IMAGES OF BILINEAR SYMMETRIC AND SKEW-SYMMETRIC FUNCTIONS[1]

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1. Introduction

Let U, V and W be vector spaces over a field F and let $\varphi : U \times V \to W$ be a bilinear function. We define the *image* of φ to be the set of all vectors in W of the form $\varphi(x, y)$, $x \in U$, $y \in V$ and denote it by $\operatorname{Im} \varphi$. It is not generally the case that $\operatorname{Im} \varphi$ is a subspace of W. In the paper [2] the following result is proved by the first author.

THEOREM 1. *Let V_1 and V_2 be vector spaces of dimensions n_1 and n_2 respectively, $n_1 \leq n_2$. If φ is a bilinear function on $V_1 \times V_2$ such that $\operatorname{Im} \varphi$ is a vector space then*

$$\dim (\operatorname{Im} \varphi) \leq n_1(n_2 - 1) - [\tfrac{1}{2} - \sqrt{(n_1 + 5/4)}]$$

where $[x]$ denotes the greatest integer function.

In this paper we consider this problem for bilinear symmetric and skew-symmetric functions. The main results follow.

THEOREM 2. *Let F be an algebraically closed field of characteristic 0 and let V be an n-dimensional vector space over F. If φ is a bilinear symmetric function defined on $V \times V$ such that $\operatorname{Im} \varphi$ is a vector space U then*

$$\dim (U) \leq n(n + 1)/2 - [\tfrac{1}{2}(n + 1 - \sqrt{(n + 3)})]. \tag{1}$$

THEOREM 3. *Let φ be a bilinear skew-symmetric function defined on $V \times V$, where V is an n-dimensional vector space over a field F of characteristic 0. If $\operatorname{Im} \varphi$ is a vector space then*

(i) $\operatorname{Im} \varphi = \{0\}$ *if* $n = 1$, *and*

(ii)

$$\dim (\operatorname{Im} \varphi) \leq n(n - 1)/2 - [\tfrac{1}{2}(n - \sqrt{(n + 2)})] \quad \text{if} \quad n \geq 2. \tag{2}$$

Some examples follow that show that if φ is a bilinear, symmetric or skew-symmetric function then the image of φ may or may not be a vector space.

Example 1. Let U and V be vector spaces over a field F and let $T : V \to U$ be a linear transformation. Let $f \in V^*$ be a non-zero linear functional. Define $\varphi : V \times V \to U$ by

$$\varphi(x, y) = f(x)Ty + f(y)Tx, \quad x, y \in V.$$

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It is obvious that φ is a bilinear, symmetric function and that $\operatorname{Im}\varphi = \operatorname{Im} T$, a subspace of U.

Example 2. Let U, V and f be as in Example 1. Define a bilinear skew-symmetric function $\varphi : V \times V \to U$ by

$$\varphi(x, y) = f(x)Ty - f(y)Tx, \quad x, y \in V.$$

Then $\operatorname{Im}\varphi$ is a subspace of U. Since $\varphi(x, y) = T(f(x)y - f(y)x)$ it suffices to show that the set

$$W = \{f(x)y - f(y)x : x, y \in V\}$$

is a subspace of V. Since $f \neq 0$ extend it to a basis $f, f_2, \cdots, f_n$ of V^* which is dual to some basis $e_1, \cdots, e_n$ of V. Let $x = \sum_{i=1}^n a_i e_i$ and $y = \sum_{i=1}^n b_i e_i$. Then

$$f(x)y - f(y)x = \sum_{i=2}^n (a_1 b_i - a_i b_1)e_i \in \langle e_2, \cdots, e_n\rangle,$$

the subspace spanned by $e_2, \cdots, e_n$. Conversely if $z = \sum_{i=2}^n c_i e_i$ then $z = f(e_1)z - f(z)e_1$ and hence $W = \langle e_2, \cdots, e_n\rangle$.

Example 3. Let $v_1, \cdots, v_n$ be a basis of a vector space V over F, $n > 2$ and let $M_n(F)$ be the space of n-square matrices over F. Define a bilinear symmetric function $\varphi : V \times V \to M_n(F)$ by

$$\varphi(x, y) = \tfrac{1}{2}([a_i b_j] + [a_i b_j]^T), \tag{3}$$

where $x = \sum_{t=1}^n a_t v_t$, $y = \sum_{t=1}^n b_t v_t$, $[a_i b_j]$ denotes the matrix whose (i, j) entry is $a_i b_j$ and the superscript T denotes the transpose. We observe that if $A \in \operatorname{Im}\varphi$ then rank $(A) \leq 2$. Let E_{ij} denote the n-square matrix with 1 in the position (i, j) and 0 elsewhere. Then $B = \frac{1}{2}(E_{12} + E_{21}) = \varphi(v_1, v_2)$ and $C = E_{33} = \varphi(v_3, v_3)$ but rank $(B + C) = 3$ and hence $\operatorname{Im}\varphi$ is not a subspace of $M_n(F)$.

Example 4. Let $n = 4$ in Example 3 and let $U = \bigwedge^2 V$, the second Grassmann space over V. Define a bilinear skew-symmetric function $\varphi : V \times V \to U$ by $\varphi(x, y) = x \wedge y$. It is easily seen that there do not exist x and y in V such that $\varphi(x, y) = v_1 \wedge v_2 + v_3 \wedge v_4$. Thus $\operatorname{Im}\varphi$ is not a vector space.

2. Proofs

We first consider certain subspaces of the m^{th} completely symmetric space $V^{(m)}$ [1, Ch. VII, §1] and the m^{th} Grassmann space $\bigwedge^m V$ over V. We denote the symmetric product of two vectors x and y by $x \cdot y$ and their Grassmann product by $x \wedge y$. We say that $z \in V^{(m)}$ has *symmetric length* k and write $\tau(z) = k$ if z is a sum of k *decomposable* elements (i.e., elements of the form $v_1 \cdot \cdots \cdot v_m$) but no fewer. We define $\tau(0) = 0$. If $z_1, \cdots, z_r$ are arbitrary elements of $V^{(m)}$ then it is obvious that

$$\tau\left(\sum_{i=1}^r c_i z_i\right) \leq \sum_{i=1}^r \tau(z_i), \tag{4}$$

for any scalars $c_1, \cdots, c_r$.

We define the *skew length*, $\mu(z)$, for $z \in \wedge^m V$, in a similar way. An inequality similar to (4) also holds for μ.

LEMMA 1. *If $\varphi : \times_1^m V \rightarrow U$ is a symmetric (skew-symmetric) multilinear onto mapping then there exists a subspace K of the m^{th} completely symmetric space $V^{(m)}$ (m^{th} Grassmann space $\wedge^m V$) such that each non-zero coset in the quotient space $V^{(m)}/K$ ($\wedge^m V/K$) contains a non-zero decomposable element. Conversely if K is a subspace of $V^{(m)}$ ($\wedge^m V$) such that each non-zero coset in $V^{(m)}/K$ ($\wedge^m V/K$) contains a non-zero decomposable element then there exists a multilinear symmetric (skew-symmetric) mapping φ defined on $\times_1^m V$ such that the image of φ is a vector space.*

The proof of the above lemma is analogous to that of Lemma 1 in [2] and is omitted. In view of this lemma the problem of finding a necessary and sufficient condition in order that the image of a symmetric (skew-symmetric) multilinear function φ be a vector space is reduced to investigating those subspaces K of $V^{(m)}$ ($\wedge^m V$) which have the property that a system of distinct representatives for the non-zero cosets in $V^{(m)}/K$ ($\wedge^m V/K$) can be chosen from the non-zero decomposable elements in $V^{(m)}$ ($\wedge^m V$).

The proof of the following lemma is analogous to that of Lemma 2 in [2] and is also omitted.

LEMMA 2. *Let K be a subspace of $V^{(m)}$ (of $\wedge^m V$), $\dim K = p$, such that the cosets in $V^{(m)}/K$ ($\wedge^m V/K$) can be represented by nonzero decomposable elements. Then given any $p + 1$ elements of $V^{(m)}$ ($\wedge^m V$) there exists a non-trivial linear combination of these of symmetric (skew) length at most $p + 1$.*

Now let $v_1, \cdots, v_n$ be a basis of a vector space V over a field F and let $\mathcal{S}_n(F)$ and $\mathfrak{I}_n(F)$ denote the spaces of all $n \times n$ symmetric and skew-symmetric matrices respectively over F. Define $\varphi : V \times V \rightarrow \mathcal{S}_n(F)$ as in (3) and define $f : V \times V \rightarrow \mathfrak{I}_n(F)$ by

$$(5) \qquad f(x, y) = \tfrac{1}{2}([a_i b_j] - [a_i b_j]^T),$$

where $x = \sum_{t=1}^n a_t v_t$ and $y = \sum_{t=1}^n b_t v_t$. It is routine to verify that $(\mathcal{S}_n(F), \varphi)$ is a second completely symmetric space and $(\mathfrak{I}_n(F), f)$ is a second Grassmann space over V. Since any two m^{th} completely symmetric (Grassmann) spaces over V are canonically isomorphic we can regard a matrix in $\mathcal{S}_n(F)$ ($\mathfrak{I}_n(F)$) to be an element of $V^{(2)}$ ($\wedge^2 V$). The following lemma gives a relationship between the rank of a symmetric matrix and its symmetric (skew) length.

LEMMA 3. (i) *Let A be an n-square symmetric matrix over an algebraically closed field F of characteristic zero. Then*

$$\tau(A) = [\tfrac{1}{2}(\text{rank } (A) + 1)].$$

(ii) *Let B be an n-square skew-symmetric matrix over a field F of character-*

istic zero. Then

$$\mu(B) = \tfrac{1}{2} \operatorname{rank}(B).$$

Proof. It is well known that A is congruent to

$$D = \operatorname{diag}(I_{2p}, \varepsilon, O_{n-2p-1})$$

where ε is 0 or 1 and B is congruent to

$$E = \operatorname{diag}(J, \cdots, J, O_{n-2k}),$$

where $J = \operatorname{antidiag}(1, -1)$. It is easily verified that $\tau(A) = \tau(D)$ and $\mu(B) = \mu(E)$. Since $x \cdot y$ and $x \wedge y$ have rank at most 2 we have

$$\operatorname{rank}(A) \leq 2\tau(A) \quad \text{and} \quad \operatorname{rank}(B) \leq 2\mu(B).$$

We note that

$$\operatorname{diag}(I_2 + O_{n-2}) = (v_1 + iv_2) \cdot (v_1 - iv_2)$$

and

$$\operatorname{diag}(J + O_{n-2}) = (v_1 + v_2) \wedge (-v_1 + v_2),$$

where $i = \sqrt{(-1)}$. This leads us to define

$$x_t = v_{2t-1} + iv_{2t}, \qquad y_t = v_{2t-1} - iv_{2t},$$
$$u_t = v_{2t-1} + v_{2t} \quad \text{and} \quad w_t = -v_{2t-1} + v_{2t}.$$

Then $D = \sum_{t=1}^{p} x_t \cdot y_t + \varepsilon v_{2p+1} \cdot v_{2p+1}$ and $E = \sum_{t=1}^{k} u_t \wedge w_t$. Thus it follows that if $\varepsilon = 0$ then

$$\tau(A) = \tau(D) \leq \tfrac{1}{2} \operatorname{rank}(A) \leq \tau(A)$$

and if $\varepsilon = 1$ then

$$\tau(A) = \tau(D) \leq \tfrac{1}{2}(\operatorname{rank}(A) + 1) \leq \tau(A) + \tfrac{1}{2}.$$

Also $\mu(B) = \mu(E) \leq \frac{1}{2} \operatorname{rank}(B) \leq \mu(B)$. These inequalities prove the lemma.

LEMMA 4. *Let V be a vector space over a field F of characteristic 0, $\dim V = n \geq 3$. Let k be any positive integer satisfying $1 < 2k + 1 \leq n$. Then there exists a subspace W of $V^{(2)}$ such that*

$$\dim W = \tfrac{1}{2}(n - 2k)(n - 2k + 1)$$

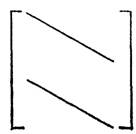
and every non-zero element of W has symmetric length at least $k + 1$.

Proof. Let p be an integer $1 \leq p < n$. For an integer r, $p + 1 \leq r \leq n$, consider the r-tuples

$$\beta_i = (1, 2^{i-1}, 3^{i-1}, \cdots, r^{i-1}), \quad i = 1, \cdots, r - p. \tag{6}$$

Any non-trivial linear combination of the vectors (6) must have at least $p + 1$ non-zero entries. For, suppose that the components $j_1, \cdots, j_{r-p}$ of $\sum_{j=1}^{r-p} d_j \beta_j$ are 0, i.e., $\sum_{i=1}^{r-p} d_i j_t^{i-1} = 0$, $t = 1, \cdots, r - p$. But the $(r - p)$-square matrix $[j_t^{i-1}]$, $i = 1, \cdots, r - p$, $t = 1, \cdots, r - p$, is a Vandermonde and hence is non-singular. Thus $d_i = 0$, $i = 1, \cdots, r - p$.

For a fixed r, $p < r \leq n$ construct $r - p$ matrices by inserting the vectors $\beta_1, \cdots, \beta_{r-p}$ along the partial diagonals of length r indicated in the diagram below:



$\in \mathcal{S}_n(F)$.

The remaining entries of the above matrix are taken to be 0. For

$$r = p + t \leq n$$

we have t such matrices. Hence the total number of such matrices is

$$1 + 2 + \cdots + n - p = \tfrac{1}{2}(n - p)(n - p + 1).$$

These symmetric matrices are obviously linearly independent. Let W be the subspace of $\mathcal{S}_n(F)$ spanned by these matrices. If $A \in W$, $A \neq 0$ then starting from the lower left corner of A there is a first non-zero partial diagonal of length r, say, containing entries $b_1, \cdots, b_r$ such that not all b_i's are 0. But then starting from the upper right corner of A the first non-zero partial diagonal is also $b_1, \cdots, b_r$. These partial diagonals are a non-trivial linear combination of the vectors (6) and hence have at least $p + 1$ non-zero entries. It follows that rank $(A) \geq p + 1$. In particular if $p = 2k$, $1 < 2k + 1 \leq n$, then we have proved the existence of a subspace W of $\mathcal{S}_n(F)$ (and hence of $V^{(2)}$) such that dim $W = \frac{1}{2}(n - 2k)(n - 2k + 1)$, and every non-zero element of W has rank at least $2k + 1$ and hence, by Lemma 3, has symmetric length at least $k + 1$.

LEMMA 5. *Let V be a vector space over F,* dim $V \geq 3$. *Let K be a subspace of $V^{(2)}$ such that every non-zero coset in $V^{(2)}/K$ contains a non-zero decomposable element. Then,* dim $K \geq k_0$, *where k_0 is the largest integer satisfying*

(i) $1 < 2k_0 + 1 \leq n$, *and*
(ii) $\frac{1}{2}(n - 2k_0)(n - 2k_0 + 1) \geq k_0 + 1$.

Proof. Suppose that dim $K = p < k_0$. Then

$$p + 1 < k_0 + 1 \leq \tfrac{1}{2}(n - 2k_0)(n - 2k_0 + 1) = q_0.$$

By Lemma 4 there exists a subspace W of dimension q_0 such that every non-zero element in W has symmetric length at least $k_0 + 1$. Since $p + 1 < q_0$, we can find $p + 1$ linearly independent vectors in W, say $w_1, \cdots, w_{p+1}$. Then

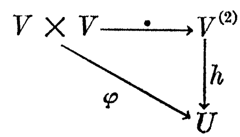
$$\tau\left(\textstyle\sum_{j=1}^{p+1} c_j w_j\right) \geq k_0 + 1, \tag{7}$$

for any choice of scalars $c_1, \cdots, c_{p+1}$ not all of which are 0. On the other hand by Lemma 2 there exists a non-trivial linear combination, $\sum_{j=1}^{p+1} d_j w_j$, such that $\tau(\sum_{j=1}^{p+1} d_j w_j) \leq p + 1 < k_0 + 1$, in contradiction to (7). This completes the proof of the lemma.

It is easily seen that if k_0 is the largest integer satisfying the conditions (i) and (ii) of the preceding lemma then

$$k_0 = [\tfrac{1}{2}(n + 1 - \sqrt{(n + 3)})]. \tag{8}$$

Proof of Theorem 2. From the universal factorization property of the completely symmetric space $V^{(2)}$ we find the unique linear map $h : V^{(2)} \rightarrow U$ such that the diagram



is commutative. We observe that h is onto because φ is onto. Therefore

$$\dim U = \dim (\operatorname{Im} h) = \dim V^{(2)} - \dim (\ker h) = \tfrac{1}{2}n(n + 1) - \dim (\ker h). \tag{9}$$

We notice that for $n = 1$ or 2 the inequality (1) reduces to

$$\dim U \leq \tfrac{1}{2}n(n + 1), \tag{10}$$

which, in view of (9) is obviously true. If $n \geq 3$ then it follows from Lemma 1, Lemma 5 and (8) that $\dim (\ker h) \geq [\frac{1}{2}(n + 1 - \sqrt{(n + 3)})]$ and the result follows from (9).

Remark. If n is 1 or 2 then the inequality (10) cannot be improved. Suppose $n = \dim V = 1$ and $\varphi \neq 0$ then it is easily verified that $\operatorname{Im} \varphi$ is a 1-dimensional vector space and the equality holds in (10). Next let $\{e_1, e_2\}$ be a basis of V. Then $\{e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2\}$ is a basis of $V^{(2)}$. Let

$$\varphi : V \times V \rightarrow V^{(2)}$$

be a symmetric bilinear function defined by $\varphi(x, y) = x \cdot y$, $x, y \in V$. Then it is easily seen that each element of $V^{(2)}$ is decomposable. Hence $\operatorname{Im} \varphi = V^{(2)}$ is a vector space and again the equality holds in (10).

LEMMA 6. *Assume $n \geq 2$ and let p be an odd integer, $1 \leq p < n$. Then there is a subspace W of $\mathfrak{Z}_n(F)$ such that every non-zero matrix in W has rank at least $p + 1$ and* $\dim W = \frac{1}{2}(n - p)(n - p + 1)$.

Proof. For any integer r, $p \leq r \leq n - 1$, consider the r-tuples

$$\beta_i = (1, 2^{i-1}, 3^{i-1}, \cdots, r^{i-1}), \quad i = 1, \cdots, r - p + 1. \tag{11}$$

Then using a similar argument as in the proof of Lemma 4 we conclude that any non-trivial linear combination of the vectors (11) has at least p non-zero entries. For a fixed r, $p \leq r \leq n - 1$, construct $r - p + 1$ matrices in $\mathfrak{Z}_n(F)$ by inserting β_i and $-\beta_i$, $i = 1, \cdots, r - p + 1$ along the partial diagonals of length r as shown in the diagram below:

$$(12) \qquad \begin{bmatrix} \ddots & \beta_i \\ \ddots & -\beta_i \end{bmatrix}$$

The remaining entries of the matrix (12) are taken to be 0. There are a total of $\frac{1}{2}(n - p)(n - p + 1)$ such matrices and they are linearly independent. Let W be the subspace of $\mathfrak{S}_n(F)$ spanned by them. If $A \in W$, $A \neq 0$ then rank $(A) \geq p$. Since A is skew-symmetric and p is odd we have

$$\text{rank } (A) \geq p + 1.$$

LEMMA 7. *Let V be a vector space over F,* $\dim V = n \geq 2$. *Let k be an integer satisfying* $2 \leq 2k + 2 \leq n$. *Then there is a subspace W of $\bigwedge^2 V$ such that* $\dim W = \frac{1}{2}(n - 2k - 1)(n - 2k)$ *and if $z \in W$, $z \neq 0$ then $\mu(z) \geq k + 1$.*

This is an immediate consequence of Lemmas 3 and 6.

LEMMA 8. *Let V be as in Lemma 7. Let K be a subspace of $\bigwedge^2 V$ such that every non-zero coset in $\bigwedge^2 V/K$ contains a non-zero decomposable element. Then* $\dim K \geq [\frac{1}{2}(n - \sqrt{(n + 2)})]$.

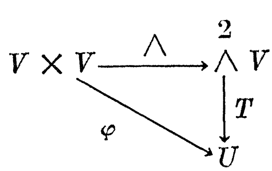
Proof. We assert that $\dim K \geq k_0$, where k_0 is the largest integer satisfying:

(i) $2k_0 + 2 \leq n$, and

(ii) $\frac{1}{2}(n - 2k_0)(n - 2k_0 - 1) \geq k_0 + 1$.

The rest of the argument is analogous to the proof of Lemma 5.

Proof of Theorem 3. If $n = 1$ then it is trivial that $\varphi = 0$ and hence $U = \{0\}$. Next consider the diagram



Since φ is onto U, T is onto U and hence

$$(13) \quad \dim U = \dim (\textstyle\bigwedge^2 V) - \dim (\ker T) = n(n - 1)/2 - \dim (\ker T).$$

Thus for $n \geq 2$ we use (13) and Lemma 8 to obtain the inequality (2).

Remark. If $\dim V$ is 2 or 3 then define φ on $V \times V$ by $\varphi(x, y) = x \wedge y$. Then each element in $\bigwedge^2 V$ is decomposable. Thus Im φ is a vector space and (2) becomes an equality in these cases.

REFERENCES

1. W. H. GREUB, *Multilinear algebra*, Springer-Verlag, New York, 1967.

2. MARVIN MARCUS, "*A dimension inequality for multilinear functions*," in *Inequalities-*

III, Proceedings of the Third Symposium on Inequalities Held at The University of California, Los Angeles, September 1–9, 1969, Academic Press, New York, 1972, pp. 217–224.

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