SECONDARY OPERATIONS IN THE COHOMOLOGY OF H-SPACES

BY

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0.1 Introduction

The study of the cohomology of H-spaces had several stages—from the purely algebraic study of Hopf algebras—the Hopf and Borel theorems through the study of Hopf algebras over $A(p)$ to the study of higher order operations such as the higher Bockstein operations and the Bockstein spectral sequence as in [2]. Lately K-theory has been used (by J. R. Hubbuck for example) to provide more information regarding the structure of the cohomology of H-spaces.

In this study we use some secondary operations to investigate $H^*(X, Z_p)$, X an H-space. It is difficult to state all the consequences of this study. As a sample we have the following.

1. (Proposition 4.1). Let ϕ' be a secondary operation associated with the relation

$$(1) \qquad \beta P^m = 1/m - 1(P^1\beta P^{m-1} - P^m\beta) \qquad m \not\equiv 1 \bmod p.$$

(For $p = 2$, put $P^m = Sq^{2m}$, $\beta = Sq^1$.) Let (X, μ) be an H-space. Suppose $\beta H^{\text{even}}(X, Z_p) = 0$. If $0 \neq x \in QH^{2m}(X, Z_p)/\text{im } P^1$ then x can be represented by an element $x' \in H^{2m}(X, Z_p)$ so that $\phi'(x')$ will have a nonzero image in $QH^{2mp}(X, Z_p)/\text{im } P^1$. Moreover, there exists a submodule A, $A \subset PH_{2m}(X, Z_p) \cap \ker P_1^*$ (P_1^* the dual of P^1) so that $x \in A^*$ and for every $y \in A$

$$\langle x, y\rangle = \langle \phi(x'), y^p\rangle, \qquad y^p = y(y \cdots (y\cdot y))\cdots).$$

2. (Corollary 4.2). Let (X, μ) be an H-space, $\beta H^*(X, Z_p) = 0$. ϕ' induces a morphism $Q\phi'$

$$Q\phi' : QH^{2m}(X, Z_p) \to QH^{2mp}(X\{ Z_p)/\text{im } \rho^1 \quad (m \not\equiv 1 \bmod p)$$

which satisfies $\ker Q\phi' \subset \text{im } P^1$.

3. (Proposition 4.3). Let (X, μ) be an H-space, $H_*(X, Z_p)$ associative and commutative. Suppose $\beta H^{\text{even}}(X, Z_p) = 0$. If a class $y \in PH_{2k}(X, Z_p)$ has a finite height, then $y^p = 0$ and for every $\lambda \geq 0$ for which $P_\lambda^* y \in \ker P_1^*$ we have $\lambda + k \equiv 1 \bmod p$.

Similar results hold for a secondary operation ϕ'' associated with $\beta P^m = \beta P^m$ (defined on $\ker \rho^m \cap H^{2m}(X, Z_p)$).

All spaces considered are of the homotopy type of a CW complex of finite type. All Hopf algebras are graded connected and of finite type. We use

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the notations in [7] of QA, PA and ξ for indecomposables primitives and p^{th} power in a Hopf algebra A. We write $Q : A \to QA$ as the projection.

1. On Cartan formulas for secondary operations

In [5] and [6] the existence of a Cartan formula for high order cohomology operations is studied. It was found, for example, that if ϕ is a secondary operation, x, y vectors of cohomology classes so that $x \cdot y = \sum_i x_i \cdot y_i$ is in the domain of ϕ and x, y and ϕ satisfy certain stability conditions then $\phi(xy) = \phi' x \cdot a' y + a'' x \cdot \phi'' y$ for some primary operations a' and a'' and some operations (that might be combinations of primary and secondary operations) ϕ' and ϕ''. The limitations on x, y and ϕ are essential if an attempt to compute a', a'', ϕ' and ϕ'' explicitly is to be made. However, if a more general property is sought these conditions can be relaxed. Let p be a prime and let ϕ be the secondary operation defined by the universal example (E, u, v) (in the sense of Adams, see [1, page 55]). By this we mean the following: Given $B = K(Z_p, m)$, $B' = \prod_j K(Z_p, n_j)$ and $b : B \to B'$. E and $r : E \to B$ are obtained as the fibration induced by b from the path fibration

$$\Omega B' \to \mathcal{L} B' \to B'.$$

Hence we have

$$\begin{array}{ccccc} & & \Omega B' & & \\ & \swarrow^{j} & & \searrow^{\bar{b}} & \\ E & & \longrightarrow & & \mathcal{L} B' \\ \downarrow r & & & & \downarrow \\ B & & \xrightarrow{\quad b \quad} & & B'. \end{array}$$

Let $u = r^* \iota_m$ where $\iota_m \in H^*(B, Z_p)$ is the fundamental class, and let v be a class in $H^*(E, Z_p)$ (usually assumed not to be in im r^*).

The operation ϕ is then defined on the natural subset $S \subset H^*(K, Z_p)$ of the cohomology of an arbitrary CW complex K consisting of those elements s with $b \circ f_s \sim *$ whenever $f_s : K \to B$ and $f_s^* \iota_m = s$. $\phi(s) \subset H^*(K, Z_p)$ is then defined to be the set

$$\{\tilde{f}_s^* v \mid \tilde{f}_s : K \to E,\ r \circ \tilde{f}_s = f_s\}.$$

We shall assume that $b = \Omega b_1$ is a loop map; however, v is not necessarily primitive. We shall refer to S as the domain of ϕ.

1.1 Proposition. *Let X and Y be CW complexes. Suppose $z = \sum_i x_i \otimes y_i \in H^m(X \times Y, Z_p)$ is in the domain of ϕ, $\dim x_i > 0$, $\dim y_i > 0$. Let $\rho(x)$ and $\rho(y)$ be the $A(p)$ ideal generated by the x_i's and y_i's respectively. Then*

$$\phi(z) \cap [\rho(x) \otimes H^*(Y, Z_p) + H^*(X, Z_p) \otimes \rho(y)] \neq \emptyset.$$

Proof. Let $K_i' = K(Z_p, \dim x_i)$, $K_i'' = K(Z_p, \dim y_i)$. Let $X_0 = \prod_i K_i'$, $Y_0 = \prod_i K_i''$ and let $f : X \to X_0$ and $g : Y \to Y_0$ be given by $f^* r_i'^* \iota_i' = x_i$,

$g^* r_i''^* \iota_i'' = y_i$ ($r_i' : X_0 \to K_i'$ and $r_i'' : Y_0 \to K_i''$ are the projections, ι_i' and ι_i'' are the fundamental classes in $H^*(K_i', Z_p)$ and $H^*(K_i'', Z_p)$ respectively). We have the following sequence:

$$X \times Y \xrightarrow{f \times g} X_0 \times Y_0 \xrightarrow{h} B \xrightarrow{b} B'.$$

$h^* \iota_m = z_0 = \sum_i \iota_i' \otimes \iota_i''$, $h \mid X_0 \vee Y_0 = *$ and $b \circ h \circ (f \times g) \sim *$, hence,

$$\begin{aligned} h^* \circ b^*(PH^*(B', Z_p)) \subset {} & [PH^*(X_0, Z_p) \otimes PH^*(Y_0 Z_p)] \cap \ker (f^* \otimes g^*) \\ = {} & [\ker f^* \cap PH^*(X_0, Z_p)] \otimes PH^*(Y_0, Z_p) \\ & + PH^*(X_0, Z_p) \otimes [\ker g^* \cap PH^*(Y_0, Z_p)] \end{aligned}$$

and there are $w_1, w_2 : X_0 \times Y_0 \to B'$, $w_i \mid X_0 \vee Y_0 = *$ so that

$$b \circ h = w_1 * w_2 = \mu_{B'} \circ (w_1 \times w_2) \circ \Delta$$

($\mu_{B'}$ the loop addition, Δ the diagonal) and

$$w_1^* PH^*(B', Z_p) \subset [\ker f^* \cap PH^*(X_0 Z_p)] \otimes PH^*(Y_0, Z_p)$$

and

$$w_2^* PH^*(B', Z_p) \subset PH^*(X_0, Z_p) \otimes (\ker g^* \cap PH^*(Y_0, Z_p)).$$

Let X_0' and Y_0' be generalized Eilenberg McLane spaces,

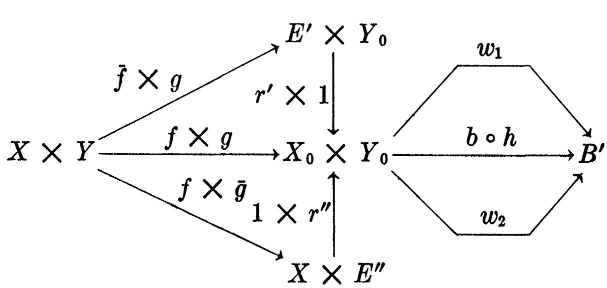
$$b' : X_0 \to X_0', \qquad b'' : Y_0 \to Y_0'$$

be H-mappings with

$$b'^*(PH^*(X_0', Z_p)) = \ker f^* \cap PH^*(X_0, Z_p),$$
$$b''^*(PH^*(Y_0', Z_p)) = \ker g^* \cap PH^*(Y_0, Z_p).$$

Let $r' : E' \to X_0$ and $r'' : E'' \to Y_0$ be the fibrations induced by b' and b'' from $\Omega X_0' \to \mathfrak{L} X_0' \to X_0'$ and $\Omega Y_0' \to \mathfrak{L} Y_0' \to Y_0'$ respectively. Put $j' : \Omega X_0' \to E'$, $j'' : \Omega Y_0' \to E''$.

Now $f : X \to X_0$ and $g : Y \to Y_0$ can be lifted to $\bar{f} : X \to E'$ and $\bar{g} : Y \to E''$. Consider now the following diagram:



As

$$w_1^* PH^*(B', Z_p) \subset \operatorname{im} \bar{b}'^* \otimes \bar{H}^*(Y_0, Z_p),$$
$$w_2^* PH^*(B', Z_p) \subset \bar{H}^*(X_0, Z_p) \otimes \operatorname{im} \bar{b}''^*,$$

we have $w_1(r' \times 1) \sim *$ (rel $E' \vee Y_0$) and $w_2(1 \times r'') \sim *$ (rel $X_0 \vee E''$).

Hence, there exist

$$v' : E' \times Y_0, E' \vee Y_0 \to \mathcal{L}B',*$$
$$v'' : X_0 \times E'', X_0 \vee E'' \to \mathcal{L}B',*$$

with $\mathcal{E}_1 \circ v' = \omega_1(r' \times 1), \mathcal{E}_1 \circ v'' = \omega_2(1 \times r'')$ where $\mathcal{E}_1 : \mathcal{L}B' \to B'$ is the evaluation at 1.

Define $w : E' \times E'' \to E \subset B \times \mathcal{L}B'$ by

$$w(\bar{x}, \bar{y}) = h(r'(\bar{x}), r''(\bar{y})), v'(\bar{x}, r''(\bar{y})) * v''(r'(\bar{x}), \bar{y})$$

where $v' * v'' = \mathcal{L}(\mu_{B'}) \circ (v' \times v'')$.

Now, $(\bar{f}^* \otimes \bar{g}^*) \circ w^*(v) \in \phi(z)$ but as $w \circ (j' \times j'') = *$ and therefore,

$$(j'^* \otimes j''^*) \circ w^*(v) = 0,$$

by [9] Proposition 5.5 III, $w^*(v)$ is in the ideal generated by im $(\overline{r'^* \otimes r''^*})$, or equivalently, in the ideal generated by

$$\text{im } \bar{r}'^* \otimes H^*(E'', Z_p) + H^*(E', Z_p) \otimes \text{im } \bar{r}'',$$

hence

$$\bar{f}^* \otimes \bar{g}^* \circ w^*(v) \in \rho(x) \otimes H^*(Y, Z_p) + H^*(X, Z_p) \otimes \rho(y).$$

2. Coproducts and indecomposables

If N is a Hopf algebra over $A(p)$ then QN is an $A(p)$-module. In general the product in N does not induce a nontrivial product in QN. On the other hand the coproduct ψ induces some operation on QN.

Let $\bar{\Psi} : \bar{N} \to \bar{N} \otimes \bar{N}$ be given by $\bar{\psi}x = \psi x - 1 \otimes x - x \otimes 1$ and let ψ^k and $\bar{\psi}^k$ be defined inductively by

$$\psi^0 = 1, \qquad \psi^1 = \psi, \qquad \psi^k = (\psi^{k-1} \otimes 1) \circ \psi, \qquad k > 1.$$
$$\bar{\psi}^0 = \varepsilon, \text{ the augmentation}, \qquad \bar{\psi}^1 = \bar{\psi}, \qquad \bar{\psi}^k = (\bar{\psi}^{k-1} \otimes 1)\bar{\psi}, \quad k > 1.$$

2.1 DEFINITION. Let A be a graded module over a ring R and let A^k be the k-fold tensor product $A^2 = A \otimes A, A^k = A \otimes A^{k-1}$. Define

$$\text{shuff}_j^k : A^j \otimes A^{k-j} \to A^k$$

by

$$\text{shuff}_j^k[(a_1 \otimes \cdots \otimes a_j) \otimes (a_{j+1} \otimes \cdots \otimes a_k)] = \sum \varepsilon_\alpha \, a_{\alpha(1)} \otimes \cdots \otimes a_{\alpha(k)},$$

summations over all permutations

$$\alpha : (1, 2, \cdots k) \to (\alpha(1), \cdots \alpha(k))$$

which satisfy $\alpha(r) < \alpha(s)$ if either $r < s \le j$ or $j < r < s$; $\varepsilon_\alpha = \prod (-1)^{|a_r||a_s|}$ $(a_r \in A_{|a_r|})$, the product taken over all r, s with $r \le j < s$ and $\alpha(r) > \alpha(s)$. $\text{shuff}_0^k = \text{shuff}_k^k = 1$.

A tedious but straightforward calculation yields

2.2 PROPOSITION. *If N is a Hopf algebra, $x, y \in N$ and $DN = \bar{N} \cdot \bar{N} = \ker Q$ then*

$$\bar{\psi}^{k-1}(x \cdot y) = \textstyle\sum_{j=1}^{k-1} \text{shuff}_j^k \, \bar{\psi}^{j-1}x \otimes \bar{\psi}^{k-j-1}y + d$$

where $d \in \sum_{n=0}^{k-1} N^n \otimes DN \otimes N^{k-n-1}$.

As a consequence we have

2.3 COROLLARY. *$\psi : N \to N \otimes N$ induces a morphism*

$$\tilde{Q}\bar{\psi}^{k-1} : QN \to (QN)^k / \textstyle\sum_{j=1}^{k-1} \text{im } (\text{shuff}_j^k).$$

3. Coproducts and secondary operations

Let (X, μ) be an H-space, ϕ a nonstable secondary operation associated with a relation

(1) $$\beta P^m = \textstyle\sum_i a_i b_i, \qquad a_i, b_i, \in A(p), e(b_i) < 2m$$

(e the excess, i.e., if $0 \neq \iota \in H^n(K(Z_p, n), Z_p)$ then $a\iota = 0$ if and only if $e(a) > n$).

Thus, in terms of universal examples $\phi = \phi_n$ is defined by (E_n, u, v) where E_n is given by the fibration

$$\Omega B_0 \xrightarrow{j_0} E_n \xrightarrow{r} K(Z_p, n), \qquad n \leq 2m,$$

induced by the mapping

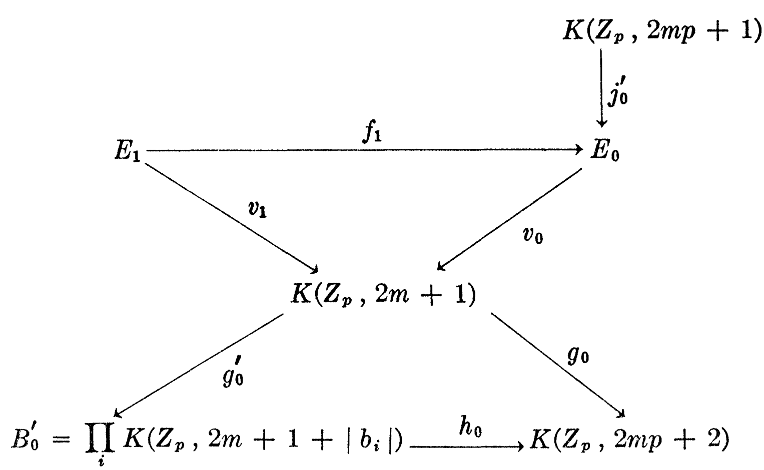
$$g : K(Z_p, n) \to B_0 = \textstyle\prod_i K(Z_p, n + |b_i|), \qquad g^* r_i^* \iota_{n+|b_i|} = b_i \iota_n$$

($r_i : B_0 \to K(Z_p, n + |b_i|)$ is the projection). Here $u = r^* \iota_n$ and $v \in H^*(E_n, Z_p)$ satisfies $j_0^* v = \sum_i a_i \sigma^* (r_i^* \iota_{n+|b_i|})$. If v is appropriately chosen then ϕ is additive in dimension $< 2m$ — $\phi(x + y) = \phi(x) + \phi(y)$ (dim x = dim $y < 2m$) and in dimension $2m$ we have

3.1 PROPOSITION.

$$\phi(x + y) = \phi(x) + \phi(y) + \textstyle\sum_{a=1}^{p-1} 1/p \binom{p}{a} x^a \cdot y^{p-a}.$$

Proof. For the case $p = 2$, ($P^k = Sq^{2k}, \beta = Sq^1$) this is being proved in [4]. Consider the following diagram:



where

$$g_0'^* r_i^* \iota_{2m+1+|b_i|} = b_i \, \iota_{2m+1}$$

$(r_i : B_0 \to K(Z_p, 2m + 1 + |b_i|)$ is the projection).

$$g_0^* \iota_{2mp+2} = \beta P^m \iota_{2m+1}, \qquad h_0^* \iota_{2mp+2} = \textstyle\sum_i a_i(r_i^* \iota_{2m+1+|b_i|}).$$

Then $\Omega E_1 = E = E_{2m}$, $\Omega g_0' = g$ and $\Omega E_0 \approx K(Z_p, 2m) \times K(Z_p, 2mp)$. The latter equivalence is not unique, but we claim that a representation

$$\Omega E_0 \approx K(Z_p, 2m) \times K(Z_p, 2mp)$$

can be chosen so that

$$\bar{\mu}_{\Omega E_0}^*(1 \otimes \iota_{2mp}) = \sum_{a=1}^{p-1} \frac{1}{p}\binom{p}{a} (\iota_{2m} \otimes 1)^a \otimes (\iota_{2m} \otimes 1)^{p-a}.$$

Indeed, if $0 \neq t \in H_{2m}(\Omega E_0, Z_p)$ satisfies $t^p = 0$, then $H^*(\Omega E_0, Z_p)$ is primitively degenerated in dim $\leq 2mp + 1$ and $1 \otimes \iota_{2mp}$ is primitive. By [3, Theorem 5.14], this implies that $1 \otimes \iota_{2mp} \in \operatorname{im} \sigma^*$ which is a contradiction as $\iota_{2mp+1} \notin \operatorname{im} j_0'^*$. Hence, $t^p \neq 0$ and there exists a class $v_0 \in QH^{2mp}(\Omega E_0, Z_p)$ dual to $t^p \in PH_{2mp}(\Omega E_0, Z_p)$.

Choosing the representation $\Omega E_0 \approx K(Z_p, 2m) \times K(Z_p, 2mp)$ appropriately we may assume that $v_0 = 1 \otimes \iota_{2mp}$ and it has the desired coproduct expression. Choosing $v = \Omega f_1^* (1 \otimes \iota_{2mp})$ we have

$$\mu_E^* v = \left(v \otimes 1 + 1 \otimes v + \sum_{a=1}^{p-1} \frac{1}{p}\binom{p}{a} u^a \otimes u^{p-a}\right) \in \phi(u \otimes 1 + 1 \otimes u).$$

3.2 Main Calculation. *Let (X, μ) be an H-space. Let ϕ be a (nonstable) secondary operation associated with* (1). *Let $B \subset H^*(X, Z_p)$ be an $A(p)$ module, and let*

$$q : H^*(X, Z_p) \to H^*(X, Z_p)//B$$

be the reduction of $H^*(X, Z_p)$ by the ideal B_1 generated by $\bar{B}$. If $x \in H^{2m}(X, Z_p)$, $x \in \bigcap_i \ker b_i$ and $\bar{\mu}^* x \in B_1 \otimes B_1$ then

$$(q \otimes q)\bar{\mu}^*\phi(x) = q \otimes q\left(\sum_{a=1}^{p-1} \frac{1}{p}\binom{p}{a} x^a \otimes x^{p-a} + \sum_i \operatorname{im} a_i\right).$$

Proof.

$$\mu^*\phi(x) \subset \phi\mu^*(x) = \phi(x \otimes 1 + 1 \otimes x + \textstyle\sum_k x_k' \otimes x_k''), \qquad x_k', x_k'' \in B_1.$$

Now

$$\phi(x \otimes 1 + 1 \otimes x + \textstyle\sum_k x_k' \otimes x_k'')$$

$$\subset \phi(x) \otimes 1 + 1 \otimes \phi(x) + \sum_{a=1}^{p-1} \frac{1}{p}\binom{p}{a} x^a \otimes x^{p-a} + \phi\,(\sum_k x_k' \otimes x_k'')$$

$$+ \sum_{a=1}^{p-1} \frac{1}{p}\binom{p}{a} (x \otimes 1 + 1 \otimes x)^a (\sum_k x_k' \otimes x_k'')^{p-a}$$

Hence $\phi\mu^* x$ can be represented by an element $\bar{v}$ with

$$\bar{\mu}^*\bar{v}^* = \sum_{a=1}^{p-1} \frac{1}{p}\binom{p}{a} x^a \otimes x^{p-a} + \phi\,(\sum_k x_k' \otimes x_k'') + d$$

where $d \in B_1 \otimes H^*(X, Z_p) + H^*(X, Z_p) \otimes B_1 + \sum_i \text{im}\, a_i$. ($\sum_i \text{im}\, a_i$ accumulates all indeterminacy of ϕ.) By 1.1, $\phi(\sum_k x_k' \otimes x_k'')$ can be represented by an element in $B_1 \otimes H^*(X, Z_p) + H^*(X, Z_p) \otimes B_1$ and 3.2 follows.

4. Applications

Only few of the application 3.2 are given here. We consider two cases of (1).

(1)′

$$\beta P^m = \frac{1}{m-1}\,(P^1\beta P^{m-1} - P^m\beta), \quad m \not\equiv 1 \bmod p.$$

Here

$$b_1 = \frac{1}{m-1}\,\beta P^{m-1}, \qquad b_2 = \frac{1}{m-1}\,\beta, \qquad a_1 = P^1, \qquad a_2 = P^m$$

and the operation will be denoted by ϕ'.

(1)″ Define the operation ϕ'' on ker P^m corresponding to $b_1 = P^m$, $a_1 = \beta$, and $(\beta P^m) > 2m$.

4.1 Proposition. *Let (X, μ) be an H-space. Suppose $\beta H^{\text{even}}(X, Z_p) = 0$. If $0 \neq x \in QH^{2m}(X, Z_p)/\text{im}\, P^1$, $m \not\equiv 1 \bmod p$, then there exists a class x' representing x so that $\phi'(x')$ has nonzero image in $QH^{2mp}(X, Z_p)/\text{im}\, P^1$. Moreover, there exists a submodule M, $M \subset PH_{2m}(X, Z_p) \cap \ker P_1^*$ (P_1^* the dual of P^1) so that $x \in M^*$ and for every $y \in M$,*

$$\langle x', y\rangle = \langle \phi'(x'), y^p\rangle \quad \textit{where } y^p = y(y(\cdots(y\cdot y))\cdots).$$

Proof. Let $B(n)$ be the $A(p)$ subalgebra of $H^*(X, Z_p)$ generated by $\sum_{k\leq n} H^k(X, Z_p)$. Let

$$q_n' : H^*(X, Z_p) \to Q[H^*(X, Z_p)//B(n)]/\text{im}\, P^1.$$

Consider the cofiltration

$$H^*(X, Z_p) \to H^*(X, Z_p)//B(1) \to \cdots \to H^*(X, Z_p)//B(k) \to \cdots.$$

Let n be the smallest integer so that the image of x in

$$Q[H^*(X\{ Z_p)//B(n+1)]/P^1$$

vanishes. Hence, x can be represented by an element in the ideal generated by $B(n+1)$ but as an indecomposable the representative can be chosen to be a nondecomposable element x' in $B(n+1)$, $x' = \sum_i a_i x_i' + d$ where $a_i \in A(p)$, $x_i' \in H^{n+1}(X, Z_p)$ and $d \in B(n)$, hence $\bar{\mu}^* x \in B(n) \otimes B(n)$, $q_n' x' \neq 0$. The rest of the proof follows from 3.2 with $B = B(n)$: Take μ_0^* to be the coproduct induced by μ^* on $H^*(X, Z_p)//(B(n) + \text{im}\, P^1)$ and we have

$$\tilde{Q}\bar{\mu}_0^{*p-1} q_n'\, \phi'(x') = q_n' x' \otimes \cdots \otimes q_n' x' \quad (\text{mod } \textstyle\sum_j \text{im shuff}_j^k)$$

($\tilde{Q}\bar{\mu}_0^{*p-1}$ as in 2.3). Choose $M = M(n)$ to be

$$\{Q[H^*(X, Z_p)//B(n)]/\text{im } P^1\}^*,$$

and 4.1 follows.

If $\beta H^*(X, Z_p) = 0$ then it follows from 1.1 that

$$\phi'(DH^*(X, Z_p)) \cap DH^*(X, Z_p) \neq 0,$$

hence ϕ' induces an operation

$$Q\phi' : QH^{2m}(X, Z_p) \to QH^{2mp}(X, Z_p)/\text{im } P^1, \quad m \not\equiv 1 \bmod p$$

and we have

4.2. COROLLARY. *Let (X, μ) be an H-space, $\beta H^{\text{even}}(X, Z_p) = 0$. Then* $\ker Q\phi' \subset \text{im } P^1$.

A dualization of 4.1 yields

4.3 PROPOSITION. *Let (X, μ) be an H-space, $H_*(X, Z_p)$ associative and commutative. Suppose $\beta H^{\text{even}}(X, Z_p) = 0$. If a class $y \in PH_{2k}(X, Z_p)$ has a finite height then $y^p = 0$ and for every $\lambda \geq 0$ for which $P_\lambda^* y \in \ker P_1^*$ we have $\lambda + k \equiv 1$ mod p.*

4.4 *Remark.* There are quite a few examples of torsion free homology algebras where a two-dimensional class has height p: that of $K(Z, 2)$, $\Omega E_j - E_j$ the exceptional Lie groups, $j = 6, 7, 8 \bmod 3$, $j = 8 \bmod 5$. An example where $z^p = 0$, $\dim z \neq 2$, is the homology of $X = B_U(6, 8 \cdots \infty)$. $H^*(X, Z)$ is odd-torsion free while the primitive element of dim $2p + 2$ has height p. (See [8] for an exact computation.)

4.5 PROPOSITION. *Let X, μ be an H-space. Let $x \in H^{2m}(X, Z_p)$. Suppose $Qx \notin \text{im } \beta$, $\bar{\mu}^* x \in B(n) \otimes B(n)$, $x \notin B(n)$. If $x^p = 0$ then $\phi''(x)$ has nonzero image in $QH^*(X, Z_p)/\text{im } \beta$.*

The proof is similar to that of 4.1.

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