ON THE EQUATION $f_1 g_1 + f_2 g_2 = 1$ IN H^p.

BY

JOSEPH A. CIMA AND GERALD D. TAYLOR

1. Introduction and definition

Let D denote the unit disk in the complex plane and $\bar{D}$ its closure. We shall say that f is in H^p of the disk, $p \geq 1$, if f is holomorphic in D and satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta < M < +\infty$$

for all $r < 1$. It is known that H^p is a complete normed linear space with

$$\| f \|_p = \lim_{r \to 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \, d\theta \right)^{1/p}.$$

In this paper we investigate the following equation

$$f_1(z)g_1(z) + f_2(z)g_2(z) = 1, \qquad z \in D \tag{1.1}$$

in the following sense. Given f_1 and f_2 in H^p and H^r respectively, what conditions are necessary to guarantee the existence of the pair g_1 and g_2 in some Hardy spaces satisfying (1.1). We show by examples one cannot always hope for solutions. We study the structure of the class of the given function pairs f_1 and f_2 and also the structure of the solution pairs g_1 and g_2.

Our study is motivated by the classical results of W. Rudin, D. J. Newman and L. Carleson. Since we use their results we state them here. Let H^∞ denote the space of bounded holomorphic functions in D with the sup norm. The closed subalgebra of H^∞ consisting of those functions which are also continuous on $\bar{D}$ is denoted by A (of $\bar{D}$). In [5] Rudin showed that if f_1 and f_2 are in A and $| f_1 | + | f_2 | > 0$ on $\bar{D}$ then the ideal generated by f_1 and f_2 is A, or there exist solutions g_1 and g_2 in A satisfying (1.1) on $\bar{D}$. Moreover, D. J. Newman has indicated that proving for f_1 and f_2 in H^∞ with $| f_1 | + | f_2 | \geq \delta > 0$ we can find g_1 and g_2 in H^∞ satisfying (1.1) on D is equivalent to showing that the point evaluations on D are dense in the maximal ideal space of H^∞. Carleson's [1] solution of this (Corona) problem has completed the H^∞ phrase of the problem.

We wish to make the following convention. If $S = \{z; | z - z_0 | < \rho\}$ is a disk then A of $\bar{S}$ means those functions continuous in $\bar{S}$ and holomorphic in S.

2. The basic solution

The following result is known but we have not found a proof in the literature, therefore we include our proof not only for completeness but also because it gives us valuable information about the pairs of solutions of (1.1).

Received June 10, 1966.

THEOREM 1. *Let $a(z)$ and $b(z)$ be holomorphic in D and assume*

$$|a(z)| + |b(z)| > 0$$

in D. Then there exist two holomorphic functions in D, $H_1(z)$ and $H_2(z)$, satisfying the equation

$$H_1(z)a(z) + H_2(z)b(z) = 1, \qquad z \in D. \tag{2.1}$$

Proof. In the course of the proof we find it necessary to subject the argument of our functions to certain magnifications and for this reason we use the following notations. Let

$$D_k(z) = \{z : |z| < (k-1)/k\}, \qquad k = 2, 3, 4, \cdots$$

and

$$D_k(\zeta) = \{\zeta = (k/(k-1))z; z \in D_k(z)\} = D.$$

Then $a(z)$ and $b(z)$ are in A of $\bar{D}_k(z)$ for each k. We can find solutions $h_{1,k}(z)$ and $h_{2,k}(z)$ of (2.1) valid in $\bar{D}_k(z)$, where the $h_{i,k}(z)$ are also in A of $\bar{D}_k(z)$. It is sufficient then to prove the theorem to show that the functions $h_{i,k}(z)$ can be chosen so that $\lim_{k\to\infty} h_{i,k}(z) = H_i(z)$ exists uniformly on compact subsets of D for each $i = 1, 2$. It is clear that for a fixed compact subset $\hat{A}$ there is a positive integer K such that all the $h_{i,k}(z)$ are well defined on $\hat{A}$ for $k \geq K$. The existence of the required limit will be guaranteed if we can choose the functions $h_{i,k}(z)$ to satisfy the condition

$$|h_{i,k}(z) - h_{i,k+1}(z)| < 1/2^k \tag{2.2}$$

for $z \in \bar{D}_k(z)$. We proceed to the proof of (2.2). Assume $h_{1,k}(z)$ and $h_{2,k}(z)$ have been chosen to satisfy equations (2.1) and (2.2) on $\bar{D}_k(z)$ and $\bar{D}_{k-1}(z)$ respectively. We indicate how to obtain $h_{1,k+1}(z)$ and $h_{2,k+1}(z)$. Let $\hat{h}_{1,k+1}(z)$ and $\hat{h}_{2,k+1}(z)$ satisfy equation (2.1). Then on $\bar{D}_k(z)$ we have the following equality:

$$(\hat{h}_{1,k+1}(z) - h_{1,k}(z))a(z) = (h_{2,k}(z) - \hat{h}_{2,k+1}(z))b(z). \tag{2.3}$$

Let $B_{1,k}(\zeta)$ be the Blaschke product for $a(\zeta)$ and $B_{2,k}(\zeta)$ the Blaschke product for $b(\zeta)$ on $\bar{D}_k(\zeta)$. Then on $\bar{D}_k(z)$ we have the factorizations

$$a(z) = B_{1,k}(z)\hat{a}(z), \qquad b(z) = B_{2,k}(z)\hat{b}(z)$$

with $\hat{a}(z)$ and $\hat{b}(z)$ being non-zero in A of $\bar{D}_k(z)$. Thus $(\hat{a}(z))^{-1}$ and $(\hat{b}(z))^{-1}$ are holomorphic in $D_k(z)$ and continuous on $\bar{D}_k(z)$. From equation (2.3) and our hypotheses we deduce the equalities

$$\begin{aligned} \hat{h}_{1,k+1}(z) - h_{1,k}(z) &= B_{2,k}(z)K_1(z) \\ h_{2,k}(z) - \hat{h}_{2,k+1}(z) &= B_{1,k}(z)K_2(z) \end{aligned} \tag{2.4}$$

on $\bar{D}_k(z)$, where the $K_i(z)$ are in A of $\bar{D}_k(z)$. We rewrite the right hand sides of (2.4) as

$$B_{2,k}(z)K_1(z) = (B_{2,k}(z)\hat{b}(z))((\hat{b}(z))^{-1}K_1(z)) = b(z)\phi_1(z)$$

where $\phi_1(z) = (\hat{b}(z))^{-1}K_1(z)$ and similarly

$$B_{1,k}(z)K_2(z) = a(z)\phi_2(z)$$

where $\phi_2(z) = (\hat{a}(z))^{-1}K_2(z)$. The $\phi_i(z)$ are in A of $\bar{D}_k(z)$. We can rewrite (2.3) now as

$$b(z)\phi_1(z)a(z) = a(z)\phi_2(z)b(z)$$

and conclude that $\phi_1(z) = \phi_2(z)$ on $\bar{D}_k(z)$.

Let $M_k = \max\ (\sup_{z\epsilon\bar{D}_k(z)} \mid a(z) \mid, \sup_{z\epsilon\bar{D}_k(z)} \mid b(z) \mid)$ and choose, by Wermers Theorem, [2, Pg 93] a polynomial $P_k(z)$ satisfying the inequality

$$\mid P_k(z) - \phi_1(z) \mid < 1/M_k 2^k$$

on $\bar{D}_k(z)$. Now we choose our $h_{1,k+1}(z)$ and $h_{2,k+1}(z)$ by the following equations

$$h_{1,k+1}(z) = \hat{h}_{1,k+1}(z) - P_k(z)b(z)$$

$$h_{2,k+1}(z) = \hat{h}_{2,k+1}(z) + P_k(z)a(z).$$

The pair $h_{1,k+1}(z)$ and $h_{2,k+1}(z)$ satisfy equation (2.1) on $\bar{D}_{k+1}(z)$ and moreover on $\bar{D}_k(z)$ we have

$$\begin{aligned} \mid h_{1,k+1}(z) - h_{1,k}(z) \mid &= \mid h_{1,k+1}(z) - P_k(z)b(z) - h_{1,k}(z) \mid \\ &= \mid b(z)\phi_1(z) - P_k(z)b(z) \mid < 1/2^k. \end{aligned}$$

Similarly $\mid h_{2,k+1}(z) - h_{2,k}(z) \mid < 1/2^k$ on $\bar{D}_k(z)$. This completes the proof.

We would like to make a few comments on the collection of all solutions of equations (2.1). Assume $a(z)$ and $b(z)$ are holomorphic and

$$\mid a(z) \mid + \mid b(z) \mid > 0$$

in D. The construction shows if $H_1(z)$ and $H_2(z)$ satisfy (2.1) and $K_1(z)$ and $K_2(z)$ satisfy (2.1) also then the H's and the K's are related by the equalities

$$H_1(z) = K_1(z) - k(z)b(z)$$

$$H_2(z) = K_2(z) + k(z)a(z)$$

where $k(z)$ is holomorphic in D. However, this implies that all such solutions are obtainable from a given pair of solutions by using a suitable holomorphic function $k(z)$. This observation for H^p solutions is a useful tool in our later work.

3. H^p solutions

Let l_p denote the set of complex sequences $\{b_m\}_{m=1}^{\infty}$ with $\sum_{m=1}^{\infty} \mid b_m \mid^p < +\infty$, and l^∞ consists of the sequences $\{c_m\}_{m=1}^{\infty}$ with

$$\mid c_m \mid \leq M < +\infty, \qquad m = 1, 2, 3, \cdots.$$

Given a sequence $\{\alpha_m\}_{m=1}^{\infty}$ in D define a mapping T_p from H^p into the set of complex sequences by

$$T_p(f) = \{f(\alpha_m)(1 - |\alpha_m|^2)^{1/p}\}_{m=1}^{\infty},$$

for each $f \in H^p$. We shall need a result of H. S. Shapiro and A. L. Shields [6].

THEOREM 2. *$T_pH^p = l_p$ if and only if*

$$\prod_{i=1, i\neq N}^{\infty} |(\alpha_i - \alpha_N)/(1 - \alpha_i \bar{\alpha}_N)| \geq \delta \geq 0, \qquad N = 1, 2, 3, \cdots.$$

The inequality in this theorem shall be referred to as condition (C). We can now state and prove our first theorem.

THEOREM 3. *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in D satisfying condition* (C). *Let $B(z)$ be the Blaschke product with simple zeros at the points $\{\alpha_n\}_{n=1}^{\infty}$ and let f be in H^p. Then there exist functions h_1 in H^1 and h_2 in $H^q(1/q + 1/p = 1)$, satisfying*

$$h_1(z)B(z) + h_2(z)f(z) = 1 \qquad \text{on } D$$

if and only if

$$\left\{\frac{1}{(f(\alpha_n))}(1 - |\alpha_n|^2)^{1/q}\right\}_{n=1}^{\infty} \text{ is in } l_q.$$

Proof. Assume

$$\left\{\frac{1}{f(\alpha_n)}(1 - |\alpha_n|^2)^{1/q}\right\}_{n=1}^{\infty} \text{ is in } l_q.$$

We have $f(\alpha_n) \neq 0$ for all $n = 1, 2, 3, \cdots$. By Theorem 1, we can find holomorphic functions g_1 and g_2 such that $g_1(z)B(z) + g_2(z)f(z) = 1$ for z in D. By Theorem 2, there exists an h_2 in H^q such that

$$T_p(h_2) = \{h_2(\alpha_m)(1 - |\alpha_m|^2)^{1/q}\}_{m=1}^{\infty} = \left\{\frac{1}{f(\alpha_m)}(1 - |\alpha_m|^2)^{1/q}\right\}_{m=1}^{\infty}.$$

That is $h_2(\alpha_n) = (f(\alpha_n))^{-1}$ for all $n = 1, 2, 3, \cdots$. We also know

$$(f(\alpha_n))^{-1} = g_2(\alpha_n)$$

for $n = 1, 2, 3, \cdots$ and so conclude that $h_2(\alpha_n) = g_2(\alpha_n)$ for $n = 1, 2, 3, \cdots$. The function $h_2(z) - g_2(z)$ is holomorphic on D and $B(z)$ divides this function.

$$h_2(z) - g_2(z) = B(z)k(z), \qquad z \in D,$$

where $k(z)$ is holomorphic in D. Letting

$$h_1(z) = g_1(z) - k(z)f(z)$$

we have $h_1(z)B_1(z) + h_2(z)f(z) = 1$ on D. Since f is in H^p and h_2 is H^q the product fh_2 is in H^1. Consequently the H^1 function $1 - h_2(z)f(z)$ has a factorization of the form $B_1(z)S(z)F(z)$, where F is outer in H^1, B_1 is a Blaschke product and S is a singular function, see [2, p. 69]. Dividing $1 - h_2(z)f(z)$ by $B(z)$ shows that $h_1(z)$ is also equal to an inner function $(B_1(z)/B(z))S(z)$ times the outer function F and so h_1 is in H^1.

Conversely, let us assume that there exists h_1 in H^1 and h_2 in H^q such that $h_1(z)B(z) + h_2(z)f(z) = 1$ on D. Then $h_2(\alpha_n) = (f(\alpha_n))^{-1}$ and the result of Theorem 2 shows that

$$\left\{\frac{1}{f(\alpha_n)}(1 - |\alpha_n|^2)^{1/q}\right\}_{n=1}^{\infty} = \{h_2(\alpha_n)(1 - |\alpha_n|^2)^{1/q}\}_{n=1}^{\infty}$$

is in l_q as the sequence $\{\alpha_n\}_{n=1}^{\infty}$ satisfies condition (C).

THEOREM 4. *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of points in D satisfying condition* (C). *Let $B(z)$ be the Blaschke product with simple zeros on the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and let $f \in H^{\infty}$. Then there are functions h_1 and h_2 in H^p satisfying the equation $h_1(z)B(z) + h_2(z)f(z) = 1$ on D if and only if*

$$\left\{\frac{1}{f(\alpha_n)}(1 - |\alpha_n|^2)^{1/q}\right\}_{n=1}^{\infty} \text{ is in } l_p \qquad (1 \le p < \infty).$$

The proof is patterned after that of Theorem 1 and is omitted. We make the following comment. If we let $p = \infty$ and interpret

$$\left\{\frac{1}{f(\alpha_n)}(1 - |\alpha_n|^2)^{1/p}\right\}_{n=1}^{\infty}$$

as $\{1/f(\alpha_n)\}_{n=1}^{\infty}$ in l_∞ (i.e. $|f(\alpha_n)| \ge \delta > 0$) then we have a known result. We also give an example. Let $0 < \alpha_1 < \alpha_2 \cdots$ be a sequence of real numbers in D with $\sum (1 - \alpha_m) < +\infty$ and assume $B_1(z)$ is the Blaschke product with zeros at $\{\alpha_m\}_{m=1}^{\infty}$. Choose δ_m, $0 < \delta_m < (\alpha_{m+1} - \alpha_m)/2$, so that if $|z - \alpha_m| < \delta_m$ then $|B(z)| < (1 - \alpha_m^2)^2$. If we set $\xi_m = \alpha_m - \delta_m/2$ then it is clear $\sum (1 - \xi_m) < +\infty$ and we may form $B_2(z)$ the Blaschke product with the sequence $\{\xi_m\}_{m=1}^{\infty}$ as zeros. We show that there can be no H^p solutions to the equation $f_1(z)B_1(z) + f_2(z)B_2(z) = 1$ on D. For if f_1 and f_2 were in H^p and satisfied this equation we would have

$$|f_1(\xi_m)| = |B_1(\xi_m)|^{-1} > (1 - \alpha_m^2)^{-2} > (1 - \xi_m^2)^{-2}.$$

But a result of A. J. Macintyre and W. W. Rogosinski [4, P. 304] states that for f in H^p we have the growth condition $|f(z)| \le \|f\|_p(1 - |z|^2)^{-1/p}$. This of course is incompatible with f_1 in H^1 and so no such H^p solutions exist.

We consider now a fixed sequence $\{\alpha_m\}_{m=1}^{\infty}$ satisfying condition (C) and let

$$F^p = \left\{ f \in H^p : \left\{\frac{1}{f(\alpha_m)}(1 - |\alpha_m|^2)^{1/q}\right\}_{m=1}^{\infty} \in l_q ; \frac{1}{p} + \frac{1}{q} = 1 \right\}$$

$$G^p = \left\{ f \in H^{\infty} : \left\{\frac{1}{f(\alpha_m)}(1 - |\alpha_m|^2)^{1/p}\right\}_{m=1}^{\infty} \in l_p \right\}.$$

THEOREM 5. *Let $1 \le r < p < \infty$. Then G^p is properly contained in G^r.*

Proof.

$$\sum_{m=1}^{\infty}\left|\frac{1}{f(\alpha_m)}(1 - |\alpha_m|^2)^{1/r}\right|^r < \|f\|_\infty^{p-r}\sum_{m-1}^{\infty}\left|\frac{1}{f(\alpha_m)}(1 - |\alpha_m|^2)^{1/p}\right|^p.$$

To see that the containment is proper let

$$\lambda_n = \left(\sum_{k=n}^{\infty} (1 - |\alpha_k|^2)\right)^{-1/p}.$$

By a theorem of Dini [3, P. 293] we have

$$\sum_{n=1}^{\infty} \lambda_n^r (1 - |\alpha_n|^2) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n^p (1 - |\alpha_n|^2) = \infty.$$

Clearly $\{1/\lambda_n\}_{n=1}^{\infty}$ is in l_∞ and we have an f in H^∞ satisfying $T_\infty f = \{f(\alpha_n)\}_{n=1}^{\infty}$. Thus f is in G^r but f is not in G^p.

The inclusion F^p contained in F^r for $1 \leq r < p$ is of course false and one can obtain functions in F^p but not in F^r by using the same ideas as in the preceding paragraph. The intersection of the classes F^p for $1 \leq p$ is non-empty and contains all H^∞ functions which are bounded below in modulus by a positive number on the set $\{\alpha_n\}_{n=1}^{\infty}$. However, this intersection also contains H^p functions which are not in H^∞, for example if $\alpha_n \neq 0$, then $\log(1 + z)$ is in F^p for $p \geq 1$.

We make the following observations concerning F^p and G^p. If p decreases toward one $(1 \leq p)$ the H^p classes increase giving us more possible candidates for admission to F^p. But admission to F^p is determined not by the global behavior of such an f but by the growth of f on the sequence $\{\alpha_n\}_{n=1}^{\infty}$. As p decreases to one, the conjugate index q, $1/p + 1/q = 1$, tends to infinity and this means that for f to be in F^p, it must tend to zero more slowly on $\{\alpha_n\}_{n=1}^{\infty}$ as $q \to \infty$. Thus, it is clear that if f is "well behaved" from below on $\{\alpha_n\}_{n=1}^{\infty}$ and in H^p, then its behavior off $\{\alpha_n\}_{n=1}^{\infty}$ determines whether it will belong to F^r for $1 \leq r \leq p$. Similar remarks can be made concerning G^p.

4. H^p-solutions for Blaschke products

We assume in this section that B_1 and B_2 are Blaschke product in D satisfying $|B_1| + |B_2| > 0$ there. We wish to investigate solution pairs f_1 and f_2 in H^p which satisfy

$$f_1(z)B_1(z) + f_2(z)B_2(z) = 1 \quad \text{on } D. \tag{4.1}$$

We know by Theorem 4 that under certain conditions solution pairs do exist and we shall not consider the existence again.

THEOREM 6. *If* (4.1) *holds with f_1 in H^p then f_2 is in H^p.*

Proof. Use the factorization for the H^p function

$$1 - f_1(z)B_1(z) = B^*(z)S(z)F(z)$$

and note that $f_2(z) = T(z)F(z)$ where T is an inner function.

Theorem 6 is not in general true if the $B_i(z)$ are replaced by H^∞ functions. For example if we choose $a(z) = z$ and $b(z) = (1 - z)$ then the pair $f_1(z) = (-1)$ and $f_2(z) = (1 + z)/(1 - z)$ satisfy (4.1) for a and b, f_1 is in H^∞ but f_2 is not in H^1.

THEOREM 7. *Assume $B_1(z)$ and $B_2(z)$ are the given Blaschke products,*

$|B_1| + |B_2| > 0$, *and let*

$$K = \{f_1 \in H^p : \text{there exists } f_2 \text{ satisfying } (4.1)\}$$

then there are functions F_1 *and* F_2 *such that*

(i) F_1 *and* F_2 *satisfy* (4.1).

(ii) $0 < \delta = \| F_1 \| \leq \| f_1 \|, f_1 \in K$.

(*By* $\| F_1 \|$ *we mean of course* $\| F_1 \|_p$.)

Proof. Let $\delta = \inf \{ \| f_1 \| : f_1 \in K\}$. δ is positive. For given any f in H^p we have

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_z(\theta) f(\theta)\, d\theta$$

where $f(\theta)$ is the boundary value of f at $e^{i\theta}$ and $P_z(\theta)$ is the Poisson Kernel. $f(\theta)$ is in L^p of $(-\pi, \pi)$ and for $0 \leq |z| \leq \rho < 1$ and

$$P_z(\theta) = (1 - |z|^2)/(1 - 2|z| \cos(\theta - \phi) + |z|^2) \leq (1 + \rho)/(1 - \rho).$$

Now if $\delta = 0$ choose $f_{1,n}$ and $f_{2,n}$ satisfying (4.1) such that

$$\| f_{1,n} \| \geq \| f_{1,n+1} \| \rightarrow 0$$

as $n \rightarrow \infty$. Let β be a zero of B_2 then

$$1 = | B_1(\beta) f_{1,n}(\beta) | \leq | B_1(\beta) | \, ((1 + |\beta|)/(1 - |\beta|)) \, \| f_{1,n} \|.$$

Thus $\delta > 0$. Let $f_{1,n}$ and $f_{2,n}$ satisfy (4.1) and $\| f_{1,n} \|$ tend monotonically to δ. We may assume $\| f_{1,n} \| < 1 + \delta$. The above representation shows that if $|z| \leq \rho$ then

$$| f_{1,n}(z) | \leq ((1 + \rho)/(1 - \rho)) \, \| f_{1,n} \| \leq ((1 + \rho)/(1 - \rho))(1 + \delta).$$

Thus $\{f_{1,n}\}_{n=1}^{\infty}$ is bounded on compact subsets and so is normal. Assume that we have chosen a subsequence which converges uniformly on compact subsets of D and for simplicity of notation let us denote it again by $\{f_{1,n}\}_{n=1}^{\infty}$. Of course we select the subsequence of $\{f_{2,n}\}_{n=1}^{\infty}$ which corresponds to the $\{f_{1,n}\}_{n=1}^{\infty}$ and relabel it so that

$$f_{n,1}(z) B_1(z) + f_{2,n}(z) B_2(z) = 1.$$

We have $\lim_{n \to \infty} f_{1,n}(z) = F_1(z)$ uniformly on compact subsets of D. Thus by the Minkowski inequality for $r < 1$

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | F_1(re^{i\theta}) |^p \, d\theta \right)^{1/p}$$
$$\leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | F_1(re^{i\theta}) - f_{1,n}(re^{i\theta}) |^p \, d \right)^{1/p} + \left(\frac{1}{2} \int_{-\pi}^{\pi} | f_{1,n}(re^{i\theta}) |^p \, d\theta \right)^{1/p}.$$

Let $\varepsilon > 0$ be given. The uniform convergence of the $\{f_{1,n}\}$ to F_1 on the compact set $z = r$ implies there is a N such that $| f_{1,n}(re^{i\theta}) - F_1(re^{i\theta}) | < \varepsilon/2$

if $n \geq N$. We can also choose N so large that $\| f_{1,n} \| < \delta + \varepsilon/2$. Thus

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | F_1(re^{i\theta}) |^p \, d\theta\right)^{1/p} < \varepsilon/2 + (\delta + \varepsilon/2) = \delta + \varepsilon, \quad \text{for } n \geq N.$$

Therefore as $n \to \infty$ we have

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | F_1(re^{i\theta}) |^p \, d\theta,\right)^{1/p} \leq \delta, \qquad r < 1.$$

We have shown F_1 is in H^p and $\| F_1 \| \leq \delta$. It is easy to show that $\{f_{2,n}\}_{n=1}^{\infty}$ is normal and by choosing subsequences we may assume $\lim_{n\to\infty} f_{1,n}(z) = F_1(z)$ and $\lim_{n\to\infty} f_{2,n}(z) = F_2(z)$ where F_2 is holomorphic on D. Then it is clear that F_1 and F_2 are H^p solutions of (4.1) implying $F_1 \in K$ and hence $\| F_1 \| = \delta$.

5. Summary

We have a sufficient condition that a pair of functions in the same or different H^p spaces might possess a solution g_1 and g_2 to $g_1(z)f_1(z) + g_2(z)f_2(z) = 1$ on D where the functions g_1 and g_2 are also in various H^p classes. We would like to point out that much remains to be done here. Hopefully, a necessary and sufficient condition might be found that holds for all H^p functions. It would even be nice to find a necessary and sufficient condition for a larger class of H^p functions than exhibited here.

BIBLIOGRAPHY

1. L. CARLESON, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math., vol. 76 (1962), pp. 547–559.
2. K. HOFFMAN, *Banach spaces of analytic functions*, New York, Prentice-Hall, 1962.
3. K. KNOPP, *Theory and applications of infinite series*, Blackie and Sons, London, 1928.
4. A. J. MACINTYRE AND W. W. ROGOSINSKI, *Extremum problems in the theory of analytic functions*, Acta Math., vol. 82 (1950), pp. 275–325.
5. W. RUDIN, *The closed ideals in an algebra of continuous functions*, Canad. J. Math., vol. 9 (1957), pp. 426–434.
6. H. S. SHAPIRO AND A. L. SHIELDS, *On some interpolating problems for analytic functions*, Amer. J. Math., vol. 83 (1961), pp. 513–532.

UNIVERSITY OF ARIZONA
TUCSON, ARIZONA