ON THE SPECTRAL RADIUS OF ELEMENTS IN A GROUP ALGEBRA[1]

BY

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Let G be a discrete group, $l_1(G)$ its group algebra. In [2], it was shown that if G contains a free non-abelian subsemigroup on two generators then $l_1(G)$ is non-symmetric. The proof, highly combinatorial in nature, rested on the fact that if x in $l_1(G)$ has no left inverse then there is a non-zero f in $l_1(G)^*$ such that $f(yx) = 0$ for each y in $l_1(G)$. This note contains the same theorem, but the proof given offers more insight into the group algebra.

Let $\mathfrak{A}$ be a Banach *-algebra with identity e. $P(\mathfrak{A})$ will denote the set of linear functionals on $\mathfrak{A}$ such that $f(xx^*) \geq 0$ for each x in $\mathfrak{A}$ and such that $f(e) = 1$. Implicit in the usual proof of Raikov's Theorem (c.f. Naimark [3]) is the following:

$\mathfrak{A}$ is symmetric if, and only if, for each $x \in \mathfrak{A}$,

$$\operatorname{sp}(xx^*) \subset \{f(xx^*) \mid f \in P(\mathfrak{A})\}.$$

(Symmetry is defined as in Rickart [4].) We require this result in proving

LEMMA 1. *Let $\mathfrak{A}$ be a symmetric Banach *-algebra with identity. If x and y are normal elements of $\mathfrak{A}$ then*

$$\nu(xy) \leq \nu(x)\nu(y).$$

Proof. We first note that if z in $\mathfrak{A}$ has no right inverse then there is a f in $P(\mathfrak{A})$ such that $f(z) = 0$. Suppose z has no right inverse. Then zz^* has no right inverse, for if v were such an inverse then z^*x would be a right inverse for z. Hence

$$0 \in \operatorname{sp}(zz^*).$$

By the preceding remark there is an $f \in P(\mathfrak{A})$ such that $f(zz^*) = 0$. But then

$$|f(z)|^2 \leq f(zz^*)f(e) = 0,$$

and so

$$f(z) = 0.$$

A similar statement holds if z has no left inverse.

Suppose now that $\alpha \in \operatorname{sp}(xy)$ and $|\alpha| = \nu(xy)$. Then $xy - \alpha e$ either has no left inverse or no right inverse. Hence there is an f_0 in $P(\mathfrak{A})$ such that

$$f_0(xy - \alpha e) = f_0(xy) - \alpha = 0.$$

Hence

$$|f_0(xy)| = |\alpha| = \nu(xy).$$

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By the Schwarz inequality

$$| f_0(xy) |^2 \leq f_0(xx^*)f_0(yy^*).$$

Now, for any hermitian element z in $\mathfrak{A}$, $| f(z) | \leq \nu(z)$. Also, since x and y are normal elements of $\mathfrak{A}$,

$$\nu(xx^*) \leq \nu(x)^2 \quad \text{and} \quad \nu(yy^*) \leq \nu(y)^2$$

(actually we have equality since $\mathfrak{A}$ is symmetric). Thus

$$|f_0 (xy) |^2 \leq \nu(xx^*)\nu(yy)^* \leq \nu(x)^2\nu(y)^2.$$

Therefore

$$\nu(xy) = | f_0(xy) | \leq \nu(x)\nu(y).$$

The mapping $g \to \delta_g$, where $\delta_g(s) = 0$ if $s \neq g$ and $\delta_g(g) = 1$, is the usual embedding of G into $l_1(G)$. We will not distinguish between g and δ_g. Note that each x in $l_1(G)$ may be written as

$$x = \sum_{g\in G} x(g)g.$$

Let $N(x) = \{g \in G \mid x(g) \neq 0\}$.

LEMMA 2. *Let x and y be elements of $l_1(G)$ such that for each positive integer n*

$$s_1 t_1 \cdots s_n t_n \neq s_{n+1} t_{n+1} \cdots s_{2n} t_{2n}$$

if for some i, $1 \leq i \leq n$, $s_i \neq s_{n+i}$ or $t_i \neq t_{n+i}$, where $s_j \in N(x)$ and $t_j \in N(y)$ for $1 \leq j \leq 2n$. Then

$$\nu(xy) = \| x \| \, \| y \|.$$

Proof. We first note that if x' and y' are elements of $l_1(G)$ such that $N(x') \cap N(y') = \emptyset$ then

$$\begin{aligned} \| x' + y' \| &= \sum_{g\in G} | x'(g) + y'(g) | \\ &= \sum_{g\in G} (| x'(g) | + | y'(g) |) \\ &= \sum_{g\in G} | x'(g) | + \sum_{g\in G} | y'(g) | \\ &= \| x' \| + \| y' \|. \end{aligned}$$

Now,

$$\begin{aligned} \| (xy)^n \| &= \| \sum_{s_1,\cdots,s_n\in N(x)} x(s_1) \cdots x(s_n)s_1y \cdots s_ny \| \\ &= \sum_{s_1,\cdots,s_n\in N(x)} | x(s_1) \cdots x(s_n) | \| s_1 y_1 \cdots s_n y \|. \end{aligned}$$

The last equality derives from the preceding observation and the fact that for $(s_1, \cdots, s_n)$ and $(s_1', \cdots, s_n')$ in $(N(x))^n$ and

$$(s_1, \cdots, s_n) \neq (s_1', \cdots, s_n'), \qquad N(s_1 y \cdots s_n y) \cap N(s_1' y \cdots s_n' y) = \emptyset.$$

Also,

$$\begin{aligned} \| s_1 y \cdots s_n y \| &= \| \sum_{t_1,\cdots,t_n\in N(y)} y(t_1) \cdots y(t_n)s_1 t_1 \cdots s_n t_n \| \\ &= \sum_{t_1,\cdots,t_n\in N(y)} | y(t_1) | \cdots | y(t_n) | \\ &= \| y \|^n. \end{aligned}$$

Hence

$$\| (xy)^n \| = (\textstyle\sum_{s_1,\cdots,s_n \epsilon N(x)} | x(s_1) | \cdots | x(s_n) |) \| y \|^n = \| x \|^n \| y \|^n.$$

Therefore

$$\nu(xy) = \inf_n \| (xy)^n \|^{1/n} = \| x \| \| y \|.$$

If H is a subgroup of G then there is a canonical embedding of $l_1(H)$ into $l_1(G)$. It is well known that if $x \epsilon l_1(H)$ then

$$\mathrm{sp}_{l_1(H)}(x) = \mathrm{sp}_{l_1(G)}(x)$$

THEOREM 3. *If G contains a free non-abelian subsemigroup generated by a and b then $l_1(G)$ is non-symmetric.*

Proof. We proceed by contradiction. Suppose $l_1(G)$ is symmetric. Then by Lemma 1,

$$\nu(xy) \leq \nu(x)\nu(y)$$

for each pair of normal elements x and y in $l_1(G)$.

Let α, β, λ be complex numbers such that

$$| \alpha | = | \beta | = | \lambda | = \tfrac{1}{3} \quad \text{and} \quad \sup_{|z|=1} | \alpha + \beta z + \lambda z^2 | < 1.$$

Define x and y in $l_1(G)$ by $x = \alpha a + \beta a^2 + \lambda a^3$ and $y = \mathrm{b}$. Then x and y are normal. Further

$$\nu(x) = \nu_{l_1(\langle a \rangle)}(x) = \sup_{|z|=1} | \alpha z + \beta z^2 + \lambda z^3 | < 1$$

and $\nu(y) = 1$. By Lemma 1, $\nu(xy) < 1$. However, since the semigroup generated by a and b is free in G, x and y satisfy the hypothesus of Lemma 2, and hence

$$\nu(xy) = \| x \| \| y \| = 1.$$

If G is the group defined by setting

$$(a, b)(a', b') = (aa', ab' + b)$$

for each (a, b) and (a', b') in $\{(a, b) \mid a \epsilon R, b \epsilon R, a \neq 0\}$ then the subsemigroup generated by $c = (a, 1)$ and $d = (-a, 1)$. $0 < a < \frac{1}{2}$, is free. Also $cd^{-1} = dc^{-1}$. (See [1], for this example.) By the preceding theorem, $l_1(G)$ is not symmetric, but a more direct proof is in the following observations.

If $l_1(G)$ is symmetric then each maximal commutative self-adjoint subalgebra, $\mathfrak{M}$, of $l_1(G)$ is also symmetric. Now $\mathfrak{M}$ is symmetric if and only if

$$\nu(x)^2 = \nu(xx^*)$$

for each x in $\mathfrak{M}$ (cf. Rickart [4]).

If we let $x = c + id$ then x is normal and hence an element of some $\mathfrak{M}$. Also, since the semigroup on c and d is free, $\| x^n \| =$ cardinality of $N(x^n) = 2^n$. Hence, $\nu(x) = 2$. However,

$$\nu(xx^*) = \nu(2e) = 2.$$

Therefore,

$$\nu(xx^*) < \nu(x)^2$$

and thus $l_1(G)$ is non-symmetric.

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