ON RESULTS OF AHLFORS AND HAYMAN

BY

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1. In Ahlfors' thesis [1] the basic technique relates to the conformal mapping of certain domains, called by him strip domains. In particular he obtained estimates for certain associated geometrical quantities in terms of integrals which he called the First and Second Fundamental Inequalities. The former is now habitually called the Ahlfors Distortion Theorem. While that paper is one of the precursors of the method of the extremal metric it operates primarily with the length-area technique. Although a number of authors have presented refinements and extensions of Ahlfors' results strangely enough no one seems to have reconsidered these problems in terms of the former method although Teichmüller [5] did treat the first result more geometrically, using symmetrization techniques. In the present paper we apply the method of the extremal metric in the context of Ahlfors' fundamental inequalities. In the case of the first inequality the proof becomes virtually trivial. In the case of the second we obtain a proof simpler both conceptually and technically than Ahlfors' ingenious but complicated procedure. In addition it becomes clear that certain of Ahlfors' conditions on the domain, in particular its symmetry, were required only for the application of his particular technique.

A result which plays an important role in Hayman's treatment of mean p-valent functions [2; Theorem 2.4] has a close connection with the same concepts although it deals with regular functions rather than conformal mappings. The method of the extremal metric provides a simple proof and a technically improved version of the result.

2. Let D be a simply-connected domain in the z-plane with boundary elements P_1 and P_2 such that for an interval of values of x, $A < x < B$, $D - \{\Re z = x\}$ contains a component D_1 in $\Re z < x$ with P_1 as a boundary element and a component D_2 in $\Re z > x$ with P_2 as a boundary element. Let $\sigma(x)$ denote a maximal open subinterval of $\Re z = x$ in D such that the two components of $D - \sigma(x)$ have P_1, P_2 as respective boundary elements. Let $\theta(x)$ denote the length of $\sigma(x)$ (the possibility of infinite length is not excluded). It is well known that $\sigma(x)$ can be chosen so that $\theta(x)$ is measurable, for example as the common boundary arc in D of D_1 and the component of $D - \mathrm{Cl}\, D_1$ with P_2 as boundary element.

Let D be mapped conformally on the strip $S : 0 < \Im\zeta < a$ in the ζ-plane so that P_1 and P_2 correspond to the boundary elements of the latter determined by the point at infinity with respective neighborhoods in $\Re\zeta < 0$ and $\Re\zeta > 0$.

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Let $\tau(x)$ denote the image of $\sigma(x)$ in S. Let

$$\xi_1(x) = \operatorname{glb}_{\zeta\in\tau(x)} \Re\zeta, \qquad \xi_2(x) = \operatorname{lub}_{\zeta\in\tau(x)} \Re\zeta.$$

THEOREM 1. (Ahlfors Distortion Theorem). *With the above conditions and notations, for* $A < x_1 < x_2 < B$,

$$\begin{aligned} \int_{x_1}^{x_2} \frac{dx}{\theta(x)} &\leq \frac{1}{a}(\xi_1(x_2) - \xi_2(x_1)) + 2, \quad && \xi_2(x_1) - \xi_1(x_2) < a \\ &\leq 1, && \xi_2(x_1) - \xi_1(x_2) \geq a. \end{aligned}$$

One component of $D - (\sigma(x_1) \cup \sigma(x_2))$ becomes a quadrangle Q (possibly degenerate) on assigning $\sigma(x_1)$, $\sigma(x_2)$ as a pair of opposite sides. Let the module of the quadrangle for the family of curves joining the complementary pair of sides be denoted by M. It is well known that

$$M \geq \int_{x_1}^{x_2} dx/\theta(x) \tag{1}$$

(for example by observing that $\int_{x_1}^{x_2} dx/\theta(x)$ is the module of the family of curves $\sigma(x)$, $x_1 < x < x_2$). Now $\tau(x_1)$, $\tau(x_2)$ are a pair of opposite sides of a quadrangle $\mathfrak{Q}$ conformally equivalent to Q whose complementary pair of sides are respectively on $\Im\zeta = 0$, $\Im\zeta = a$. The module of $\mathfrak{Q}$ for the family of curves joining the latter is again M. Let, in case $\xi_2(x_1) - \xi_1(x_2) < a$,

$$\begin{aligned} \rho(\zeta) &= 1/a, \quad && \text{at points of } \mathfrak{Q} \text{ with } \xi_2(x_1) - a < \Re\zeta < \xi_1(x_2) + a \\ &= 0, && \text{elsewhere in } \mathfrak{Q}; \end{aligned}$$

in case $\xi_2(x_1) - \xi_1(x_2) \geq a$,

$$\begin{aligned} \rho(\zeta) &= 1/a, \quad && \text{at points of } \mathfrak{Q} \text{ with } \xi_1(x_2) \leq c < \Re\zeta < c + a \leq \xi_2(x_1) \\ &= 0, && \text{elsewhere in } \mathfrak{Q}. \end{aligned}$$

In each case the metric $\rho(\zeta)\,|d\zeta|$ is admissible in the L-normalization [3] of the module problem and the respective areas are not greater than $(1/a)(\xi_1(x_2) - \xi_2(x_1)) + 2$, 1. Thus

$$\begin{aligned} M &\leq (1/a)(\xi_1(x_2) - \xi_2(x_1)) + 2, \quad && \xi_2(x_1) - \xi_1(x_2) < a \\ &\leq 1, && \xi_2(x_1) - \xi_1(x_2) \geq a. \end{aligned} \tag{2}$$

Combining (1) and (2) we have the result of Theorem 1. This is the proof referred to in [4].

3. Ahlfors' Second Fundamental Inequality provides an upper bound for $\xi_2(x_2) - \xi_1(x_1)$ in terms of the integral $\int_{x_1}^{x_2} dx/\theta(x)$ plus a certain remainder term subject to a number of fairly stringent requirements. In the framework of the method of the extremal metric the key step is to provide an upper bound for M analogous to the evident bound (1). This will now be done under certain restrictions on Q and from this it will be quite easy to obtain a new version of Ahlfors' result.

THEOREM 2. *Let the quadrangle Q be such that, for $x_1 \leq x \leq x_2$, $\sigma(x)$ is represented by a segment $-\theta_1(x) < y < \theta_2(x)$, $\theta_1(x), \theta_2(x) > 0$, where θ_1, θ_2 have respective finite total variations V_1, V_2 on $[x_1, x_2]$, $\theta_1(x), \theta_2(x) \leq L$ on $[x_1, x_2]$ and $(0 <)\theta^{(m)} \leq \theta_1(x), \theta_2(x)$ on $[x_1, x_2]$. Then for the module M of Q for the family of curves joining the pair of sides complementary to $\sigma(x_1), \sigma(x_2)$ we have*

$$M \leq \int_{x_1}^{x_2} \frac{dx}{\theta(x)} + \frac{L}{2(\theta^{(m)})^2}(V_1 + V_2). \tag{3}$$

Let $[x_1, x_2]$ be divided into n consecutive closed intervals Δ_j, $j = 1, \cdots, n$, of equal length and, for $l = 1, 2$, let

$$\theta_l^{(s)}(x) = \min \theta_l(t), \quad t \in \Delta_j \text{ where } x \in \Delta_j$$

(such minima are attained since $\theta_l(x)$ is lower semicontinuous). At an end point $\tilde{x}$ of an interval Δ_j where the step function $\theta_l^{(s)}(x)$ has a positive jump we draw the ray given by $\tilde{x} + \lambda$, $\theta_l^{(s)}(\tilde{x}) + \lambda$, $\lambda \geq 0$ and at an end point $\tilde{x}$ of an interval Δ_j where the step function $\theta_l^{(s)}(x)$ has a negative jump we draw the ray given by $\tilde{x} - \lambda$, $\theta_l^{(s)}(\tilde{x}) + \lambda$, $\lambda \geq 0$. The lower envelope of these rays and the locus $y = \theta_l^{(s)}(x)$ defines on $[x_1, x_2]$ a continuous function $\theta_l^{(t)}(x)$ which determines a decomposition of $[x_1, x_2]$ into a finite number of subintervals on each of which the locus $y = \theta_l^{(t)}(x)$ has slope $+1$, -1 or 0. The domain determined by

$$-\theta_1^{(t)}(x) < y < \theta_2^{(t)}(x), \quad x_1 < x < x_2,$$

becomes a quadrangle Q^* on assigning as a pair of opposite sides the segments

$$\sigma^*(x_l) : -\theta_1^{(t)}(x_l) < y < \theta_2^{(t)}(x_l), \quad l = 1, 2.$$

For the module M^* of Q^* for the family of curves joining the pair of sides complementary to $\sigma^*(x_1)$, $\sigma^*(x_2)$ we evidently have $M \leq M^*$. Thus it is now enough to obtain an upper bound for M^*.

If we reflect Q^* in $\sigma^*(x_2)$, form the union of the two domains and $\sigma^*(x_2)$ and assign $\sigma^*(x_1)$ and its reflection as a pair of opposite sides we obtain a quadrangle Q^{**} of module $2M^*$ for the family of curves joining the complementary pair of sides. If we map Q^{**} by $w = \exp(\pi z/i(x_2 - x_1))$ its image is a doubly-connected domain Δ slit along a radius. The module of Δ for the family of curves joining the boundary components is $2M^*$. On the other hand this module is equal to the Dirichlet integral of the harmonic measure with respect to Δ of one of its boundary components. An upper bound for this quantity is given by the Dirichlet integral of any piecewise differentiable continuous function in Δ taking continuously the value 1 on one boundary component and the value 0 on the other. Returning to Q^* we see that an upper bound for M^* is given by the Dirichlet integral of a function continuous on the closure of Q^*, piecewise differentiable and taking the value 0 on the side given by $y = -\theta_1^{(t)}(x)$, the value 1 on the side given by $y = \theta_2^{(t)}(x)$. Such a function is given for example by

$$(\theta^{(t)}(x))^{-1}(y + \theta_1^{(t)}(x)) \quad \text{where } \theta^{(t)}(x) = \theta_1^{(t)}(x) + \theta_2^{(t)}(x).$$

The estimate obtained by taking its Dirichlet integral is

$$(4) \quad M^* \leq \int_{x_1}^{x_2} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3}\int_{x_1}^{x_2} \frac{(\theta_1'^{(t)}(x))^2 - \theta_1'^{(t)}(x)\theta_2'^{(t)}(x) + (\theta_2'^{(t)}(x))^2}{\theta^{(t)}(x)}\,dx.$$

To reduce this to an expression in terms of $\theta_1^{(s)}(x)$, $\theta_2^{(s)}(x)$ let the respective closed intervals on which the slope of $\theta_l^{(t)}(x)$ is ± 1 be $\Lambda_1^{(l)}, \cdots, \Lambda_{k_l}^{(l)}$, $l = 1, 2$, and let the variation of $\theta_l^{(s)}(x)$ on these intervals be $V_1^{(l)}, \cdots, V_{k_l}^{(l)}$, $l = 1, 2$. Writing $\theta^{(s)}(x) = \theta_1^{(s)}(x) + \theta_2^{(s)}(x)$ we then have

$$(5) \quad \begin{aligned} \int_{x_1}^{x_2} \frac{dx}{\theta^{(t)}(x)} - \int_{x_1}^{x_2} \frac{dx}{\theta^{(s)}(x)} &\leq \left(\sum_{j=1}^{k_1}\int_{\Lambda_j^{(1)}} + \sum_{j=1}^{k_2}\int_{\Lambda_j^{(2)}}\right)\frac{\theta^{(s)}(x) - \theta^{(t)}(x)}{\theta^{(s)}(x)\theta^{(t)}(x)}\,dx \\ &\leq \sum_{j=1}^{k_1}\frac{(2L - 2\theta^{(m)})}{(2\theta^{(m)})^2}V_j^{(1)} + \sum_{j=1}^{k_2}\frac{(2L - 2\theta^{(m)})}{(2\theta^{(m)})^2}V_j^{(2)} \\ &\leq \frac{(2L - 2\theta^{(m)})}{(2\theta^{(m)})^2}(V_1 + V_2), \end{aligned}$$

the last since the total variation of $\theta_l^{(s)}(x)$ on $[x_1, x_2]$ is not greater than that of $\theta_l(x)$, $l = 1, 2$. On the other hand

$$(6) \quad \begin{aligned} \frac{1}{3}\int_{x_1}^{x_2} &\frac{(\theta_1'^{(t)}(x))^2 - \theta_1'^{(t)}(x)\theta_2'^{(t)}(x) + (\theta_2'^{(t)}(x))^2}{\theta^{(t)}(x)}\,dx \\ &\leq \frac{1}{2}\left(\sum_{j=1}^{k_1}\int_{\Lambda_j^{(1)}} + \sum_{j=1}^{k_2}\int_{\Lambda_j^{(2)}}\right)\frac{1}{\theta^{(t)}(x)}\,dx \\ &\leq \frac{1}{2}\left(\sum_{j=1}^{k_1}\frac{1}{2\theta^{(m)}}V_j^{(1)} + \sum_{j=1}^{k_2}\frac{1}{2\theta^{(m)}}V_j^{(2)}\right) \\ &\leq \frac{1}{4}\frac{1}{\theta^{(m)}}(V_1 + V_2). \end{aligned}$$

Combining (4), (5) and (6) we have

$$M^* \leq \int_{x_1}^{x_2} \frac{dx}{\theta^{(s)}(x)} + \frac{L}{2(\theta^{(m)})^2}(V_1 + V_2).$$

As the number n of original intervals tends to infinity, $\int_{x_1}^{x_2} dx/\theta^{(s)}(x)$ tends to $\int_{x_1}^{x_2} dx/\theta(x)$ and we obtain (3).

4. Theorem 2 can be used to derive a version of Ahlfors' Second Fundamental Inequality.

THEOREM 3. *Let D be a simply-connected domain in the z-plane with boundary elements P_1, P_2 such that for every x the segment $\sigma(x)$: $-\theta_1(x) < y < \theta_2(x)$, $\theta_1(x), \theta_2(x) > 0$, separates D into subdomains with P_1, P_2 as respective boundary* *elements. Let* $\theta(x) = \theta_1(x) + \theta_2(x)$. *Let* $V_j(x', x'')$ *denote the total variation of* θ_j *on* $[x', x'']$, $j = 1, 2$. *Let* $\theta_j(x) \leq L$, *all* $x, j = 1, 2$. *Let*

$$\min_{x' \leq x \leq x''} (\theta_1(x), \theta_2(x)) = \theta^{(m)}(x', x'').$$

Let D *be mapped conformally on the strip* $S : 0 < \Im\zeta < a$ *in the* ζ*-plane so that* P_1, P_2 *correspond to the boundary elements of the latter determined by the point at infinity with respective neighborhoods in* $\Re\zeta < 0$, $\Re\zeta > 0$. *Let* $\tau(x)$ *denote the image of* $\sigma(x)$ *in* S. *Let*

$$\xi_1(x) = \mathrm{glb}_{\zeta \in \tau(x)} \Re\zeta, \qquad \xi_2(x) = \mathrm{lub}_{\zeta \in \tau(x)} \Re\zeta.$$

Then for $x_1 < x_2$,

$$\begin{aligned} \frac{1}{a}(\xi_2(x_2) - \xi_1(x_1)) \leq \int_{x_1}^{x_2} \frac{dx}{\theta(x)} + \frac{L}{2(\theta^{(m)}(x_1, x_2))^2}(V_1(x_1, x_2) + V_2(x_1, x_2)) \\ + \frac{2L}{\theta^{(m)}(x_1 - 2L, x_1 + 2L)} + \frac{2L}{\theta^{(m)}(x_2 - 2L, x_2 + 2L)}. \end{aligned} \tag{7}$$

The rectangle $0 < \Im\zeta < a$, $\xi_1(x_1) < \Re\zeta < \xi_2(x_2)$ becomes a quadrangle $\mathfrak{Q}'$ on assigning its pairs of horizontal and vertical sides as pairs of opposite sides. The module m of $\mathfrak{Q}'$ for the family of curves joining the horizontal sides is

$$(1/a)(\xi_2(x_2) - \xi_1(x_1)).$$

Let Q' be the inverse image of $\mathfrak{Q}'$ in D. Let Q be the quadrangle with $\sigma(x_1)$, $\sigma(x_2)$ as a pair of opposite sides as before, M its module for the family of curves joining the complementary pair of sides. Let $\rho(z)\,|dz|$ be the extremal metric with L-normalization corresponding to M. Let D^* be the subdomain of D given by $-\theta_1(x) < y < \theta_2(x)$ and let

$$\rho_1(z) = 1/2\theta^{(m)}(x_1 - 2L, x_1 + 2L)$$

for

$$z \in D^* \cap \{x_1 - 2\theta^{(m)}(x_1 - 2L, x_1 + 2L) < \Re z < x_1 + 2\theta^{(m)}(x_1 - 2L, x_1 + 2L)\}$$

$$\rho_1(z) = 0, \quad \text{elsewhere in } Q';$$

$$\rho_2(z) = 1/2\theta^{(m)}(x_2 - 2L, x_2 + 2L)$$

for

$$z \in D^* \cap \{x_2 - 2\theta^{(m)}(x_2 - 2L, x_2 + 2L) < \Re z < x_2 + 2\theta^{(m)}(x_2 - 2L, x_2 + 2L)\},$$

$$\rho_2(z) = 0, \quad \text{elsewhere in } Q'.$$

Let

$$\rho^*(z) = \max\,(\rho(z), \rho_1(z), \rho_2(z)), \quad z \in Q'.$$

Then $\rho^*(z)\,|dz|$ is an admissible metric for the module problem for Q' determining m. Thus

$$\frac{1}{a}(\xi_2(x_2) - \xi_1(x_1))$$
$$< M + \frac{2L}{\theta^{(m)}(x_1 - 2L, x_1 + 2L)} + \frac{2L}{\theta^{(m)}(x_2 - 2L, x_2 + 2L)}.$$

Applying Theorem 2 we now have inequality (7).

5. We now turn to the proof of the following refined version of Hayman's result.

THEOREM 4. *Let $f(z)$ be regular for $|z| < 1$ with at most q zeros in $|z| < s$, $s \leq 1$ and have expansion about the origin*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Let $\mu_l = \max_{\nu \leq q} |a_\nu|$, $R_1 = (q+2)2^{q-1}\mu_q$, $M(r, f) = \max_{|z|=r} |f(z)|$. *Let* $n(w)$ *denote the number of roots in* $|z| < 1$ *of the equation* $f(z) = w$ *and let*

$$p(R) = \frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\varphi})\, d\varphi.$$

Then

$$\int_{R_1}^{M(r,f)} \frac{dR}{Rp(R)} < 2\log\frac{1+r}{1-r} + \pi^2 + \log\frac{1}{s}. \tag{8}$$

Consider the module of the family Γ, the individual elements of which consist of the level sets $\gamma_R : |f(z)| = R$, $R_1 < R < M(r, f)$. It is seen at once that this module is

$$\frac{1}{2\pi}\int_{R_1}^{M(r,f)} \frac{dR}{Rp(R)}.$$

Let z_r denote a point of $|z| = r$ where $M(r, f)$ is attained. Each γ_R separates 0 and z_r in $|z| < 1$. Thus each γ_R either contains an open arc separating 0 and z_r in $|z| < 1$ and tending to $|z| = 1$ in each sense or contains a simple closed curve $\tilde{\gamma}_R$ separating 0 and z_r. The unit circle can be mapped conformally onto the strip $|\Im w| < 1/2$ so that $z = 0$ goes into $w = 0$, z_r goes into $(1/\pi)\log(1+r)/(1-r)$. Let in $|\Im w| < 1/2$,

$$\mathrm{P}(w) = 1, \quad |w| < \frac{1}{2}, \quad 0 < \Re w < \frac{1}{\pi}\log\frac{1+r}{1-r}$$
$$\text{and} \quad \left| w - \frac{1}{\pi}\log\frac{1+r}{1-r} \right| < \frac{1}{2}$$
$$= 0, \quad \text{elsewhere.}$$

Let $\rho_1(z)\,|dz|$ be the metric induced in $|z| < 1$ by $\mathrm{P}(w)\,|dw|$. Then in the first instance above we have

$$\int_{\gamma_R} \rho_1(z) \, | dz | \geq 1.$$

Let

$$\rho_2(z) = 1/2s, \qquad |z| < s$$
$$= 1/2\pi |z|, \quad s \leq |z| < 1.$$

In the second instance above let

$$m_R = \min_{z \epsilon \tilde{\gamma}_R} |z|, \qquad M_R = \max_{z \epsilon \tilde{\gamma}_R} |z|$$

If $m_R \geq s$, evidently

$$\int_{\tilde{\gamma}_R} \rho_2(z) \, | dz | \geq 1.$$

If $m_R < s \leq M_R$ we readily see that

$$\int_{\tilde{\gamma}_R} \rho_2(z) \, | dz | \geq 1.$$

Now suppose $M_R < s$. We recall a result of Hayman [2; Lemma 2.2] which says that for $0 < t < s$,

$$\min_{|z|=t} |f(z)| \leq R_1 .$$

On the other hand for $M_R < t < s$

$$\max_{|z|=t} |f(z)| > R_1 .$$

Thus for these latter values of t, γ_R has at least two points of intersection with $|z| = t$. Thus again

$$\int_{\gamma_R} \rho_2(z) \, | dz | \geq 1.$$

Let now $\rho(z) = \max (\rho_1(z), \rho_2(z))$. Then $\rho(z) \, |dz|$ is admissible in the L-normalization for the module problem for Γ. Thus evidently

$$\frac{1}{2\pi} \int_{R_1}^{M(r,f)} \frac{dR}{R p(R)} < \frac{1}{\pi} \log \frac{1+r}{1-r} + \frac{\pi}{4} + \frac{\pi}{4} + \frac{1}{2\pi} \log \frac{1}{s} .$$

Since this metric is obviously not extremal we can use the inequality sign. This gives at once inequality (8).

6. We observe that the additive constant on the right-hand side of (8) is absolute, independent of q. If we apply the result [2; Lemma 2.1] in a manner similar to Hayman's proof of [2; Theorem 2.5] we obtain the following statement.

In the notation of Theorem 4, *if* $f(z)$ *is areally mean p-valent in* $|z| < 1$,

$$M(r, f) < A(p) \mu_p (1 - r)^{-2p}, \quad 0 < r < 1,$$

with $A(p) = (p + 2) 2^{3p-1} \exp (p\pi^2 + 1/2)$.

BIBLIOGRAPHY

1. LARS AHLFORS, *Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen,* Acta Societaties Scientiarum Fennicae, Nova Series A, vol. 1 (1930), pp. 1–40.
2. W. K. HAYMAN, *Multivalent functions*, Cambridge University Press, Cambridge, 1958.
3. JAMES A. JENKINS, *Univalent functions and conformal mapping*, Springer-Verlag, Berlin, 1958.
4. ———, *On the Phragmén-Lindelöf theorem, the Denjoy conjecture and related results*, A. J. Macintyre memorial volume, Ohio University Press, Athens, Ohio, 1970, pp. 183–200.
5. O. TEICHMÜLLER, *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Mathematik, vol. 3 (1938), pp. 621–678.

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