HYPERHOMOLOGY SPECTRA AND A MULTIPLICATIVE KUNNETH THEOREM

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1. Introduction

Throughout this paper we deal exclusively with chain complexes of abelian groups. If K is a torsion-free complex, the homology spectrum of K consists of: the groups

$$H(K), \qquad H(K, m) = H(K \otimes Z_m) \qquad\qquad (m > 0);$$

the coefficient homomorphisms induced by the canonical projections and injections

$$Z \to Z_m, \qquad Z_{mk} \to Z_m, \qquad Z_m \to Z_{mk} \qquad\qquad (mk > 0);$$

and the connecting homomorphisms induced by the exact sequences

$$0 \to Z \xrightarrow{m} Z \to Z_m \to 0 \qquad\qquad (m > 0).$$

The "multiplicative" Kunneth Theorem given in [2] states that for K and L torsion-free differential graded rings, the ring $H(K \otimes L)$ is completely determined by the homology spectra of K and L. A natural question then is: What is required to determine the ring $H(K \otimes L)$ if K and L are not necessarily torsion-free? The purpose of this note is to give a (partial) answer to this question. In particular, we shall show that the results of [2], suitably modified, can be extended to give a more general multiplicative Kunneth Theorem (Theorem 3.2) for which we need only require that $H(\mathrm{Tor}\,(K, L)) = 0$, instead of the condition that both K and L be torsion-free. Finally, we indicate briefly how these results can be carried over to the case of any finite number of complexes.

The chief difficulty with an arbitrary complex K is that the short exact coefficient sequence

$$0 \to Z \xrightarrow{m} Z \to Z_m \to 0$$

does not remain exact (on the left) when tensored with the complex K, and hence no connecting homomorphism $H(K, m) \to H(K)$ is defined. The basic idea needed to remedy this defect and to produce an analogue for the homology spectrum of a torsion-free complex (which reduces to the homology spectrum in case K is torsion-free) is to move from the homology spectrum to the hyperhomology spectrum of a complex.

2. The hyperhomology spectrum

We shall assume that the definition of the hyperhomology group of the complexes K and L, $\mathfrak{L}(K \otimes L)$, and the definition and properties of free double

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complex resolutions of complexes are known. This information may be found in Chapter XVII of Cartan-Eilenberg [1]. In addition we assume a familiarity with the main results of [2].

If K is a complex and G is any abelian group, G can be considered as a complex (in dimension zero). Then the hyperhomology group of K with coefficients in G is the group $\mathfrak{L}(K \otimes G)$. In particular if $G = Z_m$ $(m > 0)$, we denote by $\mathfrak{L}(K, m)$ the group $\mathfrak{L}(K \otimes Z_m)$. We shall occasionally denote $H(K) = \mathfrak{L}(K \otimes Z)$ by $\mathfrak{L}(K, 0)$.

Since $\mathfrak{L}(K, m) = H(\bar{K} \otimes Z_m)$ ($\bar{K}$ free resolution of K), the canonical maps $Z \to Z_m$, $Z_{mk} \to Z_m$ and $Z_m \to Z_{mk}$ induce coefficient homomorphisms:

$$\lambda_m^{mk} : \mathfrak{L}(K, mk) \to \mathfrak{L}(K, m) \qquad (m, k \geq 0);$$

$$\mu_{mk}^{m} : \mathfrak{L}(K, m) \to \mathfrak{L}(K, mk) \qquad (m, k > 0).$$

Since $\bar{K}$ is free, the exact sequence

$$0 \to Z \xrightarrow{m} Z \to Z_m \to 0$$

induces a connecting homomorphism of degree -1:

$$\mu_0^m : \mathfrak{L}(K, m) = H(\bar{K} \otimes Z_m) \to H(\bar{K}) \cong H(K) = \mathfrak{L}(K, 0).$$

The *hyperhomology spectrum* of the complex K consists of the groups $\mathfrak{L}(K, m)$ $(m \geq 0)$ together with the maps λ_m^{mk}, μ_{mk}^m $(m, k \geq 0)$. It is denoted $\{\mathfrak{L}(K, m)\}$. It follows from the definition of $\mathfrak{L}(K, m)$ that for K torsion-free $\mathfrak{L}(K, m) \cong H(K, m) = H(K \otimes Z_m)$; i.e. in this case the hyperhomology spectrum reduces to the homology spectrum (which was used in [2]).

If K an L are complexes, then the tensor product of their hyperhomology spectra, denoted $\{\mathfrak{L}(K, m)\} \otimes \{\mathfrak{L}(L, m)\}$, is the abelian group

$$[\textstyle\sum_{m \geq 0} \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)]/S,$$

where S is the subgroup generated by all elements of the form:

(i) $\lambda_m^{mk} u_{mk} \otimes v_m - (-1)^{\deg \mu_{mk}^m \cdot \deg u_{mk}} u_{mk} \otimes \mu_{mk}^m v_m \quad (mk \geq 0)$;
(ii) $\mu_{mk}^m u_m \otimes v_{mk} - u_m \otimes \lambda_m^{mk} v_{mk} \quad (mk \geq 0)$;

where $u_i \in \mathfrak{L}(K, i)$ and $v_j \in \mathfrak{L}(L, j)$. If $u \otimes v \in \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)$ represents an element x of $\{\mathfrak{L}(K, m)\} \otimes \{\mathfrak{L}(L, m)\}$, then the degree of x is $\deg u + \deg v - 1$ if $m > 0$ and $\deg u + \deg v$ if $m = 0$.

If K is a complex and $\bar{K}$ a free resolution of K, then the hyperhomology spectrum of K, $\{\mathfrak{L}(K, m)\}$, is by definition the homology spectrum of $\bar{K}$, $\{H(\bar{K}, m)\}$. Hence as a special case of Theorem 2.2 of [2] we have

THEOREM 2.1. *If K and L are complexes, then there is a natural isomorphism of graded groups:*

$$\{\mathfrak{L}(K, m)\} \otimes \{\mathfrak{L}(L, m)\} \cong \mathfrak{L}(K \otimes L).$$

3. Products

A *differential graded ring* K is a complex of abelian groups together with chain maps $\pi_K : K \otimes K \to K$ and $I_K : Z \to K$ such that the following diagrams are commutative:

$$\text{(3.1)}\qquad \begin{array}{ccc} K \otimes K \otimes K & \xrightarrow{\pi_K \otimes 1} & K \otimes K \\ \downarrow{\scriptstyle 1 \otimes \pi_K} & & \downarrow{\scriptstyle \pi_K} \\ K \otimes K & \xrightarrow{\pi_K} & K \end{array} \qquad \begin{array}{ccccc} Z \otimes K & = & K & = & K \otimes Z \\ \downarrow{\scriptstyle I_K \otimes 1} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle 1 \otimes I_K} \\ K \otimes K & \xrightarrow{\pi_K} & K & \xleftarrow{\pi_K} & \otimes K. \end{array}$$

The first diagram asserts that the product $uv = \pi_K(u \otimes v)$ is associative and the second that $I_K(1) = 1_K$ is a two sided identity for this product. (cf. MacLane [4], Chapter 6).

If $\bar{K}$ is a free double complex resolution of K, then the maps π_K and I_K can be lifted to double complex maps $\pi_{\bar{K}} : \bar{K} \otimes \bar{K} \to \bar{K}$ and $I_{\bar{K}} : Z \to \bar{K}$. This fact follows from Proposition 1.2 in Chapter XVII of Cartan-Eilenberg [1]. The statement of this proposition requires that both X and Y be projective resolutions; the proof, however, uses only the fact that Y is a projective resolution and that $B^I_{p,*}(X)$, $H^I_{p,*}(X)$ are free complexes over $B_p(A)$, $H_p(A)$, for each p. This is exactly the situation here: $B^I(\bar{K} \otimes \bar{K})$ is free since $\bar{K} \otimes \bar{K}$ is, and by the Kunneth Theorem

$$H^I(\bar{K} \otimes \bar{K}) \cong H^I(\bar{K}) \otimes H^I(\bar{K}),$$

which is free since $\bar{K}$ is a free double complex resolution.

In general, however, $\bar{K}$ (with the maps $\pi_{\bar{K}}$, $I_{\bar{K}}$) is not a differential graded ring. The fact that the diagrams (3.1) commute for K implies only that the corresponding diagrams for $\bar{K}$ are homotopy commutative It is true, therefore, that $H(\bar{K})$ is a graded ring and that the augmentation $\bar{K} \to K$ induces a ring isomorphism $H(\bar{K}) \cong H(K)$.

Similarly if L is a differential graded ring and $\bar{L}$ a free resolution for L, then it follows that $H(\bar{K}, m), H(\bar{L}, m), H(\bar{K} \otimes \bar{L}, m), H(K \otimes \bar{L}, m), H(\bar{K} \otimes L, m)$ ($m \geq 0$) are all graded rings. This involves showing that the diagrams (3.1) with $\bar{K} \otimes Z_m$, $\bar{L} \otimes Z_m$, $\bar{K} \otimes \bar{L}$, etc. in place of K are homotopy commutative. These facts are consequences of standard arguments about homotopic maps and their tensor products. Therefore if K and L are differential graded rings, $\mathcal{L}(K \otimes L)$ is a ring with π being the composition:

$$\begin{aligned} &\mathcal{L}(K \otimes L) \otimes \mathcal{L}(K \otimes L) \\ &\quad = H(\bar{K} \otimes \bar{L}) \otimes H(\bar{K} \otimes \bar{L}) \xrightarrow{\alpha} H(\bar{K} \otimes \bar{L} \otimes \bar{K} \otimes \bar{L}) \\ &\quad \xrightarrow{\tau} H(\bar{K} \otimes \bar{K} \otimes \bar{L} \otimes \bar{L}) \xrightarrow{(\pi_{\bar{K}} \otimes \pi_{\bar{L}})_*} H(\bar{K} \otimes \bar{L}) = \mathcal{L}(K \otimes L), \end{aligned}$$

where α is the usual homology product and τ the obvious interchange of factors. The identity in $\mathcal{L}(K \otimes L)$ is given by

$$Z = Z \otimes Z \xrightarrow{(I_{\bar{K}} \otimes I_{\bar{L}})_*} H(\bar{K} \otimes \bar{L}) = \mathfrak{L}(K \otimes L).$$

Note that essentially the same product will be defined if one uses $H(\bar{K} \otimes L)$ or $H(K \otimes \bar{L})$ in place of $H(\bar{K} \otimes \bar{L})$, since the augmentation maps $\bar{K} \to K$ and $\bar{L} \to L$ induce ring isomorphisms

$$H(K \otimes \bar{L}) \leftarrow H(\bar{K} \otimes \bar{L}) \to H(\bar{K} \otimes L).$$

If K and L are differential graded rings, then so are $\mathfrak{L}(K, m)$ and $\mathfrak{L}(L, m)$ $(m \geq 0)$. We would like to put a product structure on the spectra tensor product $\{\mathfrak{L}(K, m)\} \otimes \{\mathfrak{L}(L, m)\}$ in such a way that the isomorphism of Theorem 2.1 becomes a ring isomorphism. The fact that this can be done is again a consequence of Section 3 of [2].

For convenience we indicate briefly how this product is defined. First a product is defined on $\sum_{m\geq 0} \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)$ (it is non-associative and has other peculiarities); this induces the desired product (associative, etc.) on the quotient

$$[\textstyle\sum_m \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)]/S = \{\mathfrak{L}(K, m)\} \otimes \{\mathfrak{L}(L, m)\}.$$

If x and y are homogeneous generators of $\mathfrak{L}(K, i) \otimes \mathfrak{L}(L, i)$ and $\mathfrak{L}(K, j) \otimes \mathfrak{L}(L, j)$ respectively, then a product $*$ is given in $\sum_{m\geq 0} \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)$ by

$$\begin{aligned}
x * y &= x \cdot [(\lambda_i^0 \otimes \lambda_j^0)y], && \text{if } j = 0;\\
x * y &= (-1)^{\deg x}[(\lambda_j^0 \otimes \lambda_j^0)x] \cdot y, && \text{if } j > 0 \text{ and } i = 0;\\
x * y &= a[(\lambda_c^i \otimes \lambda_c^i)x] \cdot [(\lambda_c^j \otimes \lambda_c^j)(D_j\, y)] \\
&\quad + (-1)^{\deg D_i x} b[(\lambda_c^i \otimes \lambda_c^i)(D_i\, x)] \cdot [(\lambda_c^j \otimes \lambda_c^j)y], \\
& && \text{if } i > 0 \text{ and } j > 0,
\end{aligned}$$

where $\cdot$ is the product in (each) $\mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)$; $c = (i, j)$ and $ai + bj = c$; and

$$D_m : \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m) \to \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)$$

is the map given on $u \otimes v \in \mathfrak{L}(K, m) \otimes \mathfrak{L}(L, m)$ by

$$[(\lambda_m^0 \mu_0^m \otimes 1) + (-1)^{\deg u}(1 \otimes \lambda_m^0 \mu_0^m)](u \otimes v).$$

We can summarize these results in

THEOREM 3.2. *If K and L are differential graded rings, then there is a natural isomorphism of graded rings:*

$$\{\mathfrak{L}(K, m)\} \otimes \{\mathfrak{L}(L, m)\} \cong \mathfrak{L}(K \otimes L).$$

Thus the ring $\mathfrak{L}(K \otimes L)$ is completely determined by the hyperhomology spectra of K and L. Furthermore, if

$$H(\mathrm{Tor}\,(K, L)) = 0$$

then the theorem remains true with $H(K \otimes L)$ in place of $\mathcal{L}(K \otimes L)$.

The proof of the last statement is a consequence of the fact that there is an exact triangle (cf. [1]):

$$\begin{tikzcd} & \mathcal{L}(K \otimes L) \arrow[dl] & \\ H(K \otimes L) \arrow[rr] & & H(\mathrm{Tor}\,(K, L)). \arrow[ul] \end{tikzcd}$$

Thus we have obtained a much more general multiplicative Kunneth Theorem than the one given in [2].

4. A multiple multiplicative Kunneth Theorem

If $K^1, K^2, \cdots, K^n$ are complexes, then the tensor product of their hyperhomology spectra

$$\{\mathcal{L}(K^1, m)\} \otimes \{\mathcal{L}(K^2, m)\} \otimes \cdots \otimes \{\mathcal{L}(K^n, m)\}$$

is again a certain quotient of the group

$$(4.1) \qquad \textstyle\sum_{m \geq 0} \mathcal{L}(K^1, m) \otimes \cdots \otimes \mathcal{L}(K^n, m).$$

The precise definition is given in [3]. The only relevant fact needed here is the observation that if $K^1, \cdots, K^n$ are differentially graded rings, a $*$ product (analogous to the one defined in Section 3) can be defined on (4.1) and induces a product in the spectra tensor product.

It is clear that all the other products defined in Section 3 extend without difficulty to the case of n differential graded rings; in particular,

$$\mathcal{L}(K^1 \otimes \cdots \otimes K^n)$$

is a graded ring. If we so denote by $\mathrm{Mult}_i\,(A^1, \cdots, A^n)$ the i-th left derived functor of the functor $A^1 \otimes A^2 \otimes \cdots \otimes A^n$, then we have

THEOREM 4.2. *If $K^1, \cdots, K^n$ are differential graded rings, then there is a natural isomorphism of graded rings:*

$$(4.3) \qquad \{\mathcal{L}(K^1, m)\} \otimes \cdots \otimes \{\mathcal{L}(K^n, m)\} \cong \mathcal{L}(K^1 \otimes \cdots \otimes K^n).$$

Thus the ring $\mathcal{L}(K^1 \otimes \cdots \otimes K^n)$ is completely determined by the hyperhomology spectra of $K^1, \cdots, K^n$. Futhermore, if

$$(4.4) \qquad H(\mathrm{Mult}_i\,(K^1, \cdots, K^n)) = 0 \qquad \text{for} \quad i > 0,$$

then the theorem remains true if $\mathcal{L}(K^1 \otimes \cdots \otimes K^n)$ is replaced by

$$H(K^1 \otimes \cdots \otimes K^n).$$

The existence of an isomorphism (4.3) of the additive groups is just a special case of Theorem 3.1 of [3]. The proof that (4.4) implies that

$$\mathcal{L}(K^1 \otimes \cdots \otimes K^n) = H(K^1 \otimes \cdots \otimes K^n)$$

is given in the proof of Corollary 1.2 of [3]. It might also be noted that it is shown there that (4.4) holds if $n - 1$ of the complexes $K^1, \cdots, K^n$ are torsion-free. Finally, the fact that the isomorphism (4.3) does in fact preserve the product structure and is thus a ring isomorphism, follows as in Theorem 3.2 from Section 3 of [2] (where the case $n = 3$ is treated; but all of the arguments are valid for any n).

References

1. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, Princeton University Press, 1956.
2. T. W. Hungerford, *Bockstein spectra*, Trans. Amer. Math. Soc., vol. 115(1965), pp. 225-241.
3. ———, *Multiple Kunneth formulas for abelian groups*, Trans. Amer. Math. Soc., vol. 118(1965), pp. 257–276.
4. S. MacLane, *Homology*, Berlin, Gottingen, and Heidelberg, Springer, 1963.

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