ON SEMIGROUPS GENERATED BY TOPOLOGICALLY NILPOTENT ELEMENTS

BY

LARS INGELSTAM

In a Banach algebra A one can define, for each element x,

$$\operatorname{ixp} x = -\sum_{n=1}^{\infty} x^n/n!$$

or, if A has identity e, $\exp x = e - \operatorname{ixp} x$. Then, for a given x, $\alpha \rightarrow \operatorname{ixp} \alpha x$ ($\alpha \rightarrow \exp \alpha x$) is a one-parameter semigroup under circle composition (group under multiplication). In this paper we deal with the following question:

How "bounded" can such a semigroup be when the generator x is "topologically very nilpotent"?

In [4, pp. 248, 259] the following results are given:

I. If $\lim_{n\to\infty} \| x^n \|^{1/n} = 0$ and $x \neq 0$, then $\operatorname{ixp} \alpha x$ can not be bounded in norm for all α.

II. If $\lim n \| x^n \|^{1/n} = 0$ and $x \neq 0$, then $\operatorname{ixp} \alpha x$ can not be bounded in norm for $\alpha > 0$.

This type of problem has arisen in two different contexts. In the paper by Bohnenblust and Karlin [1] on the geometry of the unit sphere in Banach algebras they make a conjecture, which can be shown to be equivalent to

"assumption of I $\Rightarrow$ conclusion of II".

This is not true, as has been shown in [7], but apparently I and II are the affirmative results corresponding to that conjecture. On the other hand, I is a key result in the author's classification of real Banach algebras [4].

In improving these results in several different directions, we use methods from the theory of functions, in particular conditions for perfectly regular growth of an entire function. The sharpest result in the relevant direction seems to a recent theorem by Essén [2], which we quote here in the form most suitable for the application.

Let f be an entire function with $f(0) = 1$, maximum modulus M and minimum modulus m. ˥or a number λ, $0 < \lambda < 1$, let

$$K_\lambda(R) = R^{-\lambda} \log M(R)$$

$$I_\lambda(R) = \int_0^R r^{-1-\lambda} [\log m(r) - \cos \pi\lambda \log M(r)]\, dr$$

Received June 19, 1967.

If $K_\lambda(R)$ and $I_\lambda(R)$ are both bounded from above, then $I_\lambda(R)$ is bounded from below and $\lim_{R\to\infty} K_\lambda(R)$ *exists if and only if* $\lim_{R\to\infty} I_\lambda(R)$ *exists.*

In the following theorem, which contains most of the technical material needed in the sequel, a condition in form of a weak integral bound (3) on the semigroup is introduced. The space of continuous linear functionals on A is denoted A'. In Theorem 1 and all the proofs we take A complex; the results for real scalars are easily obtained by complexification.

THEOREM 1. *Assume $0 < \lambda \leq \frac{1}{2}$ and that $x \in A$ has the properties*

(1) $\limsup n^{(1/\lambda)-1} \| x^n \|^{1/n} < \infty$

(2) $\liminf n^{(1/\lambda)-1} \| x^n \|^{1/n} = 0$.

If, for every $x' \in A'$,

(3) $\displaystyle\int_1^\infty \frac{\log^+ | x'(\operatorname{ixp} \alpha x) |}{\alpha^{1+\lambda}} \, d\alpha < \infty$

then

(4) $\| \operatorname{ixp} zx \| = O(e^{\delta |z|^\lambda})$, $| z | \to \infty$, *for arbitrary $\delta > 0$.*

Proof. With

$$f(\alpha) = 1 + x'(\operatorname{ixp} \alpha x) = 1 - \sum_{n=1}^{\infty} \frac{x'(x^n)\alpha^n}{n!},$$

f is an entire function and it follows from (1) that K_λ is bounded from above. But since

(5) $\log m(\alpha) - \cos \pi\lambda \log M(\alpha) \leq 2 \log^+ | x'(\operatorname{ixp} \alpha x)| + 1$

it follows from (3) that $I_\lambda(R)$ is bounded from above. But it is then a consequence of Essén's result that $I_\lambda(R)$ is bounded from below and this together with (3) and (5) shows that $\lim_{R\to\infty} I_\lambda(R)$ exists. Hence also $\lim_{R\to\infty} K_\lambda(R) = B$ exists. If $B = 0$ for each x', $| x'(\operatorname{ixp} zx)| = O(e^{\delta |z|^\lambda})$ for any given $\delta > 0$. By the uniform boundedness principle, (4) holds.

If $B \neq 0$ for some x', we can use a theorem by Valiron [9, p. 44] asserting in this case that if $\lim_{R\to\infty} R^{-\lambda} \log M(R) = B$ there exists a sequence n_p of integers with the properties $n_p \to \infty$,

(6) $\lim_{p\to\infty} n_p | x'(x^{n_p})/n_p ! |^{\lambda/n_p} = Be\lambda$

and

(7) $\lim_{p\to\infty} n_{p+1}/n_p = 1$.

For given $\varepsilon > 0$, p large enough and $B_1 = e^{-1}(Be\lambda)^{1/\lambda}$,

$$| x'(x^{n_p}) |^{1/n_p} \geq (B_1 - \varepsilon)^{\lambda/(1-\lambda)} \frac{1}{n_p^{(1/\lambda)-1}}$$

and

$$\| x^{n_p} \|^{1/n_p} \geq \frac{1}{\| x' \|^{1/n_p}} | x'(x^{n_p}) |^{1/n_p} \geq (B_1 - 2\varepsilon)^{\lambda/(1-\lambda)} \frac{1}{n_p^{(1/\lambda)-1}} = K_0 \cdot n_p^{-(1/\lambda)+1}.$$

For a given n, let $n_p \leq n < n_{p+1}$ and $s = n_{p+1} - n$. It is a consequence of (7) that $\lim_{n\to\infty} s/n = 0$. Condition (1) tells that, if $\sigma = (1/\lambda) - 1$,

$$\| x^s \| \leq (K_1/s^\sigma)^s$$

for all s and a constant K_1. Hence, using the submultiplicativity of the norm, one gets

$$\| x^n \| \geq \frac{\| x^{n_{p+1}} \|}{\| x^s \|} \geq \left(\frac{s^\sigma}{K_1}\right)^s K_0^{n_{p+1}} \frac{1}{(n_{p+1}^\sigma)^{n_{p+1}}}$$

or

$$n^\sigma \| x^n \|^{1/n} \geq K_0^{n_{p+1}/n} \frac{(s^\sigma)^{s/n}}{K_1^{s/n}} \frac{n^\sigma}{(n_{p+1}^\sigma)^{n_{p+1}/n}} = \frac{K_0^{1+s/n}}{K_1^{s/n}} \left(\frac{s}{n}\right)^{\sigma s/n} \frac{1}{(1+s/n)^{\sigma(1+s/n)}}.$$

Hence

$$\liminf_{n\to\infty} n^\sigma \| x^n \|^{1/n} \geq K_0 = [(Be\lambda)^{1/\lambda} e^{-1} - 2\varepsilon]^{\lambda/(1-\lambda)}$$

for each $\varepsilon > 0$, and

$$\liminf_{n\to\infty} n^\sigma \| x^n \|^{1/n} \geq e(B\lambda)^{1/(1-\lambda)} > 0,$$

But this conflicts with (2) and so we must have $B = 0$.

For the case $\lambda = \frac{1}{2}$ we can now formulate a result that is a direct improvement of II.

THEOREM 2. *Assume that* $x \in A$ *satisfies* (1) *and* (2) *for* $\lambda = \frac{1}{2}$. *If* $x \neq 0$, ixp αx *is unbounded in norm for* $\alpha > 0$.

Proof. Assume ixp αx bounded for $\alpha > 0$. (Clearly, it is equivalent to assume $x'(\text{ixp}\ \alpha x)$ bounded for each $x' \in A'$.) The integrand in (3) is less than const. $\cdot\ \alpha^{-3/2}$. From Theorem 1 it follows that $x'(\text{ixp}\ \alpha x)$ is an entire function of order $\frac{1}{2}$, minimum type, and a Phragmén-Lindelöf theorem, originally due to Wiman [10], shows that $x'(x) = 0$. But then $x = 0$, which is a contradiction.

It is easy to see from an example (see [5, sec. 9]) that, if condition (2) is removed, the conclusion is no longer true.

Theorem 2 could have been obtained without reference to the integral condition (3), using instead the basic result by Heins [3]. In combination with other conditions, however, (3) has a decisive importance. As a sample result of this nature, we give the following different generalization of II.

THEOREM 3. *Assume that* $x \in A$ *satisfies* (1) *and* (2), *with* $\lambda = \frac{1}{2}$. *Then the conditions* (3) *and*

(8) $y_n = \text{ixp}\ n^2 x$, $n = 1, 2, \cdots$, *is a bounded set*

can not both be satisfied, unless $x = 0$.

Proof. If (1), (2) and (3) are satisfied it follows from Theorem 1 that f is a function of order $\frac{1}{2}$, minimum type. If (8) is also true, it follows from a theorem by Polya [8] that it must be a constant. Then $x'(x) = 0$ for every x' and $x = 0$.

Results similar to that of Theorem 3 exist even for $0 < \lambda < \frac{1}{2}$. In those cases the situation is even more favourable since different types of boundedness conditions are available: e.g., boundedness in certain sequences of points or bounds on the logarithmic density of sets where the function is large (see [6]). The case $\frac{1}{2} < \lambda < 1$ can partly be reduced to $0 < \lambda < \frac{1}{2}$ and one is led to consider, e.g., symmetric sets and their density instead of just sets on the positive axis. For $\lambda = 1$, i.e., direct generalizations of I, there is no point in relaxing $\lim_{n\to\infty} \| x^n \|^{1/n} = 0$ to (1) and (2), since the limit always exists. It is clear, however, that it can suffice with weaker boundedness conditions than a uniform bound on the norm on the entire axis.

References

1. H. F. Bohnenblust and S. Karlin. *Geometrical properties of the unit sphere of Banach algebras*, Ann. of Math., vol. 62 (1955), pp. 217–229.
2. M. Essén, *A theorem on the maximum modulus of entire functions*. Math. Scand., vol. 17 (1965), pp. 161–168.
3. M. Heins, *Entire functions with bounded minimum modulus; subharmonic functions analogues*, Ann. of Math. (2), vol. 49 (1948), pp. 200–213.
4. L. Ingelstam, *Real Banach algebras*. Ark. Mat., vol. 5 (1964), pp. 231–270.
5. ———, *Convolution multiplication on Banach spaces*, Mimeographed report, Stockholm.
6. B. Kjellberg, *On certain integral and harmonic functions*, Thesis, Uppsala, 1948.
7. G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math., vol. 11 (1961), pp. 679–698.
8. G. Polya, *Aufgabe* 105, Jahresbericht d. Deutschen Math. Ver., vol. 40 (1931).
9. Valiron, G., *Integral functions*, Chelsea Publishing Co., New York, 1949 (1923).
10. Wiman, A., *Über die angenäherte Darstellung von ganzen Funktionen*, Ark. Mat. Astr. Fys., vol. 1 (1903).

Royal Institute of Technology
Stockholm, Sweden