A CHARACTERIZATION OF SELF-INJECTIVE RINGS

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Eilenberg and Nakayama [2] showed that if a ring R is left and right Noetherian, R is quasi-Frobenius if and only if R is self-injective. Self-injective rings have been considered by a number of other authors usually in connection with rings of quotients; see, for instance, Gentile [3], Lambek [5] and Wong and Johnson [6]. In this note we shall give a characterization of self-injective rings in terms of function topologies on modules as defined in S. U. Chase's paper [1].

Throughout this paper we shall assume that R is a ring with identity and with the discrete topology. The identity of R acts like identity on R-modules.

Let M be a left R-module with the discrete topology. We shall denote $\mathrm{Hom}_R(M, R)$ by M^*; M^* is called the dual of M. M^* is given the structure of a right R-module by $(fr)(x) = f(x)r$ where $f \in M^*$, $r \in R$ and $x \in M$.

If M and N are two R-modules and $\theta : M \to N$ is an R-homomorphism, applying $(^*)$ gives $\theta^* : N^* \to M^*$ where for $f \in N^*$, $\theta^*(f)$ acts on elements of M like $f \circ \theta$. For further properties of $(^*)$ see [4, Chapter 4].

The following definition is due to S. U. Chase.

DEFINITION 1. Let T be a right R-submodule of M^*. The T-topology on M is defined to be the weak topology induced on M by T. That is, it is the coarsest topology on M such that every element of T is a continuous homomorphism from M to R.

The T-topology on M is equivalent to the topology whose base for the neighborhood system of zero consists of all subsets of M of the form $\bigcap_{i=1}^{n} \mathrm{Ker}\, \tau_i$, $\tau_i \in T$, $i = 1, \cdots, n$. See [1, Prop. 1.2].

It is easy to see that the T-topology for M is Hausdorff if and only if for each $m \in M$, $m \neq 0$, there exists $f \in T$ such that $f(m) \neq 0$. In this situation we say that "T separates points of M".

DEFINITION 2. We shall say T is separating if the T-topology (for M) is Hausdorff.

We shall denote by $C_T \mathrm{Hom}_R(M, R)$ the right R-submodule of M^* consisting of all continuous R-homomorphisms from M to R where M has the separating T-topology. From the definition of T-topology it follows that $T \subseteqq C_T \mathrm{Hom}_R(M, R)$. We shall be interested in the case that

$$T = C_T \mathrm{Hom}_R(M, R).$$

Received August 5, 1964.
[1] Research supported in part by a National Science Foundation Grant.

The following theorem gives a characterization of self-injective rings.

THEOREM. *A necessary and sufficient condition that R be left self-injective is that $T = C_T \operatorname{Hom}_R(M, R)$ for all left R-modules M and all separating right R-submodules T of M^*.*

Proof that the condition is sufficient. Suppose R is not left self-injective. Then there exists a left ideal L such that the exact sequence

$$0 \to L \xrightarrow{j} R \to \frac{R}{L} \to 0$$

induces the exact sequence

$$0 \to \operatorname{Hom}_R\left(\frac{R}{L}, R\right) \to \operatorname{Hom}_R(R, R) \xrightarrow{j^*} \operatorname{Hom}_R(L, R)$$

$$\to \operatorname{Ext}_R^1\left(\frac{R}{L}, R\right) \to 0 \qquad \text{with } \operatorname{Ext}_R^1\left(\frac{R}{L}, R\right) \neq 0.$$

(See [4] for the notation "Ext".) That is, j^* is not an epimorphism, and we conclude that $\operatorname{Hom}_R(L, R) \supsetneq j^*(\operatorname{Hom}_R(R, R))$.

Let T be the right R-submodule of $\operatorname{Hom}_R(L, R)$ generated by j. We note that T separates points of L, so L is Hausdorff in the T-topology. Since j is a monomorphism, L is discrete in the T-topology; that is,

$$C_T \operatorname{Hom}_R(L, R) = \operatorname{Hom}_R(L, R).$$

We shall show that $T = j^*(\operatorname{Hom}_R(R, R))$. Since $T = jR$, for any $t \in T$ there exists an element r in R such that $t = jr$. Therefore $t(x) = jr(x) = j(x)r$ for all x in L. Now $j(x)r = (f_r \circ j)(x)$ where f_r is the right multiplication by r. Thus we have $t = f_r \circ j = j^*(f_r)$, $f_r \in \operatorname{Hom}_R(R, R)$. This shows that $T \subseteq j^*(\operatorname{Hom}_R(R, R))$. To show the reverse containment, we note that every homomorphism from R to R is given by the right multiplication of an element in R. Then it is easy to see that $j^*(\operatorname{Hom}_R(R, R)) \subseteq T$. Therefore $T = j^*(\operatorname{Hom}_R(R, R))$ and hence $C_T \operatorname{Hom}_R(L, R) \supsetneq T$.

Proof that the condition is necessary. Suppose R is left self-injective. As we noted above, it is always true that $T \subseteq C_T \operatorname{Hom}_R(M, R)$. We wish to show that $C_T \operatorname{Hom}_R(M, R) \subseteq T$ for all left R-modules M and right R-submodules T of M^*.

Let $f \in C_T \operatorname{Hom}_R(M, R)$. Since R is discrete and f is continuous, $\operatorname{Ker} f$ is open in M. Then there exist $t_1, t_2, \cdots, t_n \in T$ such that

$$\bigcap_{i=1}^n \operatorname{Ker} t_i \subseteq \operatorname{Ker} f.$$

Let $K = \bigcap_{i=1}^{n} \operatorname{Ker} t_i$, $F_n = R \oplus R \oplus \cdots \oplus R$, ($n$ copies of R). Define

$$\theta : M \to F_n$$

by $\theta(m) = (t_1(m), t_2(m), \cdots, t_n(m))$, $m \in M$. Then $\operatorname{Ker} \theta = K$.

Consider the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & M \arrow[rr, "\theta"] \arrow[dr, "\beta"'] & & F_n \arrow[r] & \dfrac{F_n}{\operatorname{Im}\theta} \arrow[r] & 0 \\
& & & \operatorname{Im}\theta \arrow[ur, "j"'] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where β is θ with restricted image and $j =$ the injection of $\operatorname{Im}\theta$ into F_n.

$(^*)$ the above diagram and we have the exact commutative diagram

$$\begin{tikzcd}
& & & & & & & {} \\
& & & & & & \operatorname{Ext}_R^1(\operatorname{Im}\theta, R) \arrow[ur, dotted] & \\
& & & & & K^* \arrow[ur] & & \\
0 \arrow[r] & \left(\dfrac{F_n}{\operatorname{Im}\theta}\right)^* \arrow[r] & F_n^* \arrow[rr, "\theta^*"] \arrow[dr, "j^*"'] & & M^* \arrow[ur] & & & \\
& & & (\operatorname{Im}\theta)^* \arrow[ur, "\beta^*"'] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & \operatorname{Ext}_R^1\left(\dfrac{F_n}{\operatorname{Im}\theta}, R\right). \arrow[dr] & & & \\
& & & & & 0 & &
\end{tikzcd}$$

Since R is left self-injective, $\operatorname{Ext}_R^1(\cdot, R) = 0$ and the above diagram reduces to the following exact commutative diagram:

$$\begin{tikzcd}
 & & & & & & 0 \\
 & & & & & K^* \arrow[ur] & \\
0 \arrow[r] & \left(\frac{F_n}{\mathrm{Im}\ \theta}\right)^* \arrow[r] & F_n^* \arrow[rr, "\theta^*"] \arrow[dr, "j^*"'] & & M^* \arrow[ur] & & \\
 & & & (\mathrm{Im}\ \theta)^* \arrow[ur, "\beta^*"'] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0. & &
\end{tikzcd}$$

Now we shall show $\mathrm{Im}\ \theta^*$ is contained in the right R-submodule of M^* generated by $t_1, t_2, \cdots, t_n$. Let $g \in \theta^*(F_n^*)$, say $g = \theta^*(\alpha)$, $\alpha \in F_n^* = \mathrm{Hom}_R(F_n, R)$. Let α_i be the restriction of α on the i^{th} summand of F_n. Since each α_i is given by the right multiplication by an element in R, say r_i, and $\alpha = \sum_{i=1}^n \alpha_i$; for $m \in M$ we have

$$\begin{aligned} g(m) = [\theta^*(\alpha)](m) = \alpha[\theta(m)] &= \alpha(t_1(m), t_2(m), \cdots, t_n(m)) \\ &= \textstyle\sum_{i=1}^n t_i(m) r_i = \sum_{i=1}^n [t_i r_i](m). \end{aligned}$$

This shows that $g = \sum_{i=1}^n t_i r_i$, $r_i \in R$. Hence $\mathrm{Im}\ \theta^*$ is contained in the right R-submodule of M^* generated by $t_1, t_2, \cdots, t_n$.

Now by exactness and commutativity of the above diagram $\beta^*[(\mathrm{Im}\ \theta)^*] = \mathrm{Im}\ \theta^*$. Therefore $\beta^*[(\mathrm{Im}\ \theta)^*] \subseteqq T$.

Considering the following diagram

$$\begin{tikzcd}
M \arrow[r, "\beta"] \arrow[d, "f"'] & \mathrm{Im}\ \theta \arrow[dl, dotted] \\
R &
\end{tikzcd}$$

where $\mathrm{Ker}\ f \supseteqq K = \mathrm{Ker}\ \theta = \mathrm{Ker}\ \beta$, we see that there exists $f' \in (\mathrm{Im}\ \theta)^*$ such that $f' \circ \beta = f$. Therefore it follows that $f \in \beta^*([\mathrm{Im}\ \theta]^*)$ and since we have shown that $\beta^*([\mathrm{Im}\ \theta]^*)$ is contained in T, $f \in T$. This completes the proof of the theorem.

The following lemma [1, Lemma 1.6] is an easy consequence of Definition 1.

LEMMA. *Let M be a topological R-module, and N be an R-module with the S-topology, where S is an R-submodule of N^*. Then a homomorphisn $f : M \to N$ is continuous if and only if $f^*(S)$ is contained in the submodule of M^* consisting of all continuous homomorphisms from M to R.*

COROLLARY. *Let R be a left self-injective ring. Let M and N be R-modules with the separating T-, S-topologies respectively, where $T \subseteqq M^*$, $S \subseteqq N^*$. Then a homomorphism $f : M \rightarrow N$ is continuous if and only if $f^*(S) \subseteqq T$.*

The proof is a direct consequence of the theorem and the lemma.

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