CONVEX POLYTOPES IN LINEAR SPACES

BY

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1. Introduction

H. Weyl [7] defines a convex polyhedron as a subset P, of a finite dimensional space E^n, which can be expressed as the intersection of a finite number of half spaces. Here we generalize this definition in a meaningful way to infinite dimensional spaces. In Section 1, it is shown that such sets have "faces" and in general enjoy many of the geometric properties of their finite dimensional counterparts. However, as Theorem 2.4 points out, nondegenerate bounded convex polytopes in infinite dimensional spaces do not have any extremal points. This enables us to prove that reflexive Banach spaces do not contain any bounded convex polytopes.

In Section 3, a comparison is made between out convex polytopes and those defined by Bastiani [1]. Although a direct comparison is not possible, we show that our sets essentially satisfy her definition but produce a counter example to show that the converse is not true.

Clarkson [3] and Fullerton [5] among others have characterized B-spaces by the shapes of their unit spheres. Here we extend this work showing that subspaces of the B-space (c_0) of all sequences which converge to zero are, and are the only, separable B-spaces whose unit spheres are convex polytopes. The space (c_0) itself is the only B-space whose unit sphere is a "paralleletope" (a generalized parallelepiped). This generalizes and reproves a result of Klee [6]. Namely that every symmetric convex polytope can be realized as the central section of the unit ball of (c_0).

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2. Definition and properties of convex polytopes

Throughout this section, X will denote a real locally convex linear topological space which is Hausdorff and P a convex subset of X such that the origin θ is in the interior P^0, of P. If $\{E_\alpha \mid \alpha \in \mathrm{A}\}$ is a collection of closed half spaces such that $P = \bigcap \{E_\alpha : \alpha \in \mathrm{A}\}$ and if for each $x \in X$ there exists a finite subcollection $\alpha_1, \alpha_2, \cdots, \alpha_k$ of A having the property that

$$x \in \bigcap \{E_\alpha \mid \alpha \in \mathrm{A}, \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k\},$$

then P will be called a *convex polytope.* For the rest of this section, we will

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assume that P is a convex polytope and $\{E_\alpha \mid \alpha \in \mathrm{A}\}$ is such a collection of half spaces, each of whose bounding hyperplanes and defining continuous linear functional is H_α and f_α respectively, i.e.,

$$E_\alpha = \{x \in X \mid f_\alpha(x) \leq c_\alpha\}$$
$$H_\alpha = \{x \in X \mid f_\alpha(x) = c_\alpha\},$$

where c_α is a real number. Since $\theta \in P^0$ we lose no generality by assuming $c_\alpha = 1$ for each $\alpha \in \mathrm{A}$.

PROPERTY 2.1. *The intersection of two convex polytopes is a convex polytope.*

PROPERTY 2.2. *For each $x \in X$ there exist $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathrm{A}$ such that*

$$x \in (\bigcap E_\alpha)^0 \qquad (\alpha \in \mathrm{A},\ \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k).$$

Proof. Let $y \in P^0$. There exists $z \in X$ such that x belongs to the half open line segment $(z, y]$ (i.e., $x = az + (1 - a)y$ for $0 \leq a < 1$). From the definition of P, we can find $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathrm{A}$ such that

$$z \in \bigcap \{E_\alpha \mid \alpha \in \mathrm{A},\ \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k\}.$$

Let $C(P, z) = \bigcup (z, v)\ (v \in P)$. This cone is contained in

$$\bigcap \{E_\alpha : \alpha \in \mathrm{A},\ \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k\}$$

and contains the point x in its interior. Therefore

$$x \in (\bigcap E_\alpha)^0 \qquad (\alpha \in \mathrm{A},\ \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k).$$

Since the interior of the intersection of a collection of sets is always contained in the intersection of their interiors, Property 2.2 also implies

$$x \in \bigcap (E_\alpha)^0 \qquad (\alpha \in \mathrm{A},\ \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k)$$

or equivalently

PROPERTY 2.3. *Each x in the boundary, $b(P)$, of P is a member of at most finitely many hyperplanes of the collection $\{H_\alpha \mid \alpha \in \mathrm{A}\}$.*

Conversely we have

PROPERTY 2.4. *Each $x \in b(P)$ is a member of at least one hyperplane of the collection $\{H_\alpha \mid \alpha \in \mathrm{A}\}$.*

Proof. From Property 2.2, there exists $\alpha_1, \alpha_2, \cdots, \alpha_k$ such that

$$x \in (\bigcap E_\alpha)^0 \qquad (\alpha \in \mathrm{A},\ \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k).$$

If $x \notin H_{\alpha_i}$ for any integer $i = 1, 2, \cdots, k$ then $x \in (\bigcap E_{\alpha_i})^0$ $(i = 1, 2, \cdots, k)$ so that we reach the contradiction that

$$x \in (\textstyle\bigcap_{i=1}^k E_{\alpha_i})^0 \cap (\bigcap E_\alpha)^0 = P^0 \qquad (\alpha \in \mathrm{A},\ a \neq \alpha_1, \alpha_2, \cdots, \alpha_k).$$

PROPERTY 2.5. *If $\Gamma \subset \mathrm{A}$ then $\bigcup \{H_\gamma \mid \gamma \in \Gamma\}$ is closed.*

Proof. Since P is a convex polytope it follows that $P' = \bigcap \{E_\gamma \mid \gamma \in \Gamma\}$ is also a convex polytope. If $x \notin \bigcup \{H_\gamma \mid \gamma \in \Gamma\}$ then Property 2.2 implies the existence of $\gamma_1, \gamma_2, \cdots, \gamma_k \in \Gamma$ such that $x \in (\bigcap E_\gamma)^0$ $(\gamma \neq \gamma_1, \gamma_2, \cdots, \gamma_k)$. Thus we can find an open set V_x containing x such that $V_x \cap H_\gamma = \emptyset$ for each $\gamma \neq \gamma_1, \gamma_2, \cdots, \gamma_k$. Clearly there exists an open set W_x containing x such that $W_x \cap H_\gamma = \emptyset$ for each $\gamma = \gamma_1, \gamma_2, \cdots, \gamma_k$. Thus $(V_x \cap W_x) \cap H_\gamma = \emptyset$ for each $\gamma \in \Gamma$ and consequently x is not a limit point of $\bigcup \{H_\gamma : \gamma \in \Gamma\}$ so that this set must be closed.

If G is a closed nondegenerate $(G^0 \neq \emptyset)$ convex subset of X such that x is a boundary point of G and H is the only hyperplane which is tangent to G at x then H will be referred to as a *smooth* (*hyperplane*) *support* of G and x as a *smooth point* of the boundary of G. The half space E which contains G and whose boundary is H will be called a smooth (*half space*) *support* of G.

PROPERTY 2.6. *Every boundary point x of P is contained in a smooth* (*hyperplane*) *support of P. Moreover the collection $\{E_\alpha \mid \alpha \in \mathrm{A}\}$ contains all of the smooth supports of P.*

The proof of this property depends on the following lemma.

LEMMA 2.1. *If G is a closed convex set with interior point x and E is a half space whose boundary H contains x, then H is a smooth support of $E \cap G$.*

Proof. Clearly $E \cap G$ has a nonvoid interior. Since x is an interior point of G there exists an open set V_x about x which is contained in G. Thus the *disk* $H \cap V_x$ is a subset of $E \cap G$. Let H' be a support hyperplane of $G \cap E$ such that $x \in H'$ but $H' \neq H$. It is easily seen that H is the only hyperplane which contains the disk $H \cap V_x$. Thus there exists y in the disk such that $y \notin H'$. But then we can also find z in the disk such that x is a member of the open line segment (z, y). Hence H' separates the point z from y which contradicts the fact that both z and y belong to $G \cap E$ and H' supports this set.

Proof of Property 2.6. Property 2.4 and the definition of a smooth support assures the validity of the second assertion. For the first let H_{α_1}, $H_{\alpha_2}, \cdots, H_{\alpha_k}$ be the collection of all hyperplanes in $\{H_\alpha : \alpha \in \mathrm{A}\}$ which contain the boundary point x (this collection is finite from Property 2.3). For notational convenience order A so that α_i will be its i^{th} element and all nonsubscripted elements of A will follow α_k. There exists at least one index i, such that

$$H_{\alpha_i} \cap (\textstyle\bigcap_{\alpha > \alpha_i} E_\alpha)^0 \neq \emptyset \quad \text{and} \quad P = \textstyle\bigcap_{\alpha \geq \alpha_i} E_\alpha .$$

Indeed for if the inequality did not hold for $i = 1$ and if we denote the closure of G by cl (G), then we would have

$$E_{\alpha_1} = \mathrm{cl}\,(E_{\alpha_1}) \supset \mathrm{cl}\,(\textstyle\bigcap_{\alpha > \alpha_1} E_\alpha)^0 = \mathrm{cl}\,(\textstyle\bigcap_{\alpha > \alpha_1} E_\alpha) = \textstyle\bigcap_{\alpha > \alpha_1} E_\alpha .$$

The second equality being valid since if G is any convex set with a nonvoid interior then $\mathrm{cl}\ (G^0) = \mathrm{cl}\ (G)$. Thus $P = \bigcap \{E \mid \alpha \geq \alpha_2\}$ and the argument can be extended by induction until an appropriate index i is found. Clearly one such i must be found before the $k + 1$ inductive step is reached; for if not then we would see that $P = \bigcap \{E_\alpha, \alpha \geq k + 1\}$ which contradicts Property 2.4. Lemma 2.1 now implies that H_{α_i} is a smooth support of P.

The above proof underscores the possibility of a convex polytope being represented by more than one collection of half spaces which satisfies our definition and, as can easily be verified by considering a 2-dimensional polyhedron, this is in fact the case. However, as Property 2.7 will indicate, there is a unique minimal such representation.

In general a collection $\{H_\beta \mid \beta \in \mathrm{B}\}$ of support hyperplanes will be called a representation of a nondegenerate closed convex set G if each $x \in b(G)$ is a member of some hyperplane H_β.

PROPERTY 2.7. *The subcollection $\{H_\beta \mid \beta \in \mathrm{B}\}$ of all smooth (hyperplane) supports of P is a representation of P which is contained in all other representations of P. Moreover, the collection $\{E_\beta \mid \beta \in \mathrm{B}\}$ of smooth (half space) supports of P satisfies the definition of a convex polytope and*

$$P = \bigcap \{E_\beta \mid \beta \in \mathrm{B}\}.$$

Proof. Property 2.6 and the definition of a smooth (hyperplane) support imply the first assertion. The fact that P is a convex polytope and that the collection $\{E_\beta \mid \beta \in \mathrm{B}\}$ is contained in every representation of P then implies that the set $P' = \bigcap \{E_\beta \mid \beta \in \mathrm{B}\}$ is a convex polytope. It remains only to show that $P = P'$. Clearly $P' \supset P$. Suppose $x \in P' \setminus P$. Let $y \in P^0$. Then the convexity of P implies the existence of $z \in (x, y) \cap b(P)$. But since $\{H_\beta \mid \beta \in \mathrm{B}\}$ is a representation of P there exists $\beta_0 \in \mathrm{B}$ such that $z \in H_{\beta_0}$. Thus we reach the contradiction that H_β separates y from x or that $x \notin P'$.

Hereafter we will assume that the collection $\{H_\beta \mid \beta \in \mathrm{B}\}$ is the minimum representation of P. For each $\beta \in \mathrm{B}$ we will define $F_\beta = H_\beta \cap P$ to be a *face* of P. In Corollary 2.1.1 we will show that the faces of P are precisely the maximal convex subsets of its boundary. Note that Property 2.7 implies that each face F_{β_0} of P contains a point x which is not in any other face of P. For if this were not the case then the collection $\{H_\beta \mid \beta \in \mathrm{B}, \beta \neq \beta_0\}$ would be a representation of P which does not contain the minimum representation. Thus each face is uniquely determined by some point x. In Corollary 2.1.2 we will show that the points of $b(P)$ which determine faces in this way are the smooth points of P. We can generalize this notion by defining a set F_y to be the *edge determined by* y $(y \in b(P))$ if F_y is the intersection of the collection of all faces which contain y. Since F_y is also the intersection of P with all support hyperplanes H_β $(\beta \in \mathrm{B})$ which contain y, it is clear from Property 2.3 that the set of all faces which contain y is finite. In Theorem 2.1, we show that the edges of P are precisely its facets, where a facet is as defined in [2]. In fact, if we define an *exposed set* of P to be a subset of its boundary which can be expressed as the intersection of a support hyperplane with P then we can show that the edges and exposed sets are also synonymous. To do this we will need the following lemma.

LEMMA 2.2. *If the edge F_x is represented as*

$$P \cap [\bigcap_{i=1}^{k} H_{\beta_i}] \qquad (\beta_i \in \mathrm{B})$$

then there exists an open set V_x containing x such that the disk

$$V_x \cap [\bigcap_{i=1}^{k} H_{\beta_i}]$$

is contained in F_x. Thus x is an interior point of F_x with respect to the relative topology on $\bigcup_{i=1}^{k} H_{\beta_i}$.

Proof. Suppose that the assertion is false. Then for each open set V_x there exists y such that y is a member of the disk determined by V_x but not a member of the edge. Each such y can be strictly separated from P by a smooth (hyperplane) support H_β such that $\beta \neq \beta_1, \beta_2, \cdots, \beta_k$. But since x is not a member of $\bigcap \{H_\beta \mid \beta \neq \beta_1, \beta_2, \cdots, \beta_k\}$ we see that this union is not closed, thereby contradicting Property 2.5.

THEOREM 2.1. *If P is a convex polytope which contains θ in its interior then the following statements are equivalent.*

1. *F is an exposed set of P.*
2. *F is an edge of P.*
3. *F is a facet of P.*
4. *F is an extremal subset of P.*

Proof. (i) To prove that statement 1 implies statement 2, we note from Property 2.3 that there exists $x \in F$ such that x belongs to no more faces than any other point of F. From Lemma 2.2, x is an interior point of the edge F_x with respect to the relative topology imposed on $\{H_\beta \mid \beta \in \mathrm{B}, x \in H_\beta\}$. Thus x is a relative internal point [4, *p.* 413] of F_x (with respect to itself). Therefore, if H supports P at x it follows that H supports P at each point of the edge F_x. For if this were not the case we could find y, $z \in F_x$ such that H separated y from z. Therefore $F \supset F_x$.

Conversely suppose $y \in F$ such that $y \neq x$. Let $z \in (x, y)$. If $z \in H_\beta$ then clearly $x \in H_\beta$. But then the definition of x implies that $\{H_\beta \mid z \in H_\beta\} = \{H_\beta \mid x \in H_\beta\}$. But since $z \in H_\beta$ also implies $y \in H_\beta$ we see that

$$\{H_\beta \mid y \in H_\beta\} \supset \{H_\beta \mid x \in H_\beta\}.$$

Hence

$$y \in \bigcap \{H_\beta \mid y \in H_\beta\} \subset \bigcap \{H_\beta \mid x \in H_\beta\} = F_x$$

or $F_x \supset F$. Thus statement 1 implies statement 2.

(ii) To prove that statement 2 implies statement 3, we will represent the edge F as F_x where x determines F in the sense of the definition of an edge. From the lemma and the remarks of (i), it is apparent that x is a relative internal point of F_x (with respect to itself). Thus the facet K_x of P determined by x must contain F_x. To see that $K_x \subset F_x$ we observe (as we did in (i)) that since x is a relative internal point of K_x, if H supports P at x, then H must support P at each point of K_x.

(iii) To show that statement 3 implies statement 1, suppose x is a relative internal point of F. From Property 2.3, the collection

$$\{H_{\alpha_i} \mid i = 1, 2, \cdots, k\}$$

of all smooth supports which support P at x is finite. Let

$$\{f_{\alpha_i} \mid i = 1, 2, \cdots, k\}$$

be the associated support functionals. Since $\theta \in P^0$ we will assume, without loss of generality, that $f_{\alpha_i}(x) = 1$ for each index i. Let f be any convex sum of the form $\sum_{i=1}^{k} a_i f_{\alpha_i}$ where $a_i > 0$ for each index i. If $H = \{y \in X \mid f(x) = 1\}$ then clearly H supports P at x. But since x is a relative internal point of F, H must support P at each point of F. Thus the exposed set $H \cap P$ contains F. Now if the containment were proper we could find $y \in H \cap P$ such that $f_i(y) < 1$ for some index i. But then the definition of f would imply the contradiction that $f(y) < 1$.

(iv) Since we have already shown the equivalence of 1 through 3 and since every exposed set is obviously extremal we can complete the proof by showing that 4 implies 2. For this let x belong to the extremal set F such that every $y \in F$ belongs to at least as many smooth supports of P as x. Then from 3, the edge F_x determined by x, is a subset of F. On the other hand suppose $y \in F$ $(y \neq x)$. Then if H is a smooth support which supports P at $z \in (x, y)$, it follows that H must support P at both endpoints x and y. But then the definition of x implies that the smooth supports which support P at x are precisely those which support P at z and therefore are contained in those which support P at y. Thus $y \in F_x$ so that $F_x = F$.

Since a face is merely a special type of edge, Theorem 1 implies

COROLLARY 2.1.1. *A set F is a face of P if and only if it is a maximal convex subset of its boundary.*

Proof. Since P has a nonvoid interior, it is well known that every maximal convex subset F of the boundary of P is an exposed set of P. Thus Theorem 2.1 implies that F is an edge of P, and therefore is a subset of some face F'. However if F' properly contains F then we see that F is not a maximal convex subset of $b(P)$. Therefore $F = F'$.

To prove the converse we note that Zorn's lemma implies that every face F is contained in a maximal convex subset F'. But as noted above, Property 2.7 implies that F must contain a smooth point x. Thus if H supports P at each point of F' then H must be the defining hyperplane for F, so that $F = F'$.

COROLLARY 2.1.2. *A point $x \in b(P)$ is a smooth point of P if and only if it belongs to no more than one face F.*

Proof. It is clear that no smooth point can belong to more than one face. The converse is a corollary to (i) in the proof of the theorem. For there we saw that whenever x is a member of an exposed set F which is contained in no more faces than any other member of F, then x is a relative internal point of F. Thus if H supports P at x, H must support P at each point in F. But since the exposed set F in this case is a face, by definition there is only one support which supports each point of F. Therefore x is a smooth point of P.

THEOREM 2.2. *A closed bounded convex set P which has a nonvoid interior in a finite dimensional space E^n is a convex polyhedron if and only if it is a convex polytope.*

Proof. Since a closed bounded nondegenerate convex set in a finite dimensional space X is a convex polyhedron if and only if it can be expressed as the intersection of a finite number of half spaces, we need only prove that each bounded convex polytope P in E^n has at most a finite number of faces.

Let x_β be a smooth point of P which lies in the face F_β. Since P is bounded and E^n is finite dimensional, P is compact. Therefore if the set $\{F_\beta \mid \beta \in \mathrm{B}\}$ is infinite then the set $\{x_\beta \mid \beta \in \mathrm{B}\}$ has a limit point, say $x_0 \in b(P)$. But from Property 2.3, x_0 is a member of at most finitely many smooth (hyperplane) supports $H_{\beta_1}, H_{\beta_2}, \cdots, H_{\beta_k}$. Therefore we reach a contradiction to Property 2.5 since the set $\bigcup \{H_\beta \mid \beta \in \mathrm{B}, \beta \neq \beta_1, \beta_2, \cdots, \beta_k\}$ is not closed.

An elementary consideration of convex sets in the plane reveals that Theorem 2.2 is not valid if the boundedness restriction is removed. However, as a corollary to the following theorem it is evident that convex polytopes of finite dimensional spaces can have at most a countable number of faces.

THEOREM 2.3. *If P is a convex polytope in a separable space X then P has at most a countable number of faces.*

Proof. Let A be a countably dense subset of X and $\{E_\beta \mid \beta \in \mathrm{B}\}$ be the minimum representation of P. We will prove the assertion by showing that the indexing set B is countable. From the definition of P it is clear that there exists a countable subcollection B' of B such that $A \subset \bigcap \{E_\beta \mid \beta \in \mathrm{B} \setminus \mathrm{B}'\}$. If $\mathrm{B}' \neq \mathrm{B}$ then there exists $\beta' \in \mathrm{B} \setminus \mathrm{B}'$ such that $A \subset E_{\beta'}$. But this is a contradiction since $X \setminus E_{\beta'}$ is a nonvoid open set and A is dense in X. Thus $\mathrm{B} = \mathrm{B}'$ so that B must be countable.

In the foregoing we have emphasized several of those geometric properties which are shared by both the finite dimensional convex polyhedron and its infinite dimensional counterpart. As indicated by the following theorem and corollaries however, not all of these properties seem to have a convenient generalization.

THEOREM 2.4. *If P is a convex polytope in an infinite dimensional space then P has no extremal points.*

Proof. The assertion follows easily from Theorem 2.1, Lemma 2.2 and the fact that the intersection of a finite set of hyperplanes is never a single point in an infinite dimensional space.

COROLLARY 2.4.1. *If X is infinite dimensional, P is a convex polytope and τ is any locally convex Hausdorff topology which is coarser than the initial topology on X, then P is not τ-compact.*

Proof. The proof is an elementary consequence of the theorem and the Krien Milman theorem.

COROLLARY 2.4.2. *If X is an infinite dimensional space which is either semi-reflexive or the adjoint of a barrelled space then X does not contain any bounded convex polytopes.*

Proof. Since the adjoint of a semi-reflexive space is barrelled [2], the definition of semi-reflexive [2] implies that we need only consider the case where X is the adjoint of a barrelled space Y. If we denote the weakest locally convex linear topology which can be imposed on X such that every member of Y is a continuous linear functional by w^* then clearly every bounded subset of X is also w^*-bounded. But since every w^*-bounded subset of X is also w^*-compact [2], the assertion follows from the preceding corollary.

COROLLARY 2.4.3. *No infinite dimensional space which is the adjoint of a Banach space (or which is itself a reflexive B-space) can contain any bounded convex polytopes.*

3. The convex polytope of A. Bastiani

If K is an arbitrary convex set in a real locally convex space Y, then for each $y \in K$ the *cone of support* $C(K, y)$ is defined as the union of all half rays originating at y and passing through a point of K. If τ represents the linear topology on Y, Bastiani [1] defines a convex cone P' to be a τ-convex pyramid if its cone $C(K, y)$ of support is closed for each $y \in P'$. She then defines a τ-convex polytope as the intersection $P' \cap H$ of a τ-convex pyramid with a hyperplane H. Since the convex polytopes we have been considering have nonvoid interiors and a τ-convex polytope does not, a direct comparison of these definitions is impossible. However, it is natural to ask if a convex polytope P can be imbedded into a space by an imbedding map λ such that $\lambda(P)$ is a τ-convex polytope. In Theorem 3.1 we answer this question in the affirmative for bounded convex polytopes. In this section as before, X will denote a locally convex linear space which is Hausdorff and P will denote a convex polytope of X such that $\theta \in P^0$. The symbol $\mathcal{R}$ will denote the real numbers and $X \times \mathcal{R}$ will symbolize the product space of all pairs $(x * r)$ where $x \in X$ and $r \in \mathcal{R}$. Then $X \times \mathcal{R}$ is a locally convex space with respect to the product topology τ, and coordinate-wise scalar multiplication and addition. In Theorem 3.1, we will be concerned primarily with the linear homeomorphism λ as defined by $\lambda(x) = (x * 0)$ which imbeds the space X into the space $X \times \mathcal{R}$.

THEOREM 3.1. *If P is a bounded convex polytope in X then its image $\lambda(P)$ is a τ-convex polytope in $X \times \mathcal{R}$.*

Proof. We first observe that $H' = \lambda(X)$ is a closed hyperplane of $X \times \mathcal{R}$. Let $\{f_\beta \mid \beta \in \mathrm{B}\}$ be the collection of all smooth (functional) supports of P. For each $\beta \in \mathrm{B}$ define the continuous linear functional f'_β on $X \times \mathcal{R}$ by

$$f'_\beta(x * r) = f_\beta(x) + r.$$

Recall that $P = \bigcap_{\beta \in \mathrm{B}} \{x \in X \mid f_\beta(x) \leq 1\}$. Define P' by

$$P' = \bigcap_{\beta \in \mathrm{B}} \{(x * r) \mid f'_\beta(x * r) \leq 1\}$$

and note that P' is a convex cone with vertex $(\theta * 1)$ such that $\lambda(P) = P' \cap H'$. We will complete the proof by showing that P' is a τ-convex pyramid.

Let $(x_0 * r_0) \in P'$ and $C(P', (x_0 * r_0))$ be its corresponding cone of support. Let $\mathrm{B}_0 = \{\beta \in \mathrm{B} \mid f'_\beta(x_0 * r_0) = 1\}$ (B_0 possibly void). Since each half space is a closed set we may prove the cone is closed by showing that

$$C(P', (x_0 * r_0)) = \bigcap_{\beta \in \mathrm{B}_0} \{(x * r) \mid f'_\beta(x * r) \leq 1\}.$$

In the event that $\mathrm{B}_0 = \emptyset$ this intersection is by definition the entire space. In any case it is clear that the intersection contains the cone. To show the converse, suppose $(x * r)$ is in the intersection but not in the cone. Thus for each $(y * t)$ in the open line segment $((x * r), (x_0 * r_0))$ there is $\beta \in \mathrm{B} \setminus \mathrm{B}_0$ such that $f'_\beta(y * t) > 1$. But since $f'_\beta(x_0 * r_0) < 1$ we can find a sequence $(x_n * r_n)$ of points on the line which converges to $(x_0 * r_0)$ and a sequence of functionals f'_{β_n}, $\beta_n \in \mathrm{B} \setminus \mathrm{B}_0$ such that $f'_{\beta_n}(x_n * r_n) = 1$. Thus $f_{\beta_n}(x_n) = 1 - r_n$ for each n. Now if $r_0 \neq 1$ then we can assume without loss of generality that $r_n \neq 1$ for each n so that $f_{\beta_n}(x_n/(1 - r_n)) = 1$ for each index n. But since the sequence $\{x_n/(1 - r_n) \mid n = 1, 2, \cdots\}$ converges to $x_0/(1 - r_0)$ and this point is not a member of $\bigcup_n \{x \in X \mid f_{\beta_n}(x) = 1\}$, we see that the latter set is not closed which contradicts Property 2.5.

If however $r_0 = 1$, then since $(x_0 * 1) \in P'$, we see that

$$1 \geq f'_\beta(x_0 * 1) = f_\beta(x_0) + 1$$

so that $f_\beta(x_0) \leq 0$ for all $\beta \in \mathrm{B}$. But since P is bounded this implies $f_\beta(x_0) = 0$ and hence $f'_\beta(x_0 * 1) = 1$ for all $\beta \in \mathrm{B}$ so that $\mathrm{B}_0 = \mathrm{B}$. Thus the cone of support at $(x_0 * 1)$ is P' and is therefore closed.

As a corollary to the proof of the previous theorem, we may state

COROLLARY 3.1. *If P is a convex polytope then the cone of support at each of its points is closed.*

From Bastiani's work it seems reasonable to define a *nondegenerate τ-convex polytope* as closed convex set with a nonvoid interior such that the cone of support at each of its points is closed. The above corollary then implies that every convex polytope is a nondegenerate τ-convex polytope. As seen by the following example, however, the converse is false.

Example 3.1. Not every nondegenerate τ-convex polytope is a convex polytope.

Consider the set P' as defined in the proof of the previous theorem. For simplicity we will assume X is normed, each smooth (functional) support, f_β has norm 1 and the collection B is infinite. It will be seen in Section 4 that the unit ball of the B-space (c_0) has this property. We will norm the space $X \times \mathcal{R}$ by $\| x * r \| = \| x \| + | r |$ and note that this norm is indeed compatible with the product topology τ on $X \times \mathcal{R}$.

To see that $(\theta * 0)$ is an interior point of P', suppose $\| (x * r) \| < 1$. Then using the notation of the proof of the theorem,

$$\begin{aligned} f'_\beta(x * r) &= f_\beta(x) + r \leq \| f_\beta \| \cdot \| x \| + | r | \\ &\leq \| (x * r) \| \leq 1 \qquad \text{for each } \beta \in \mathrm{B}, \end{aligned}$$

so that $(x * r) \in P'$. Since we have already observed P' to be a τ-convex pyramid, the above shows that P' is a nondegenerate τ-convex polytope.

We will now show that each f'_β is a smooth (functional) support of P'. Indeed for each $\beta \in \mathrm{B}$ there exists a smooth point $x_\beta \in P$ such that $f_\beta(x_\beta) = 1$. Let g'_β support P' at $(x_\beta * \theta)$. Then clearly g'_β agrees with f'_β on

$$\{(x * \theta) \mid x \in X\}.$$

Moreover $g'_\beta(\theta * 1) = 1$. For suppose $g'_\beta(\theta * 1) < 1$. Then for any $a > 1$ we have

$$g'_\beta[a(x_\beta * \theta) + (1 - a)(\theta * 1)] > 1.$$

But since P' is a cone with vertex $(\theta * 1)$ we reach the contradiction that g'_β does not support P'. Therefore g'_β agrees with f'_β on the set $\{(x * \theta) \mid x \in X\}$ and on the point $(\theta * 1)$. Since the point and set generate all of $X \times \mathcal{R}$ we see that $g'_\beta = f'_\beta$ or that f'_β is the only continuous linear functional which supports P' at $(x_\beta * \theta)$. Thus, if P' is a convex polytope each f'_β is a facial functional. This of course contradicts Property 2.3 since the point $(\theta * 1)$ is in infinitely many faces.

4. Convex polytopes and (c_0)

In this section we will characterize the class of all separable B-spaces which contain bounded nondegenerate convex polytopes, as closed subspaces of the B-space (c_0) of all sequences which converge to zero.

THEOREM 4.1. *The unit ball S of the separable B-space (c_0) is a convex polytope.*

Proof. If $E_i = \{(x_1, x_2, \cdots) \in (c_0) \mid x_i \leq 1\}$ for each $i = 1, 2, \cdots$ then

$$S = (\bigcap_{i=1}^{\infty} E_i) \cap (\bigcap_{i=1}^{\infty} - E_i).$$

The collection $\{\pm E_i \mid i = 1, 2, \cdots\}$ can easily be seen to satisfy the definition of a convex polytope. Moreover each half space $\pm E_i$ is a smooth support of P.

COROLLARY 4.1.1. *The unit sphere $S \cap M$ of every subspace M of (c_0) is a convex polytope in M.*

In his paper *Polyhedral sections of convex bodies* [6], V. L. Klee, Jr., defines a B-space X to be polyhedral provided that the unit ball of every finite dimensional subspace of X is a convex polyhedron. He then proves the corollary stated below by an altogether different technique than that presented here.

COROLLARY 4.1.2. *If M is a closed subspace of (c_0) whose unit ball $S \cap M$ has an extremal point, then M is finite dimensional and $S \cap M$ is a convex polyhedron. Thus a finite dimensional B-space X is isometric to a closed linear subspace of (c_0) if and only if X is polyhedral.*

Proof. It follows immediately from Theorem 2.4 that M is finite dimensional and M is polyhedral. The "if" part of the second assertion is a consequence of the following theorem and Theorem 2.2.

THEOREM 4.2. *If the unit ball S of a normed linear space X is a convex polytope with a countable number of faces, then X is isometric to a closed subspace of (c_0).*

Proof. Let $\{\pm E_i \mid i = 1, 2, \cdots\}$ be the class of smooth (half space) supports of S, and for each i let f_i be the continuous linear functional defined by $E_i = \{x \in X \mid f_i(x) \leq 1\}$. If the indexing set is finite, say k in number, define a map λ from X into the B-space l_∞ of all bounded sequences by

$$\lambda(x) = (f_1(x), f_2(x), \cdots, f_k(x), 0, 0, \cdots).$$

If this collection is infinite, define λ by $\lambda(x) = (f_1(x), f_2(x), \cdots)$. In either case λ is linear.

To show that the range of λ is a subset of (c_0), assume the contrary. Then there exists $y \in X$ and $\varepsilon > 0$ such that $| f_i(y) | > \varepsilon$ for infinitely many indices i. But since S is a convex polytope there exists an integer k such that

$$(2/\varepsilon)y \in \bigcap_{i>k} \{x \in X \mid | f_i(x) | \leq 1\},$$

and therefore we reach the contradiction that $f_i(y) \leq \varepsilon/2$ for all but finitely many indices i.

To prove that λ is an isometry, suppose $x \in X$. The point $x/\| x \|$ is in some face of P. Thus $| f_j(x/\| x \|) | = 1$ for some integer j and $| f_i(x/\| x \|) | \leq 1$ for all indices i. Therefore

$$\| \lambda(x/\| x \|) \| = \sup_i | f_i(x/\| x \|) | = 1$$

so that $\| \lambda(x) \| = \| x \|$.

Corollary 4.2.1. *A separable normed linear space X contains a bounded convex polytope P such that $\theta \in P^0$ if and only if it is linearly homeomorphic to a subspace of (c_0).*

Proof. The set $P \cap (-P)$ can easily be seen to be a symmetric non-degenerate bounded convex polytope. The space X can be renormed with an equivalent norm in such a way that $P \cap (-P)$ is its unit ball. This renormed space is then, by Theorem 4.2, isometric to a subspace of (c_0). The converse is a consequence of Corollary 4.1.1.

Corollary 4.2.2. *If P is a convex polytope which is symmetric about the origin in a separable normed linear space, then it can be realized as a central section of the unit ball in (c_0). If P is a symmetric convex polyhedron with $2k$ faces then it can be realized as the central section of a k-dimensional cube.*

Proof. If we define a central section of the unit ball S of (c_0) as the intersection $S \cap Y$ of S and a subspace Y of (c_0), then the first part of the assertion is an elementary consequence of Theorems 4.2 and 2.3.

The second assertion has already been proved by Klee [6]. Here we prove it by noting that the map λ defined in the proof of Theorem 4.2 maps X into a k-dimensional space such that $\lambda(P)$ is the central section of the unit cube.

We have characterized subspaces (c_0) in terms of convex polytopes to within an isometry (Corollary 4.1.1) and (Theorem 4.2) and to within an isomorphism (Corollary 4.2.1). We will now characterize (c_0) itself by generalizing the definition of a cube to an infinite dimensional space and making the characterization in terms of it.

Let $\{H_\beta \mid \beta \in \mathrm{B}\}$ be a collection of all smooth (hyperplane) supports of a bounded convex polytope P which is symmetric about the origin. For each β let N_β be a translate of the hyperplane H_β such that $\theta \in N_\beta$. The polytope P will be called a paralleletope if for each $\beta_0 \in \mathrm{B}$, $\bigcap \{N_\beta \mid \beta \in \mathrm{B}, \beta \neq \beta_0\} \neq \{\theta\}$. The following three properties are sufficient to convince the reader that if the indexing set B is finite, say k in number, then P is a k-dimensional cube (or parallelepiped, since the concepts are indistinguishable in an arbitrary linear space) centered about the origin.

Property 4.1. *The intersection $\bigcap \{N_\beta \mid \beta \in \mathrm{B}\} = \{\theta\}$*

Proof. If the assertion were false, P would not be bounded.

Property 4.2. *If for each $\beta \in \mathrm{B}$, $\pm E_\beta$ denotes the pair of corresponding smooth (half space) supports of P, then for each $\beta_0 \in \mathrm{B}$ the intersections*

$$[\textstyle\bigcap_{\beta \neq \beta_0} E_\beta] \cap [\bigcap_{\beta \in \mathrm{B}} - E_\beta] \quad \text{and} \quad [\bigcap_{\beta \in \mathrm{B}} E_\beta] \cap [\bigcap_{\beta \neq \beta_0} E_\beta]$$

are not bounded.

Property 4.3. *If for each $\beta \in \mathrm{B}$, f_β is defined by $H_\beta = \{x \in X \mid f_\beta(x) = 1\}$ then the collection $\{f_\beta \mid \beta \in \mathrm{B}\}$ is a linearly independent class of continuous linear functionals.*

Proof. If this collection were not linearly independent then one of the null spaces N_{β_0} would contain the intersection $\bigcap \{N_\beta \mid \beta \in \mathrm{B}, \beta \neq \beta_0\}$ of the remaining null spaces. But then by Property 4.1 we would have

$$\bigcap \{N_\beta \mid \beta \in \mathrm{B}, \beta \neq \beta_0\} = \bigcap \{N_\beta \mid \beta \in \mathrm{B}\} = \{\theta\}$$

which contradicts the definition of a paralleletope.

THEOREM 4.3. *A necessary and sufficient condition that a B-space X be linearly isometric to (c_0) is that the closed unit sphere S of X be a paralleletope with a countably infinite number of faces.*

Proof. For each integer i, let $H_i = \{(x_1, x_2, \cdots) \in (c_0) \mid x_i = 1\}$. The collection $\{\pm H_i \mid i = 1, 2, \cdots\}$ is then the minimal representation of the unit sphere S (which has previously been seen to be a convex polytope) and it can easily be verified that S is a paralleletope.

To prove the converse let λ be the map from X into (c_0) defined in the proof of Theorem 4.2. If P is a paralleletope with a countably infinite number of faces, then for each integer i there exists $u_i \in X$ such that $\lambda(u_i) = (\delta_{1i}, \delta_{2i}, \cdots)$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. But since the closed linear span of the "unit vectors" $\lambda(u_i)$ is dense in (c_0), we have $\lambda(X) \supset (c_0)$. The assertion then follows from the proof of Theorem 4.2.

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