A TRANSPLANTATION THEOREM FOR JACOBI SERIES

BY
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1. Introduction

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, order (α, β), defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}],$$

$$\alpha, \beta > -1.$$

These polynomials are orthogonal on $(-1, 1)$ with respect to $(1-x)^\alpha(1+x)^\beta$ and

$$\int_{-1}^{1}[P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta\,dx$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

$$= [t_n^{(\alpha,\beta)}]^{-2}.$$

Then the functions

$$\varphi_n^{(\alpha,\beta)}(\theta) = t^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2}\,2^{(\alpha+\beta+1)/2}$$

are orthonormal functions on $(0, \pi)$ with respect to Lebesgue measure. The functions $\varphi_n^{(1/2,1/2)}(\theta)$ are $(2/\pi)^{1/2}\sin(n+1)\theta$, $n = 0, 1, \cdots$ and $\varphi_n^{(-1/2,-1/2)}(\theta) = (2/\pi)^{1/2}\cos n\theta$, $n = 1, 2, \cdots$, $\varphi_0^{(-1/2,-1/2)}(\theta) = \pi^{-1/2}$. Fourier series with respect to these two sets of functions have been studied extensively. Fourier series for Jacobi polynomials have not been as extensively studied and many fewer results are known. The one type of result that has been studied in any detail deals with equiconvergence theorems. This type of result comes from asymptotic formulas for $\varphi_n^{(\alpha,\beta)}(\theta)$ which are valid for $\varepsilon \leqq \theta \leqq \pi - \varepsilon$. However, for many of the results in Fourier analysis we want to use all the values of θ, $0 \leqq \theta \leqq \pi$. In this paper we show how to set up a bounded mapping between Fourier series with respect to Jacobi polynomials which allows one to read off many of the deep results for Fourier-Jacobi expansions from the corresponding results for ordinary Fourier series.

I haven't discussed this work with Stephen Wainger, but many of the ideas that are used arose in connection with other work we have done together and I would like to acknowledge my indebtedness to these discussions.

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Let $f(\theta)$ be integrable on $(0, \pi)$ and define

$$a_n = \int_0^\pi f(\theta)\varphi_n^{(\alpha,\beta)}(\theta)\, d\theta. \tag{1}$$

Then formally

$$f(\theta) \sim \sum a_n \varphi_n^{(\alpha,\beta)}(\theta).$$

Keeping the same a_n we consider the series

$$g_r(\theta) = T_r f(\theta) = \sum a_n r^n \varphi_n^{(\gamma,\delta)}(\theta), \qquad 0 < r < 1. \tag{2}$$

Our main theorem is as follows

THEOREM 1. *Let* $1 < p < \infty$, $\alpha, \beta, \gamma, \delta \geqq -\frac{1}{2}$, $-1 < \sigma < p - 1$, $-1 < \tau < p - 1$. *Let* $(\sin \theta/2)^\sigma (\cos \theta/2)^\tau \,|f(\theta)|^p \in L^1(0, \pi)$ *and let* a_n *and* $g_r(\theta)$ *be defined by* (1) *and* (2). *Then*

$$\int_0^\pi |g_r(\theta)|^p (\sin\theta/2)^\sigma (\cos\theta/2)^\tau\, d\theta \leqq A \int_0^\pi |f(\theta)|^p (\sin\theta/2)^\sigma (\cos\theta/2)^\tau\, d\theta.$$

Also there is $g(\theta)$ *so that* $g_r(\theta) \to g(\theta)$ *a.e.*,

$$\int_0^\pi |g_r(\theta) - g(\theta)|^p (\sin\theta/2)^\sigma (\cos\theta/2)^\tau\, d\theta \to 0,$$

and

$$\int_0^\pi |g(\theta)|^p (\sin\theta/2)^\sigma (\cos\theta/2)^\tau\, d\theta \leqq A \int_0^\pi |f(\theta)|^p (\sin\theta/2)^\sigma (\cos\theta/2)^\tau\, d\theta.$$

In [1] Wainger and I prove the same theorem for ultraspherical series, i.e. for the case $\alpha = \beta$, $\gamma = \delta$. We used an integral representation which is unknown for Jacobi polynomials and we also use some work of Muckenhoupt and Stein [7] which is also unproven for Jacobi series. The techniques we develop in this paper allow one to prove the analogue of some of the results of Muckenhoupt and Stein for Jacobi series. In particular we get a proof of the Marcinkiewicz multiplier theorem which was first obtained for ultraspherical series by them. As a special case of the Marcinciewicz theorem we obtain a new proof of the mean convergence theorem for Jacobi series which was first obtained by Pollard [8].

II. Preliminary information

We need a number of facts about Jacobi polynomials.

$$|\varphi_n^{(\alpha,\beta)}(\theta)| \leqq A, \quad n = 0, 1, \cdots, 0 \leqq \theta \leqq \pi, \qquad [9, (7.32.8)] \tag{3}$$

We also need an important asymptotic formula of Darboux [4].

$$\begin{aligned}(4)\quad & \varphi_n^{(\alpha,\beta)}(\theta) \\ &= \left(A + \frac{B}{n}\right)\cos\left[\left(n + \frac{\alpha+\beta+1}{2}\right)\theta - \frac{\pi}{2}\left(\alpha + \frac{1}{2}\right)\right] \\ &\quad + \left[\frac{C}{n}\cot\frac{\theta}{2} + \frac{D}{n}\tan\frac{\theta}{2}\right]\sin\left[\left(n + \frac{\alpha+\beta+1}{2}\right)\theta - \frac{\pi}{2}\left(\alpha + \frac{1}{2}\right)\right] \\ &\quad + O((n\sin\theta)^{-2}), \qquad n = 0, 1, \cdots, 0 < \theta < \pi.\end{aligned}$$

We must sum a series of Jacobi polynomials by parts, and we need

$$\frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)} - \frac{P_{n+1}^{(\alpha,\beta)}(\cos\theta)}{P_{n+1}^{(\alpha,\beta)}(1)} = \frac{(2n+\alpha+\beta+2)}{2(\alpha+1)}(1-\cos\theta)\frac{P_n^{(\alpha+1,\beta)}(\cos\theta)}{P_n^{(\alpha+1,\beta)}(1)}, \qquad [9,(4.5.4)].$$

We also need the estimate

$$(6)\qquad |P_n^{(\alpha,\beta)}(\cos\theta)| \leqq P_n^{(\alpha,\beta)}(1) \leqq An^{\alpha} \quad \text{for} \quad 0<\theta<\pi/2 \qquad [9,(7.32.2)].$$

Finally from Fourier series we need

$$D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n}\cos k\theta = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} = O\left(\frac{1}{\theta}\right).$$

Also

$$(7)\qquad \frac{1}{2} + \sum_{n=1}^{\infty} r^n\cos n\theta = \frac{1-r^2}{2(1-2r\cos\theta+r^2)} = P(r,\theta).$$

$$(8)\qquad \sum_{n=1}^{\infty} r^n\sin n\theta = \frac{r\sin\theta}{2(1-2r\cos\theta+r^2)} = Q(r,\theta).$$

(7) is the Poisson kernel and (8) is the conjugate Poisson kernel. We have the following inequalities for integrals involving P and Q. See [10, Vol. I].

If

$$f(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)P(r,\varphi)\,d\varphi$$

and

$$\tilde{f}(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)Q(r,\varphi)\,d\varphi$$

then

$$\left[\int_{-\pi}^{\pi}|f(r,\theta)|^p\,d\theta\right]^{1/p} \leqq \left[\int_{-\pi}^{\pi}|f(\theta)|^p\,d\theta\right]^{1/p}, \qquad 1 \leqq p \leqq \infty,$$

and

$$\left[\int_{-\pi}^{\pi}|\tilde{f}(r,\theta)|^p\,d\theta\right]^{1/p} \leqq A_p\left[\int_{-\pi}^{\pi}|f(\theta)|^p\,d\theta\right]^{1/p}, \qquad 1 < p < \infty,$$

uniformly in r, $0 \leqq r < 1$.

There are also versions of these inequalities with weighted norms, e.g.

$$\left[\int_{-\pi}^{\pi} | f(\theta) |^p \mid \sin \theta/2 |^\sigma \mid \cos \theta/2 |^\tau \, d\theta\right]^{1/p}.$$

For results of this type see [6].

III. Proof

The proof of Theorem 1 is dual to the proof of the dual result which is given in [4]. It was originally thought that results of this type are substantially harder than are the dual results. This is just another example of Littlewood's dictum, look for the dual result and dualize the proof.

We first estimate what should be the harder term the one where $\theta/2 \leqq \chi \leqq \theta$. Since

$$\varphi_n^{(\alpha,\beta)}(\theta) = (-1)^n \varphi_n^{(\beta,\alpha)}(\pi - \theta)$$

we may also assume that $0 \leqq \theta \leqq \pi/2$. We have $g_r(\theta) = \int_0^\pi K_r(\theta, \chi) f(\chi)\, d\chi$ where

$$(9) \quad K_r(\theta, \chi) = \sum_{n=0}^{\infty} r^n \varphi_n^{(\gamma,\delta)}(\theta)\varphi^{(\alpha,\beta)}(\chi) = \sum_{n=0}^{[1/\theta]-1} + \sum_{[1/\theta]}^{\infty} = I + J.$$

Since $| \varphi_n^{(\alpha,\beta)}(\chi) | < A$, see (3), we have

$$I = \sum_{n=0}^{[1/\theta]-1} r^n \varphi^{(\alpha,\beta)}(\chi)\varphi_n^{(\gamma,\delta)}(\theta) = O(\theta^{-1}).$$

To estimate J we must use an important asymptotic formula due to Darboux [4]. We can ignore the error term when we put (4) in (9). This is so because

$$\sum_{n=[1/\theta]}^{\infty} \frac{1}{(n\theta)^k} = O(1/\theta) \quad \text{for} \quad k = 2, 3, 4.$$

Similarly terms which arise from products with $n^{-1}\theta^{-1}$ can also be ignored. Then we get

$$A \sum_{n[1/\theta]}^{\infty} r^n \Bigg\{\cos\left[\left(n + \frac{\alpha+\beta+1}{2}\right)\chi - \frac{\pi}{4}(2\alpha+1)\right]\cos\left[\left(n + \frac{\gamma+\delta+1}{2}\right)\chi - \frac{\pi}{2}(2\gamma+1)\right]$$

$$+ A\,\frac{\sin\left[\left(n + \frac{\alpha+\beta+1}{2}\right)\chi - \frac{\pi}{2}(2\alpha+1)\right] \cdot \cos\left[\left(n + \frac{\gamma+\delta+1}{2}\right)\chi - \frac{\pi}{4}(2\gamma+1)\right]}{n \tan\frac{\chi}{2}}$$

$$+ \text{ similar terms}\Bigg\}.$$

Then the series we need to consider are

$$\sum_{n=[1/\theta]}^{\infty} r^n \cos n(\theta - \chi), \qquad \sum_{n=[1/\theta]}^{\infty} r^n \cos n(\theta + \chi),$$
$$\sum_{n=[1/\theta]}^{\infty} r^n \sin n(\theta - \chi), \qquad \sum_{n=[1/\theta]}^{\infty} r^n \sin n(\theta + \chi),$$
$$\sum_{n=[1/\theta]}^{\infty} (r^n \cos n(\theta - \chi))/n\theta, \qquad \sum_{n=[1/\theta]}^{\infty} (r^n \sin n(\theta - \chi))/n\theta$$

and so forth.
The summation in each of the first four terms we can extend to $\sum_{n=0}^{\infty}$, since the added terms give a contribution of $O(1/\theta)$. Then we have either a Poisson or a conjugate kernel and each of these gives a bounded operator on the spaces in question. The hardest of the remaining terms is

$$\sum_{n=[1/\theta]}^{\infty} (r^n \cos n(\theta - \chi))/n\theta.$$

This is

$$\sum_{n=[1/\theta]}^{[1/(\theta-\chi)]} \frac{r^n \cos n(\theta - \chi)}{n\theta} + \frac{1}{\theta} \sum_{n=[1/(\theta-\chi)]}^{\infty} \frac{r^n \cos n(\theta - \chi)}{n}.$$

The second term is $O(1/\theta)$ since we can sum the series by parts and get

$$\frac{1}{\theta} \sum_{n=[1/(\theta-\chi)]}^{\infty} \frac{1}{n^2} D_n(\theta - \chi) = O\left(\frac{1}{\theta}\frac{1}{\theta - \chi} \sum_{n=[1/(\theta-\chi)]}^{\infty} \frac{1}{n^2}\right) = O\left(\frac{1}{\theta}\right).$$

In the first term we replace $\cos n(\theta - \chi)$ by $1 + O(n^2(\theta - \chi)^2)$ and get

$$\frac{1}{\theta} \sum_{n=[1/\theta]}^{[1/(\theta-\chi)]} \frac{r^n}{n} + O\left(\frac{1}{\theta} \sum_{n=1}^{[1/(\theta-\chi)]} n(\theta - \chi)^2\right).$$

The second term is clearly $O(1/\theta)$. The first term is

$$O\left(\frac{1}{\theta} \log \frac{\theta}{\theta - \chi}\right)$$

since $\sum_{n=1}^{1/k} 1/n = \log k + O(1)$. In [2] Wainger and I showed that the discrete analogue of this type of kernel is a bounded operator on weighted l^p spaces, and we remarked that the same argument works in the continuous case. Thus all the expressions which arise give rise to bounded operators. This completes the term where $\theta/2 \leqq \chi \leqq \theta$.

We now assume that $\chi \leqq \theta/2$. Then break the sum into $\sum_0^{1/\theta} + \sum_{1/\theta}^{\infty} = J_1 + J_2$. $J_1 = O(1/\theta)$ as above. In J_2 we replace $\varphi_n^{(\gamma,\delta)}(\theta)$ by its asymptotic formula. Then we get

$$(\sin \chi/2)^{\alpha+1/2} \sum_{n=1/\theta}^{\infty} r^n n^{1/2} P_n^{(\alpha,\beta)}(\cos \chi)[\cos (n_1\theta + c)$$
$$+ (\sin(n_1\theta + c))/n\theta + O(n^{-2}\theta^{-2})]$$

where $n_1 = n + (\gamma + \delta + 1)/2$, $c = (2\gamma + 1)\pi/4$ and where we ignore terms that are the same as ones we write multiplied by n^{-1}. Again the error term is negligible since it is $O(\theta^{-1})$. For the term involving $\sin (n_1\theta + c)$, we write it as

$$\theta^{-1}\chi^{\alpha+1/2}\sum_{n=1/\theta}^{\infty} r^n n^{-1/2+\alpha}\frac{P_n^{(\alpha,\beta)}(\cos\chi)}{P_n^{(\alpha,\beta)}(1)}\sin(n_1\theta+c).$$

Summing by parts and using $\sum_{n=1}^{N}\sin(n_1\theta+c)=O(\theta^{-1})$ we have the estimate

$$O\left\{\theta^{-2}\sum_{n=1/\theta}^{\infty}\Delta\left[r^n n^{-1/2+\alpha}\frac{P_n^{(\alpha,\beta)}(\cos\chi)}{P_n^{(\alpha,\beta)}(1)}\right]\chi^{\alpha+1/2}\right\}.$$

Taking differences in order we get the estimate

$$\theta^{-2}(1-r)\sum_{n=1/\theta}^{\infty} r^n n^{-1} + \theta^{-2}\sum_{n=1/\theta}^{\infty} n^{-2}$$
$$+\theta^{-2}\sum_{n=1/\theta}^{\infty} n^{-1/2+\alpha}\left[\frac{P_n^{(\alpha,\beta)}(\cos\chi)}{P_n^{(\alpha,\beta)}(1)}-\frac{P_{n+1}^{(\alpha,\beta)}(\cos\chi)}{P_{n+1}^{(\alpha,\beta)}(1)}\right]\chi^{\alpha+1/2}.$$

We have used (3). We have to be a little more careful with the third term. The part $\sum_{n=1/\theta}^{1/\chi}$ is bounded since

$$\frac{P_n^{(\alpha,\beta)}(\cos\chi)}{P_n^{(\alpha,\beta)}(1)}-\frac{P_{n+1}^{(\alpha,\beta)}(\cos\chi)}{P_{n+1}^{(\alpha,\beta)}(1)}$$
$$=\frac{(2n+\alpha+\beta+2)}{2(\alpha+1)}(1-\cos\chi)\frac{P_n^{(\alpha+1,\beta)}(\cos\chi)}{P_n^{(\alpha+1,\beta)}(1)}=O(n\chi^2)$$

by (5) and (6). Thus the sum we estimate here is

$$\theta^{-2}\chi^{\alpha+1/2+2}\sum_{n=1/\theta}^{1/\chi} n^{-1/2+\alpha+1}=O(\chi\theta^{-2})=O(\theta^{-1}).$$

However, to handle the other part we must go back a step and see that what we must sum is essentially

$$\chi^{\alpha+1/2}\frac{\chi^2}{\theta^2}\sum_{n=1/\chi}^{\infty} r^n n^{+1/2+\alpha}\frac{P_n^{(\alpha+1,\beta)}(\cos\chi)}{P_n^{(\alpha+1,\beta)}(1)}\sin(n_2\theta+c)$$

since we had a Dirichlet kernel when we summed $\sin(n_1\theta+c)$.

Using one term plus an error term in the asymptotic formula for $P_n^{(\alpha+1,\beta)}(\cos\theta)$ we have

$$(\chi/\theta^2)\sum_{n=1/\chi}^{\infty} r^n n^{-1}[\cos(n_3\chi+c_1)+O(n^{-1}\chi^{-1})]\sin(n_2\theta+c)]$$
$$=O[(\chi/\theta^2)\sum_{n=1/\chi}^{\infty} r^n n^{-1}\cos[n(\chi-\theta)]+O(\chi/\theta^2)]$$

Summing the first term by parts we get the estimate

$$(\chi/\theta^2)\chi/(\chi-\theta)=O(\chi^2/\theta^3)=O(1/\theta).$$

So we may consider

$$\sum_{n=1/\chi}^{\infty} r^n n^{1/2}P_n^{(\alpha,\beta)}(\cos\chi)\chi^{\alpha+1/2}\cos n\theta.$$

We take the liberty of dropping the subscript on n and of dropping the phase angle as they only add to the confusion without adding any difficulty. Then as before we sum by parts to get a higher power of χ. We then have

$$1/\theta \sum_{n=1/\theta}^{\infty} \Delta(r^n n^{1/2} P_n^{(\alpha,\beta)}(\cos\chi)\chi^{\alpha+1/2})e^{in\theta}$$
$$= O\left(\frac{1-r}{\theta}\sum_{1/\theta}^{\infty} r^n + \frac{1}{\theta}\sum_{n=1/\theta}^{\infty} n^{-1/2}P_n^{(\alpha,\beta)}(\cos\chi)\chi^{\alpha+1/2}e^{in\theta}\right.$$
$$\left. + \frac{\chi}{\theta}\sum_{n=1/\theta}^{\infty} r^n n^{1/2+\alpha+1}\frac{P_n^{(\alpha+1,\beta)}(\cos\chi)}{P_n^{(\alpha+1,\beta)}(1)} e^{in\theta}\chi^{\alpha+3/2}\right)$$

where we use $e^{in\theta}$ instead of either $\cos n\theta$ or $\sin n\theta$, to simplify the writing. The first term is $O(1/\theta)$. In each of the second and third terms we may use

$$\frac{P_n^{(\alpha,\beta)}(\cos\chi)}{P_n^{(\alpha,\beta)}(1)} = O(1)$$

and sum only to $1/\chi$ and get terms that are $O(1/\theta)$. Now in the sums from $1/\chi$ we can use the asymptotic formula for $P_n^{(\alpha,\beta)}(\cos\chi)$ to get

$$\frac{1}{\theta}\sum_{n=1/\chi}^{\infty} r^n \frac{\cos n\chi \cos n\theta}{n} + \frac{1}{\theta\chi}\sum_{n=1/\chi}^{\infty}\frac{1}{n^2}$$
$$+ \frac{\chi}{\theta}\sum_{n=1/\chi}^{\infty} r^n\left[\cos n\chi + \frac{\sin n\chi}{n\chi} + O(n^{-2}\chi^{-2})\right]e^{in\theta}$$
$$= O\left(\frac{\chi}{\theta(\chi-\theta)}\right) + O\left(\frac{1}{\theta}\right) + O\left(\frac{\chi}{\theta(\theta-\chi)}\right) + O\left(\frac{1}{\theta}\right) = O\left(\frac{1}{\theta}\right)$$

by the same arguments we have used before. This takes care of the part where $\chi \leqq \theta/2$. This gives us the first part of Theorem 1. Unfortunately the Abel summability of Jacobi series in L^p is not done anywhere in the literature so we must say a little more to complete the proof of the theorem. One way is to use the type of argument given in [3] and handle the Abel summability directly. Another way is to assume that $f(\theta)$ is C^∞ with compact support in $(-1, 1)$. Then the Abel summability in L^p is easy. Then a standard approximation argument combined with our norm inequality for $g_r(\theta)$ gives the rest of the conclusion of Theorem 1.

IV. Applications and comments

From Theorem 1 we get a form of the Marcinkiewicz multiplier theorem in the same way that it was obtained in [1]. The only difference is that we must appeal to a slightly stronger form of this theorem for Fourier series, one with a weight function of the form $(\sin\theta/2)^\sigma\ (\cos\theta/2)^\tau$. Hirschman has this form of the Marcinkiewicz theorem in [6].

In connection with the work of Muckenhoupt and Stein, it is of interest to consider

$$\sum a_n P_n^{(\alpha,\beta)}(\cos\theta) \quad\text{and}\quad \sum a_n t_n P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta,$$

where $t_n = 1 + O(1/n)$. It should be possible to choose t_n so that these two series are conjugate functions in the sense that their Abel means when considered as functions of $x = r \cos \theta$, $y = r \sin \theta$, satisfy equations which generalize the Cauchy-Riemann equations. Then it should be possible to set up an H^p theory for $p \geqq 1$ and even for some $p < 1$. For $p > 1$ the methods used in this paper suffice to obtain an analogue of the M. Riesz conjugate function theorem for any $t_n = 1 + O(1/n)$.

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