N$ such that $\varphi(X_A) = 0$ for all $\varphi \in M$ and $|f(n) - \alpha| < \varepsilon$ if $n \in N \backslash A$. Thus, in some sense, $\mathfrak{M}_F$ is a very small subspace of $F$. For example, it follows from the above characterization that if $f$ is a non-constant almost periodic function on $N$ then $fF \not\subset F$.

In the last section of this paper we shall consider the generalization of the above results to groups. The author wishes to thank Professor M. M. Day for suggesting the generalization.

## 2. Preliminaries

Let $k_j$ and $n_j$ be two sequences of positive integers such that $k_j \to \infty$ as $j \to \infty$. For $j \in N$ let $\varphi_j$ be the linear functional on $m(N)$ defined as follows:

$$\varphi_j(f) = k_j^{-1} \sum_{i=0}^{k_j-1} f(n_j + i) \quad (f \in m(N)).$$

It is easily verified and is well known that the $w^*$-cluster points of the sequence $(\varphi_j)$ are Banach limits. With the above observation and the Krein-Milman theorem, Raimi [9] proved the following.

LEMMA 2.1. *For $f \in m(N)$, let*

$$\bar{d}(f) = \sup \{\varphi(f) : \varphi \in M\} \quad \textit{and} \quad \underline{d}(f) = \inf \{\varphi(f) : \varphi \in M\}.$$

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*Then*

$$\bar{d}(f) = \lim \sup_n \sup_k n^{-1} \sum_{j=k}^{k+n-1} f(j);$$

$$\underline{d}(f) = \lim \inf_n \inf_k n^{-1} \sum_{j=k}^{k+n-1} f(j).$$

If $f \in F$, then $\bar{d}(f) = \underline{d}(f)$ and we shall denote the common value by $d(f)$. The above lemma implies that $f \in F$ if and only if

$$\lim_n n^{-1} \sum_{j=k}^{k+n-1} f(j) \text{ exists uniformly in } k, \qquad \text{cf. [6].}$$

For convenience, if $A \subset N$, then $\bar{d}(X_A)$, $\underline{d}(X_A)$ and $d(X_A)$ will be denoted by $\bar{d}(A)$, $\underline{d}(A)$ and $d(A)$ respectively, where $X_A$ is the characteristic function of the set $A$ in $N$. By applying Lemma 2.1 to the function $X_A$, we see that $d(A)$ exists if and only if $A$ is "evenly distributed" in $N$ and $d(A) = 0$ if and only if $A$ is "thinly distributed" in $N$.

We shall also need the following consequence of Lemma 2.1. We quote it here for later reference.

LEMMA 2.2 (cf. [1]). *Let $A \subset N$. Then $\underline{d}(A) > 0$ if and only if there exists a positive integer $m$ such that*

$$A \cap \{k, k+1, \cdots, k+m-1\} \neq \emptyset \quad \textit{for each } k \in N.$$

Let $\beta N$ be the Stone-Čech compactification of the discrete set $N$, cf. [4]. Each $f \in m(N)$ can be extended uniquely to a continuous function $f^{-}$ on $\beta N$. The mapping $f \to f^{-}$ is an isometry of $m(N)$ onto $C(\beta N)$, the space of real-valued continuous functions on $\beta N$ with the sup norm. Therefore, each $\varphi \in m(N)^*$ corresponds to a measure $\mu_\varphi$ on $\beta N$. The correspondence is characterized by $\varphi(f) = \int_{\beta N} f^{-} d\mu_\varphi$, $f \in m(N)$.

If $A \subset N$, then $A^{-}$ denotes the closure of $A$ in $\beta N$. Sets of the form $A^{-}$, $A \subset N$, are closed-open and they form an open basis for $\beta N$. As in [10] we set

$$K^\tau = \bigcap\{A^{-} : A \subset N, d(A) = 1\}.$$

Then $K^\tau$ is a compact nowhere dense subset of $\beta N$ and

$$K^\tau = \text{cl}\,[\cup\,\{\text{suppt}\, \mu_\varphi : \varphi \in M\}].$$

## 3. The main theorem

DEFINITION. $f \in m(N)$ is said to be *$\tau$-convergent* if there is a real number $\alpha$ satisfying the following: given $\varepsilon > 0$ there exists a set $A \subset N$ such that $d(A) = 0$ and $|f(n) - \alpha| < \varepsilon$ if $n \in N \backslash A$. In this case we denote $\alpha$ by $\tau$-$\lim f$.

Clearly, every convergent sequence is $\tau$-convergent and if $\tau$-$\lim f = \alpha$ exists them $f \in F$ and $d(f) = \alpha$.

THEOREM 3.1. *Let $f \in m(N)$. Then the following three conditions are* *equivalent*:

(a) $fF \subset F$.
(b) $f$ *is* $\tau$-*convergent*.
(c) $f^{-} \equiv$ *a constant on* $K^{\tau}$.

*Proof*. (b) $\Rightarrow$ (c). Assume that $\tau$-lim $f = \alpha$ exists. Then, for a given $\varepsilon > 0$, there exists a set $A \subset N$ with $d(A) = 1$ and $|f(n) - \alpha| < \varepsilon$ for $n \in A$. Therefore $|f^{-}(w) - \alpha| \leq \varepsilon$ if $w \in K^{\tau} \subset A^{-}$. Since $\varepsilon > 0$ is arbitrary, $f^{-} \equiv \alpha$ on $K^{\tau}$.

(c) $\Rightarrow$ (b). Assume that $f^{-} \equiv \alpha$ on $K^{\tau}$ and let $\varepsilon > 0$ be given. Then since $K^{\tau}$ is compact and sets of the form $B^{-}$, $B \subset N$, form a basis for $\beta N$, we can find a set $A \subset N$ such that $A^{-} \supset K^{\tau}$ and $|f^{-}(w) - \alpha| < \varepsilon$ if $w \in A^{-}$. It follows that $d(A) = 1$ and $|f(n) - \alpha| < \varepsilon$ if $n \in A$.

(c) $\Rightarrow$ (a). Assume $f^{-} \equiv \alpha$ on $K^{\tau}$. If $g \in m(N)$ then $(fg)^{-} \equiv \alpha g^{-}$ on $K^{\tau}$. If $\varphi \in M$, then suppt $\mu_{\varphi} \subset K^{\tau}$ and hence

$$\varphi(fg) = \int_{K^{\tau}} \alpha g^{-} \, d\mu_{\varphi} = \alpha\varphi(g).$$

Thus if $g \in F$ then so is $fg$. Thus $fF \subset F$.

(a) $\Rightarrow$ (b). This is the most difficult implication. Let $f \in \mathfrak{M}_F$ be fixed. We have to show that $\tau$-lim $f$ exists. Without loss of generality, we may assume that $f \geq 0$ and $d(f) = 1$. For $\varepsilon > 0$, let

$$A(\varepsilon) = \{n \in N : f(n) \geq 1 + \varepsilon\}, \qquad B(\varepsilon) = \{n \in N : f(n) \leq 1 - \varepsilon\},$$
$$C(\varepsilon) = \{n \in N : |f(n) - 1| < \varepsilon\}.$$

Note that $N$ is the disjoint union of $A(\varepsilon)$, $B(\varepsilon)$ and $C(\varepsilon)$. We need to show that $d(A(\varepsilon)) = 0$ and $d(B(\varepsilon)) = 0$ for each $\varepsilon > 0$. For the sake of clearness, we divide the proof of this fact into several steps.

I. Let $a < b$ be real numbers. Let

$$A = \{n \in N : f(n) \geq b\} \quad \text{and} \quad B = \{n \in N : f(n) \leq a\}.$$

Then either $\underline{d}(A) = 0$ or $\underline{d}(B) = 0$.

*Notation*. For a fixed positive integer $m$, $N$ can be divided into blocks of $m$ consecutive integers $N(m, n)$, where

$$N(m, n) = \{(n-1)m + 1, (n-1)m + 2, \cdots, nm\}, \quad n \in N.$$

*Proof of I*. If both $\underline{d}(A)$ and $\underline{d}(B)$ are positive then by Lemma 2.2 there exists $m \in N$ such that $N(m, n) \cap A \neq \emptyset$ and $N(m, n) \cap B \neq \emptyset$ for $n \in N$. Choose

$$a_n \in N(m, n) \cap A \quad \text{and} \quad b_n \in N(m, n) \cap B, \quad n \in N.$$

Let $k_1, k_2, \cdots$ be an increasing sequence of positive integers such that

$k_{n+1} - k_n \to \infty$ as $n \to \infty$; let $k_0 = 0$. Define a subset $S = \{s_1, s_2, \cdots\}$ of $N$ as follows

$$s_j = a_j \quad \text{if} \quad k_{2n} < j \leq k_{2n+1}, \quad n = 0, 1, 2, \cdots,$$
$$= b_j \quad \text{if} \quad k_{2n-1} < j \leq k_{2n}, \quad n = 1, 2, \cdots.$$

Then, for each $n \in N$, $N(m, n) \cap S$ is a singleton. Thus, by Lemma 2.1

(1) $$X_S \in F \quad \text{and} \quad d(S) = 1/m.$$

On the other hand, since $k_{n+1} - k_n \to \infty$ as $n \to \infty$, we may apply Lemma 2.1 again to get the following inequalities:

$$\bar{d}(fX_S) \geq \limsup_n \frac{1}{m(k_{2n} - k_{2n-1})} \sum_{j=k_{2n-1}+1}^{k_{2n}} f(b_j)$$
$$\geq b/m, \quad \text{since } b_j \in A,$$
$$d(fX_S) \leq \liminf_n \frac{1}{m(k_{2n+1} - k_{2n})} \sum_{j=k_{2n}+1}^{k_{2n+1}} f(a_j)$$
$$\leq a/m, \quad \text{since } a_j \in B.$$

Thus,

(2) $$fX_S \notin F.$$

By (1) and (2), $f \notin \mathfrak{M}_F$. This contradicts our assumption and the proof of I is completed.

II. For a given $\varepsilon > 0$, $\underline{d}(A(\varepsilon)) = 0$ and $\underline{d}(B(\varepsilon)) = 0$.

*Proof.* Let $A = \{n \in N : f(n) \geq 1\}$. Assume that $\underline{d}(B(\varepsilon)) > 0$. Then, by I, $\underline{d}(A) = 0$. Thus there exists a $\varphi \in M$ such that

(3) $$\varphi(X_A) = 0.$$

But,

(4) $$\varphi(X_{B(\varepsilon)}) \geq \underline{d}(B(\varepsilon)) > 0.$$

Hence,

$$
\begin{aligned}
1 = d(f) &= \varphi(f) \\
&= \varphi(fX_{B(\varepsilon)}) + \varphi(fX_A) + \varphi(fX_{C(\varepsilon)\setminus A}) \\
&\leq \sup\{f(n) : n \in B(\varepsilon)\}\varphi(X_{B(\varepsilon)}) + \|f\|\varphi(X_A) \\
&\qquad + \sup\{f(n) : n \in C(\varepsilon)\setminus A\}\varphi(X_{C(\varepsilon)\setminus A}) && \text{(by (3))} \\
&\leq (1 - \varepsilon)\varphi(X_{B(\varepsilon)}) + \varphi(X_{C(\varepsilon)}) \\
&= \varphi(X_{B(\varepsilon) \cup C(\varepsilon)}) - \varepsilon\varphi(X_{B(\varepsilon)}) < 1 && \text{(by (4))}.
\end{aligned}
$$

This is impossible and, hence, $d(B(\varepsilon)) = 0$. Similarly, $d(A(\varepsilon)) = 0$.

III. For a given $\varepsilon > 0$, $\bar{d}(C(\varepsilon)) = 1$.

*Proof.* If $\bar{d}(C(\varepsilon)) < 1$ then $d(A(\varepsilon) \cup B(\varepsilon)) = t > 0$. Since, by II, $d(A(\varepsilon t/2)) = 0$, there exists $\varphi \in M$ such that

$$\varphi(X_{A(\varepsilon t/2)}) = 0. \tag{5}$$

Since $\varphi(X_{A(\varepsilon) \cup B(\varepsilon)}) \geq t$ and $_{\varphi}(X_{A(\varepsilon)}) \leq \varphi(X_{A(\varepsilon t/2)})$, we see that

$$\varphi(X_{B(\varepsilon)}) \geq t. \tag{6}$$

Thus,

$$\begin{aligned} 1 = \varphi(f) &= \varphi(fX_{B(\varepsilon)}) + \varphi(fX_{A(\varepsilon t/2)}) + \varphi(fX_{N\setminus B(\varepsilon)\setminus A(\varepsilon t/2)}) \\ &\leq (1 - \varepsilon)\varphi(X_{B(\varepsilon)}) + (1 + \varepsilon t/2)\varphi(X_{N\setminus B(\varepsilon)}) && \text{(by (5))} \\ &\leq \varphi(X_{B(\varepsilon)}) - \varepsilon t + \varphi(_{N\setminus B(\varepsilon)}) + \varepsilon t/2 && \text{(by (6))} \\ &= 1 - \varepsilon t/2 < 1. \end{aligned}$$

This is impossible. Thus, $\bar{d}(C(\varepsilon)) = 1$, as we claimed.

IV. For $n \in N$, $d(f^n) = 1$.

*Proof.* Since $f \in \mathfrak{M}_F$, $f^n \in F$. For a fixed $\delta > 0$, since, by III, $\bar{d}(C(\delta)) = 1$, there exists a $\varphi \in M$ such that $\varphi(X_{C(\delta)}) = 1$. It follows that

$$d(f^n) = \varphi(f^n) = \varphi(f^n X_{C(\delta)}). \tag{7}$$

On the other hand, since $(1 - \delta)^n < f^n X_{C(\delta)} < (1 + \delta)^n$. We see that

$$(1 - \delta)^n \leq \varphi(f^n X_{C(\delta)}) \leq (1 + \delta)^n. \tag{8}$$

Combining (7) and (8), we have $(1 - \delta)^n \leq d(f^n) \leq (1 + \delta)^n$ for each $\delta > 0$. Thus $d(f^n) = 1$.

V. For $\varepsilon > 0$, $d(A(\varepsilon)) = 0$ and $d(B(\varepsilon)) = 0$.

*Proof.* Let $\varphi \in M$. Then,

$$\begin{aligned} 1 = \varphi(f^n) &\geq \varphi(f^n X_{A(\varepsilon)}) \quad (\text{since } f \geq 0) \\ &\geq (1 + \varepsilon)^n \varphi(X_{A(\varepsilon)}). \end{aligned}$$

Since $n$ can be arbitrarily big, $\varphi(X_{A(\varepsilon)}) = 0$. Thus $\bar{d}(A(\varepsilon)) = d(A(\varepsilon)) = 0$ for each $\varepsilon > 0$.

By way of contradiction, if there exist an $\varepsilon > 0$ and a $\varphi \in M$ such that $\varphi(X_{B(\varepsilon)}) > 0$ then set $\delta = \varphi(X_{B(\varepsilon)}) \cdot \varepsilon/2$. Then, by the above, $\varphi(X_{A(\delta)}) = 0$. Thus, as in the proof of III, we have the following inequalities:

$$\begin{aligned} 1 &\leq (1 - \varepsilon)\varphi(X_{B(\varepsilon)}) + (1 + \delta)\varphi(X_{C(\varepsilon)\setminus A(\delta)}) \\ &\leq 1 - \varepsilon\varphi(X_{B(\varepsilon)}) + \delta \\ &= 1 - \delta < 1. \end{aligned}$$

This is impossible. Thus $\varphi(B(\varepsilon)) = 0$ for each $\varepsilon > 0$ and each $\varphi \in M$. Thus $d(B(\varepsilon)) = 0$ for each $\varepsilon > 0$. This completes the proof of the theorem.

*Remarks.* (1) We actually proved that if (i) $fX_A \in F$ for each $X_A \in F$ and (ii) $f^n \in F$ for each $n \in N$, then $f$ is $\tau$-convergent. In particular, let $A \subset N$. Then $X_{A \cap B} \in F$ for each $X_B \in F$ if and only if $d(A) = 0$ or 1.

(2) Let $A(N)$ be the algebra of almost periodic functions on $N$. Then it is well known that $A(N) \subset F$. But $A(N) \cap \mathfrak{M}_F$ only consists of constant functions. Indeed, if $f \in A(N) \cap \mathfrak{M}_F$, say, $\tau - \lim f = \alpha$, then $f^{\frown} \equiv \alpha$ on $K^\tau$. Thus $\varphi(|f - \alpha|) = 0$ for each $\varphi \in M$. Thus the non-negative almost periodic function $|f - \alpha|$ has mean value 0. Thus $f \equiv \alpha$ on $N$.

As an example, let $A = \{1, m + 1, 2m + 1, \cdots\}$ where $m > 2, m \in N$. Then $X_A \in A(N)$ and there exists $B \subset N$ such that $X_B \in F$ but $X_A X_B \notin F$. Thus, the almost convergent function $X_B$ is not even weakly almost periodic.

(3) The fact that $\tau - \lim f = \alpha$ exists does not imply the existence of a set $B = \{b_1, b_2, \cdots\}$ in $N$, $b_1 < b_2 < \cdots$, such that $d(B) = 1$ and $\lim_n f(b_n)$ exists.

*Example.* Let $a_n$ be an arbitrary increasing sequence of positive integers such that $a_{n+1} - a_n \to \infty$. Let $A_n = (n - 1) + \{a_1, a_2, \cdots\}, n \in N$. Then $\cup A_n = N$ and $d(A_n) = 0$ for $n \in N$. Define a function $f \in m(N)$ as follows:

$$\begin{aligned} f &\equiv 1 \qquad \text{on } A_1 \\ &\equiv 1/n \quad \text{on } A_n \backslash (A_1 \cup \cdots \cup A_{n-1}), \quad n \geq 2. \end{aligned}$$

Given $\varepsilon > 0$, choose $n_0 \in N$ such that $1/n_0 < \varepsilon$ and let $B = \cup_{k=1}^{n_0} A_k$. Then $d(B) = 0$ and $|f(n)| < \varepsilon$ if $n \in N \backslash B$. Thus $\tau$-$\lim f = 0$. On the other hand, if $B \subset N$ such that $\bar{d}(B) < 1$, then, by Lemma 2.1, there exists $n \in N$ such that $A_1 \cup \cdots \cup A_n \backslash B$ is infinite. Let $N \backslash B = \{b_1, b_2 \cdots\}$, where $b_1 < b_2 \cdots$. Then clearly $\lim_n f(b_n)$ does not exist. (A similar example is also considered by Raimi [8].)

## 4. The generalization

Let $G$ be an amenable group and denote the set of all left invariant means on $G$ by $ML(G)$ (cf. Day [3] for the basic facts concerning amenable groups.) As before, we set

$$\bar{d}(f) = \sup \{\varphi(f) : \varphi \in ML(G)\} \quad \text{and} \quad \underline{d}(f) = \inf \{\varphi(f) : \varphi \in ML(G)\},$$

where $f$ is a bounded real function on $G$. If $\bar{d}(f) = \underline{d}(f)$ then we say $f$ is almost convergent and in this case we denote the common value by $d(f)$. The space of almost convergent functions on $G$ is denoted by $F(G)$. A bounded real function $f$ on $G$ is said to be $G$-convergent if there exists a real number $\alpha$ such that for each $\varepsilon > 0$ there is a set $A \subset G$ satisfying (a) $d(A) = 0$ and (b) $|f(x) - \alpha| < \varepsilon$ if $x \notin A$. We wonder whether $fF(G) \subset F(G)$ implies that $f$ is $G$-convergent. (The other implications of Theorem 3.1 can be readily generalized.) We can only answer the above question when $G$ has an additional property:

($*$) If $A \subset G$ and $\underline{d}(A) > 0$ then there exists $B \subset A$ such that $X_B$ is almost convergent and $d(B) > 0$.
It is easy to show that finitely generated abelian groups and locally finite groups have property ($*$). We would like to conjecture that every amenable group has property ($*$).

LEMMA 4.1. *Let $G$ be an amenable group.*

(1) *If $C \subset G$ and $\underline{d}(C) > 0$ then there exist $x_1, \cdots, x_n$ in $G$ such that for each $x \in G$,*

$$C \cap \{x_1x, \cdots, x_nx\} \neq \emptyset.$$

(2) *If $x_1, \cdots, x_n$ are $n$ distinct elements of $G$ then there exists $C \subset G$ such that $\underline{d}(C) > 0$ and*

$$x_iC \cap x_jC = \emptyset, \quad i \neq j.$$

(3) *Let $C \subset G$ and $x_i \in G$, $i = 1, \cdots, n$, such that $x_iC \cap x_jC = \emptyset$ if $i \neq j$. Assume that $c \in C$ is associated with an element*

$$t(c) \in \{x_1c, \cdots, x_nc\}$$

*and set $T = \{t(c) : c \in C\}$. Then for each $\varphi \in ML(G)$, $\varphi(X_T) = \varphi(X_C)$.*

*Proof.* (1) is an easy consequence of [7, Theorem 7].

(2) Choose $C \subset G$ such that $x_iC \cap x_jC = \emptyset$ if $i \neq j$ and that $C$ is a maximal with this property. Then $\bigcup_{i,j=1}^{n} x_i^{-1}x_jC = G$. Thus $\underline{d}(C) > 0$.

(3) Let $C_i = \{c \in C : t(c) = x_ic\}$. Then $C = C_1 \cup \cdots \cup C_n$, $C_i \cap C_j = \emptyset$ if $i \neq j$ and $T = x_1C_1 \cup \cdots \cup x_nC_n$. Thus $\varphi(X_T) = \varphi(X_C)$ if $\varphi \in ML(G)$.

THEOREM 4.2. *Let $G$ be an amenable group with property ($*$). Then $fF(G) \subset F(G)$ implies that $f$ is $G$-convergent.*

*Proof.* The proof is similar to (a) $\Rightarrow$ (b) of Theorem 3.1 except step I there. Let $f$ be a multiplier of $F(G)$, $f \geq 0$; let

$$A = \{x \in G : f(x) \geq b\} \quad \text{and} \quad B = \{x \in G : f(x) \leq a\}$$

where $a < b$ are real numbers. We have to show that either $\underline{d}(A) = 0$ or $\underline{d}(B) = 0$.

Assume that both $\underline{d}(A)$ and $\underline{d}(B)$ are positive. Then, by Lemma 4.1 (1) there exist $x_1, \cdots, x_n$ in $G$ such that for each $x \in G$,

$$\{x_1x, \cdots, x_nx\} \cap A \neq \emptyset \quad \text{and} \quad \{x_1x, \cdots, x_nx\} \cap B \neq \emptyset.$$

Let $C$ be a subset of $G$ such that $x_iC \cap x_jC = \emptyset$ if $i \neq j$ and that $\underline{d}(C) > 0$, cf. Lemma 4.1 (2). Since $G$ has property ($*$), there exists $D \subset C$ such that $d(D) > 0$. Without loss of generality, we may assume that $G$ is infinite. Then there exists $E \subset G$ such that $\bar{d}(E) = 1$ and $\underline{d}(E) = 0$, cf. [2]. For

$x \in D$, choose

$$t(x) \in A \cap \{x_1x, \cdots, x_nx\} \quad \text{if } x \in D \cap E,$$

$$t(x) \in B \cap \{x_1x, \cdots, x_nx\} \quad \text{if } x \in D \backslash E.$$

Let $T = \{t(x) : x \in D\}$. Then, by Lemma 4.1 (3), $d(T) = d(D)$. It is clear that $\bar{d}(fX_T) \geq d(D) \cdot b$ and $\underline{d}(fX_T) \leq d(D) \cdot a$. This contradicts the fact that $f$ is a multiplier of $F(G)$.

*Added in Proof.* (1) We are able to show that every group in $EG$ has property (*). Cf. [3, p. 520] for the definition of $EG$. (2) J. P. Duran and the author have proved recently that Theorem 4.2 holds for countable left amenable cancellative semigroups.

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