THE ANALYTIC CONTINUATION OF EULER PRODUCTS WITH APPLICATIONS TO ASYMPTOTIC FORMULAE

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One of the most fruitful methods employed in the study of the distribution of the primes has been the consideration of those functions, such as the Riemann zeta function, which embody the Fundamental Theorem of Arithmetic. This method (sometimes referred to as the "analytic method") has also been successfully applied to the study of generalized prime number systems, an account of which may be found in [1].

Briefly, a *generalized prime number system* is a sequence

$$P = \{1 < p_1 \leq p_2 \leq \cdots\}$$

of real numbers such that $p_k \to \infty$. The multiplicative semigroup generated by P is called the *generalized integers* of the system P, which we denote by

$$N = \{1 = n_1 < n_2 \leq n_3 \leq \cdots\}.$$

Note that two integers n_i and n_j of N, of possibly equal value, are, nevertheless, to be distinguished if they arise as distinct products of the primes P.

The research in generalized number systems has involved considerable use of the zeta functions, $\zeta_P(s)$, defined by either the product or the series

$$\zeta_P(s) = \prod_{k=1}^{\infty} (1 - p_k^{-s})^{-1} = \sum_{j=1}^{\infty} n_j^{-s},$$

wherever the infinite product converges. (It is known that the product and series converge on the same half-plane, possibly empty.) In the present paper we prove that, in certain cases (Theorem 2 and Theorem 3), $\zeta_P(s)$ can be analytically continued across the abscissa of convergence of its defining series. Part of this information is used to evaluate the asymptotic distribution of the integers generated by P. Since asymptotic formulae of this type have already been obtained (with explicit error terms) by a different means in [6], we shall only outline their derivation here. We also prove (Theorem 1) that if a general Dirichlet series with non-negative coefficients has the same general properties as $\zeta(s)$, then it must be $\zeta(s)$. Finally, an example of a generalized prime number system is given for which $p_k \sim k \log k$ and such that the boundary of $\zeta_P(s)$ is the line $\mathrm{Re}(s) = 1$.

THEOREM 1. *Let* $a_j \geq 0, j = 1, 2, \cdots$, *and let*

$$f(s) = \sum_{j=1}^{\infty} a_j n_j^{-s}$$

be a general Dirichlet series which converges at some point of the complex plane.

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Assume that $f(s)$ *can be continued to the complex plane as* $f(s) = F(s)(s-1)^{-1}$, *where* $F(s)$ *is an entire function of finite order such that* $F(0) = \frac{1}{2}$ *and* $F(1) = 1$. *Further, Let*

$$g(s) = \sum_{k=1}^{\infty} b_k \lambda_k^{-s}$$

be a general Dirichlet series that is absolutely convergent for $s = 2$; *and suppose that* $f(s)$ *and* $g(s)$ *are related by the functional equation*

$$\pi^{-s/2}\Gamma(s/2)f(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)g(1-s).$$

Then $f(s) = \zeta(s)$, *the Riemann zeta function.*

Proof. Theorem 1 is similar to a theorem first proved by Hamburger and later simplified by C. L. Siegel in [9]. The essential difference in our two results is that we do not assume that the n_j are integers in Theorem 1. Our proof uses some of the ideas in [9], which we shall only outline. Additional arguments are needed to show that the n_j are all integers, and these will be given in detail. Previous attempts to generalize Hamburger's result in the direction of general Dirichlet series have all imposed some lacunarity condition on the n_j (see [2], [3], [4], [5]).

From the hypotheses, the series for $f(s)$ converges for $\sigma > 1$. This follows by a well known theorem of Landau. The convergence is uniform for

$$\sigma \geq 1 + \delta, \qquad \delta > 0.$$

Following the argument in [10, pp. 31–32], we have for $x > 0$,

$$\begin{aligned}\varphi(x) &= \eta \int_{(2)} (\pi x)^{-s/2}\Gamma(s/2)f(s)\,ds \\ &= \sum_{j=1}^{\infty} a_j\, \eta \int_{(2)} (\pi n_j^2 x)^{-s/2}\Gamma(s/2)\,ds \\ &= 2\sum_{j=1}^{\infty} a_j\, e^{-\pi x n_j^2},\end{aligned}$$

where $\eta = (2\pi i)^{-1}$. Using the functional equation, we can express $\varphi(x)$ in the form

$$\varphi(x) = \eta \int_{(2)} \pi^{-(1-s)/2}\Gamma((1-s)/2)g(1-s)x^{-s/2}\,ds.$$

We now move the line of integration from $\sigma = 2$ to $\sigma = -1$. An application of the Phragmén-Lindelöf principle and Cauchy's theorem shows that $\varphi(x)$ may be evaluated as

$$\varphi(x) = \eta \int_{(-1)} \pi^{-(1-s)/2}\Gamma((1-s)/2)g(1-s)x^{-s/2}\,ds + x^{-1/2} - 1,$$

where the two terms $x^{-1/2}$ and -1 arise from the residues at $s = 1$ and $s = 0$, respectively.

On the line $\sigma = -1$, $g(1-s)$ can be expressed in terms of its Dirichlet series. The integral can be evaluated termwise:

$$\begin{aligned}\varphi(x) &= x^{-1/2}\sum_{k=1}^{\infty} b_k\, \eta \int_{(-1)} (\pi\lambda_k^2/x)^{-(1-s)/2}\Gamma((1-s)/2)\, ds + x^{-1/2} - 1\\ &= 2x^{-1/2}\sum_{k=1}^{\infty} b_k\, e^{-\pi\lambda_k^2/x} + x^{-1/2} - 1.\end{aligned}$$

Equating the two expressions for $\varphi(x)$, we obtain

$$\tfrac{1}{2} + \sum_{j=1}^{\infty} a_j\, e^{-\pi n_j^2 x} = \tfrac{1}{2}x^{-1/2} + x^{-1/2}\sum_{k=1}^{\infty} b_k\, e^{-\pi\lambda_k^2/x}.$$

Now replace x by x^{-1} in the last equation, multiply the resulting equation by $tx^{-1/2}\exp(-\pi t^2 x)$ with $t > 0$, and integrate both sides of the product over $0 < x < \infty$. This yields the equation

$$\text{(1)} \qquad \frac{1}{2} + \sum_{j=1}^{\infty} a_j\, e^{-2\pi n_j t} = \frac{1}{2\pi t} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{b_k\, t}{t^2 + \lambda_k^2}.$$

The argument up to this point is essentially that given in [10]; and if it is assumed that the n_j are integers, this argument can be continued in the usual way to show that $f(s) = \zeta(s)$. We shall now show that, in fact, the n_j must be integers.

Note that equation (1) is valid for complex t with $\mathrm{Re}(t) > 0$, and that the right side of (1) defines a meromorphic function on $\mathbf{C}$ with simple poles only at $t = 0$ and $t = \pm i\lambda_k$, $k = 1, 2, \cdots$. Set $t = \delta + iu$, $\delta > 0$, in equation (1), multiply by $1 - |u|$, and integrate over $-1 < u < 1$. We obtain

$$\begin{aligned}&\frac{1}{2}\int_{-1}^{1} (1 - |u|)\, du + \sum_{j=1}^{\infty} a_j \int_{-1}^{1} (1 - |u|)\, e^{-2\pi n_j(\delta + iu)}\, du\\ &\qquad = \frac{1}{2\pi}\int_{-1}^{1} \frac{1 - |u|}{\delta + iu}\, du + \frac{1}{\pi}\sum_{j=1}^{\infty} b_k \int_{-1}^{1} \frac{(\delta + iu)(1 - |u|)}{\delta^2 - u^2 + 2i\delta u + \lambda_k^2}\, du\end{aligned}$$

which we write compactly in the form

$$A + B(\delta) = C(\delta) + D(\delta).$$

We see immediately that $A = 1/2$ and

$$\text{(2)} \qquad \lim_{\delta\to 0^+} C(\delta) = 1/2.$$

Performing the indicated integrations in $B(\delta)$ gives

$$B(\delta) = \sum_{j=1}^{\infty} a_j\, e^{-2\pi n_j\delta}\, \frac{1 - \cos 2\pi n_j}{2\pi^2 n_j^2}.$$

Therefore,

$$\text{(3)} \qquad \lim_{\delta\to 0^+} B(\delta) = \sum_{j=1}^{\infty} a_j\, \frac{1 - \cos 2\pi n_j}{1\pi^2 n_j^2}$$

by the monotone convergence theorem. Finally, to evaluate $\lim_{\delta\to 0^+} D(\delta)$, note that

$$\frac{1 - |u|}{|\delta^2 - u^2 + 2i\delta u + \lambda_k^2|} \leq 1 \qquad \text{for } 1 \leq \lambda_k \leq 2$$

$$\leq \frac{1}{\lambda_k^2 - 1} \quad \text{for } \lambda_k > 2$$

uniformly in $|u| < 1$ and $0 < \delta < 1$. Hence, Lebesgue's theorem on dominated convergence is applicable, and we have

$$\lim_{\delta \to 0^+} D(\delta) = \frac{1}{\pi} \sum_{k=1}^{\infty} b_k \int_{-1}^{1} \frac{iu(1 - |u|)}{\lambda_k^2 - u^2}\, du = 0 \tag{4}$$

since each integrand is an odd function of u. Putting (2), (3) and (4) together along with $A = 1/2$ yields the equation

$$0 = \sum_{j=1}^{\infty} a_j \frac{1 - \cos 2\pi n_j}{2\pi^2 n_j^2}.$$

Therefore, either $a_j = 0$ or $1 - \cos 2\pi n_j = 0$, $j = 1, 2, \cdots$, and in either case $f(s)$ has an ordinary Dirichlet series.

Following the argument in [10] again, we note that the left side of (1) is invariant under $t \to t + i$. The right side of (1) also shares this property and has the aforementioned meromorphy. Then, if some $\lambda_k \notin Z^+$, the right side of (1) has a pole at $i\lambda_k$ and, by periodicity, at $i(\lambda_k - [\lambda_k])$. But this shows that $\lambda_1 < 1$ which is forbidden behavior for a number in a general Dirichlet series. Therefore, all $\lambda_k \in Z^+$, and $g(s)$ also has an ordinary Dirichlet series. Theorem 1 now follows from that of Hamburger.

Let $U = \{(u_2, u_3, \cdots) : u_p > p^{-1}\}$, where each $u = (u_2, u_3, \cdots)$ in U is a sequence of real numbers, indexed on the rational primes, satisfying the given conditions. For each $u \in U$, define a *perturbed zeta function* $\zeta(s, u)$ by

$$\zeta(s, u) = \prod\nolimits_p (1 - (u_p p)^{-s})^{-1},$$

wherever the product converges. Further, let $\partial(u) = \partial\zeta(s, u)$ denote the natural boundary of $\zeta(s, u)$ in those cases where the product converges in a non-empty half-plane.

Now it is desirable to determine $\partial(u)$ as u varies over U. This has only been accomplished in special cases; and the extent of the difficulties encountered, in general, can be seen in Theorem 2 and Theorem 3. In view of Theorem 1, if $\zeta(s, u)$ converges in a non-empty half-plane and satisfies the other conditions of Theorem 1, then the factors u_p simply permute the primes among themselves.

The hypothesis of the functional equation in Theorem 1 is necessary to the conclusion in order to distinguish $\zeta(s)$ from other general Dirichlet series which can be continued by $F(s)(s - 1)^{-1}$, $F(1) = 1$, $F(s)$ entire. The functional equation may not be necessary in the case of $\zeta(s, u)$, however, due to the fact that $\zeta(s, u)$ has an Euler product representation. Of course it is critical that in the extension of $\zeta(s, u)$ to **C** by means of $\zeta(s, u) = F(s)(s-1)^{-1}$, we require that $F(1) = 1$.

THEOREM 2. *Let* $u = (v, v, \cdots)$, *where* $v > \frac{1}{2}$; *and let* $\mathcal{E}(u)$ *denote the comb-like set*

$$\mathcal{E}(u) = \cup_{\rho, n}\{s : s = (x\beta + i\gamma)/n; 0 < x \leq 1\} \cup (0, 1],$$

where the first union is over all non-real zeros ρ *of* $\zeta(s)$ *and over all positive intergers* n. *Then* $\zeta(s, u)$ *is analytic on* $\mathfrak{D}(u) = \{s : \mathrm{Re}(s) > 0\} - \mathcal{E}(u)$. *Moreover, if*

$$N(u) = \{1 = n_1 < n_2 \leq n_3 \leq \cdots\}$$

denotes the integers generated by $P(u) = \{vp\}$, *then*

$$\sum_{n_j \leq x} 1 = c_1(v)x(\log x)^{v^{-1}-1} + O(x(\log x)^{v^{-1}-3/2}\log\log x),$$

where

$$c_1(v) = \frac{(\zeta(2))^{-1/2v}}{\Gamma(v^{-1})}\prod_{m=2}^{\infty}\left(\frac{\zeta^2(m)}{\zeta(2m)}\right)^{g(m,1,v)},$$

and

$$g(m, s, v) = (1/2m)\sum_{d \mid m, (d,2)=1}\mu(d)v^{-ms/d}.$$

We first prove a lemma.

LEMMA. *Let* c *and* z *be complex numbers satisfying* $|cz| < 1$ *and* $|z| < 1$. *Then there exist complex numbers* $h(m, c)$, $m = 1, 2, \cdots$, *so that the equation*

$$(1 - cz)^{-1} = \prod_{m=1}^{\infty}\left(\frac{1 + z^m}{1 - z^m}\right)^{h(m,c)}$$

holds.

Proof of Lemma. Let c be a complex number, and consider the system of equations

$$\sum_{d \mid n} dh(d)(1 + (-1)^{n/d-1}) = c^n, \quad n = 1, 2, \cdots. \tag{5}$$

It is clear that if n is odd, then

$$2nh(n) = \sum_{d \mid n}\mu(d)c^{n/d}.$$

A simple calculation shows that if n is even, then

$$2nh(n) = \sum_{d \mid n, (d,2)=1}\mu(d)c^{n/d}.$$

Therefore, the last formula holds for all integers n.

Now choose z so that $|z| < 1$ and $|cz| < 1$. Multiply the n-th equation of (5) by z^n/n and sum over all n:

$$\sum_{n=1}^{\infty} c^n\frac{z^n}{n} = \sum_{n=1}^{\infty}\sum_{d \mid n} dh(d)(1 + (-1)^{n/d-1})\frac{z^n}{n}$$
$$= \sum_{d=1}^{\infty} h(d)\sum_{m=1}^{\infty}\frac{z^{dm}}{d}(1 + (-1)^{m-1}),$$

where we have put $n = md$. It follows that

$$-\log(1 - cz) = \textstyle\sum_{d=1}^{\infty} h(d)(\log(1 + z^d) - \log(1 - z^d)),$$

where the logarithms have their principal values. Exponentiating the last expression gives the lemma.

Proof of Theorem 2. With $s > 1$, $v > \frac{1}{2}$, and p a rational prime, we apply the lemma to $(1 - (vp)^{-s})^{-1}$ with $c = v^{-s}$, $z = p^{-s}$, and $h(m, c) = g(m, s, v)$. Thus,

$$(1 - (vp)^{-s})^{-1} = \prod_{m=1}^{\infty} \left(\frac{1 + p^{-ms}}{1 - p^{-ms}}\right)^{g(m,s,v)}.$$

Taking the product over all rational primes yields

$$\zeta(s, u) = \prod_{p} \prod_{m} \left(\frac{1 + p^{-ms}}{1 - p^{-ms}}\right)^{g(m,s,v)} = \prod_{m=1}^{\infty} \left(\frac{\zeta^2(ms)}{\zeta(2ms)}\right)^{g(m,s,v)},$$

where the interchange of limits is justified by the absolute convergence of the double product when $s > 1$. This is the important step to show that $\zeta(s, u)$ can be continued across the line $\mathrm{Re}(s) = 1$. Here we see in this procedure that the use of the product representation for $\zeta(s, u)$ is essential.

Now put

$$H(s) = (\zeta(2s))^{-v^{-s/2}} \prod_{m=2}^{\infty} \left(\frac{\zeta^2(us)}{\zeta(2ms)}\right)^{g(m,s,v)},$$

We specify the branches of $\log \zeta(ms)$, $m = 1, 2, \cdots$, by requiring that their values be real for $s > 1$.

Let Δ be any compact subset of $\mathfrak{D}(u)$. Then $\log \zeta(ms)$ is analytic on Δ for all positive integers m. Furthermore, for $s \in \Delta$, we have the estimates

$$\begin{aligned} |\log H(s)| &= |-\tfrac{1}{2}v^{-2} \log \zeta(2s) + \textstyle\sum_{m=2}^{\infty} g(m, s, v)(2 \log \zeta(ms) - \log \zeta(2ms))| \\ &\leq |\tfrac{1}{2}v^{-s} \log \zeta(2s)| \\ &\quad + \textstyle\sum_{m=2}^{\infty} |g(m, s, v)|(|2 \log \zeta(ms)| + |\log \zeta(2ms)|) \\ &\leq K(\Delta) + \textstyle\sum_{m \geq M} |g(m, s, v)|(|2 \log \zeta(ms)| + |\log \zeta(2ms)|), \end{aligned}$$

where M is chosen so that $\mathrm{Re}(ms) \geq 2$ for all $s \in \Delta$, when $m \geq M$. From the estimates

$$|g(m, s, v)| = \frac{1}{2m} \Big| \sum_{d|m, (d,2)=1} \mu(d) v^{-ms/d} \Big| \leq \frac{\sigma_0(m)}{2m} \max(v^{-\sigma}, v^{-m\sigma}),$$

where $\sigma_0(m) = \sum_{d|m} 1$, and from

$$|2 \log \zeta(ms)| + |\log \zeta(2ms)| \ll 2^{-\sigma m}, \quad m \geq M,$$

we see that the series defining $\log H(s)$ is uniformly convergent, and, hence, analytic for $s \in \Delta$. It follows that $\zeta(s, u)$ is analytic on $\mathfrak{D}(u)$, the analytic continuation being given by

$$(\zeta(s))^{-v^{-s}} H(s) = \zeta(s, u). \tag{6}$$

This proves the first part of Theorem 2.

We shall only adumbrate the proof of the asymptotic distribution of the number of $N(u)$, since their distribution has already been obtained in [6].

For large $x > 0$, define functions $S(x)$ and $T(x)$ by

$$S(x) = \sum_{n_j \le x} 1 \quad \text{and} \quad T(x) = \sum_{n_j \le x} (1 - n_j/x).$$

Then for $\tau > 1$, we have the integral representation

$$T(x) = \eta \int_{(\tau)} \zeta(s, u) \frac{x^s}{s(s+1)} \, ds. \tag{7}$$

We now use the representation (6) of $\zeta(s, u)$ to deduce from (7) that

$$T(x) = H(1)\eta \int_{\mathfrak{C}_1} \frac{x^s(s-1)^{-v^{-1}}}{s+1} \, ds + O(x(\log x)^{v^{-1}-2} \log \log x),$$

where $\mathfrak{C}_1$ is a horseshoe-shaped contour running counter clockwise about $s = 1$. We then deduce that

$$S(x) = H(1)\eta \int_{\mathfrak{C}_2} x^s (s-1)^{-v^{-1}} ds + O(x(\log x)^{v^{-1}-2} \log \log x),$$

where $\mathfrak{C}_2$ is a loop running counter clockwise about the line $(-\infty, 1]$. Observing that

$$\eta \int_{\mathfrak{C}_2} x^s (s-1)^{-v^{-1}} ds = x(\log x)^{v^{-1}-1}/\Gamma(v^{-1}),$$

we have the result stated in Theorem 2.

THEOREM 3. *Let q be an odd prime. For each integer k, $1 \le k \le q-1$, choose real numbers $u_k > k^{-1}$. For each prime p, define real numbers u_p by $u_p = u_k$ if $p \equiv k \pmod q$; and let $u = (u_2, u_3, \cdots)$. Then if*

$$\mathcal{E}_q(u) = \bigcup_{\rho(\chi), n} \{s : s = (x\beta(\chi) + i\gamma(\chi))/n;\ 0 < x \le 1\} \cup (0, 1],$$

where the first union is over all positive integers n and over all zeros $\rho(\chi) = \beta(\chi) + i\gamma(\chi)$, $\beta(\chi) > 0$, of all $L(s, \chi) \pmod q$, $\zeta(s, u)$ is analytic on

$$\mathfrak{D}_q(u) = \{s : \mathrm{Re}(s) > 0\} - \mathcal{E}_q(u).$$

Furthermore, the integers $N(u)$ are asymptotically distributed as

$$\sum_{n_j \le x} 1 \sim c(u_1, \cdots, u_{q-1}) x (\log x)^{\beta(u_1, \cdots, u_{q-1})}, \quad x \to \infty,$$

where

$$\beta(u_1, \cdots, u_{q-1}) = (u_1^{-1} + \cdots + u_{q-1}^{-1})/(q-1) - 1,$$

and where $c(u_1, \cdots, u_{q-1})$ is a product of L functions raised to various complex powers.

Proof. Let $(k, q) = 1$, $q =$ prime, and define

$$E = E(l, m, n, s, u_k) = \mu(n)\chi(\bar{k})g(m, s, u_k)/ln(q-1),$$

where $g(m, s, u_k)$ has its usual definition, and where $\bar{k}$ is defined by $k\bar{k} \equiv 1 \pmod{q}$. The argument used in Theorem 2 to continue $\zeta(s, u)$ to $\mathfrak{D}(u)$ can be used, *mutatis mutandis*, to continue $\zeta(s, u)$ to $\mathfrak{D}_q(u)$ in the present case. The expression

$$\zeta(s, u) = \prod_{l,m,n=1} \prod_{\chi(\mathrm{mod}\, q)} \left(\frac{L^2(lmns, \chi^n)}{L(2lmns, \chi^n)}\right)^{F(l,m,n,s,u)} \tag{8}$$

where the innermost product is over all characters modulo q, and where

$$F(l, m, n, s, u) = \sum_{k=1}^{q-1} E(l, m, n, s, u_k),$$

provides the aforementioned analytic continuation. The asymptotic distribution of the numbers of $N(u)$ are calculated in a straightforward manner using standard properties of L-functions, together with the representation of $\zeta(s, u)$ given in (8).

In Theorems 2 and 3 it was shown that certain classes of perturbed zeta functions can be continued across the abscissa of convergence of their defining products. Although a complete characterization of such functions has not been made, the next theorem (Theorem 4) shows that there exist perturbed zeta functions, close (in a sense to be described) to the Riemann zeta function, which have the line $\mathrm{Re}(s) = 1$ as their natural boundary.

THEOREM 4. *There exists an element $u \in U$ such that $u_p \to 1$ as $p \to \infty$ and*

$$\partial(u) = \{s : \mathrm{Re}(s) = 1\}.$$

Proof. The idea of the proof is due to Harold G. Diamond, conveyed to the author in private correspondence. His idea is based on the consideration of some research in [7].

Let $r_1, r_2, \cdots$ be the sequence of positive rational numbers, where we put $r_m = a_m/b_m$; $a_m, b_m \in Z^+$. Consider the sequence of complex numbers $\{(\sigma_m + ir_m)\}_{m=1}^{\infty}$, where $\sigma_m = 1 - 2^{-b_m}$. Clearly, every point of the line $\mathrm{Re}(s) = 1$, $t \geq 0$, is a limit point of this sequence. We shall now construct a generalized prime number system for which $p_k \sim k \log k$, $k \to \infty$, and such that

$$\zeta_p(s) = \prod_{k=1}^{\infty} (1 - p_k^{-s})^{-1}$$

has a fractional order at the points $(\sigma_m + ir_m)$, $(\sigma_m - ir_m)$, $m = 1, 2, \cdots$.

To do this, define a function $h(u)$, $u \geq 2$, by

$$h(u) = \sum_{m=1}^{\infty} (2^{-m}/\log u)(1 - u^{-1+\sigma_m} \cos(r_m \log u)).$$

Clearly, $h(u)$ is continuous for $u \geq 2$. For $x \geq 2$, define

$$H(x) = \int_2^x h(u)\, du.$$

Since $H(x)$ is strictly increasing to ∞ on $[2, \infty)$, a unique sequence of real numbers $2 < p_1 < p_2 < \cdots$ is defined by

$$p_k = H^{-1}(k), \quad k = 1, 2, \cdots . \tag{9}$$

We now show that the p_k defined in (9) satisfy $p_k \sim k \log k$, $k \to \infty$, by showing that

$$H(x) = li(x) + o(li(x)), \quad x \to \infty. \tag{10}$$

We have

$$\begin{aligned} H(x) &= \int_2^x \frac{du}{\log u} - \sum_{m=1}^{\infty} 2^{-m} \int_2^x \frac{u^{-1+\sigma_m} \cos(r_m \log u)}{\log u} du \\ &= li(x) - \sum_{m=1}^{\infty} 2^{-m} \int_{\log 2}^{\log x} \frac{e^{\sigma_m u} \cos(r_m u)}{u} du. \end{aligned}$$

Choose $\varepsilon > 0$, and choose M so that

$$\sum_{m>M} 2^{-m} < \varepsilon.$$

Then

$$\left| \sum_{m>M} 2^{-m} \int_{\log 2}^{\log x} \frac{e^{\delta_m u} \cos(r_m u)}{u} du \right| < \varepsilon \int_{\log 2}^{\log x} \frac{e^u}{u} du < 2\varepsilon li(x)$$

for all sufficiently large x. Also, if $\sigma^* = \max_{1 \le m \le M} \sigma_m$, then

$$\left| \sum_{1 \le m \le M} 2^{-m} \int_{\log 2}^{\log x} \frac{e^{\delta_m u} \cos(r_m u)}{u} du \right| \le x^{\sigma^*} \log \frac{\log x}{\log 2} = o(li(x)),$$

since $\sigma^* < 1$. It follows that (10) holds; and, consequently, $p_k \sim k \log k$, $k \to \infty$.

Define a function $\zeta_p(s)$ by

$$\zeta_p(s) = \prod_{k=1}^{\infty} (1 - p_k^{-s})^{-1},$$

where the p_k are given by (9). For $\operatorname{Re}(s) > 1$,

$$\log \zeta_p(s) = -\sum_{k=1}^{\infty} \log(1 - p_k^{-s}) = \sum_{k=1}^{\infty} p_k^{-s} + \Phi_1(s), \tag{11}$$

where $\Phi_1(s)$ is analytic for $\operatorname{Re}(s) > \frac{1}{2}$. Since $H(x)$ is a strictly increasing, continuously differentiable function of x, so is $H^{-1}(x)$. Thus, we may apply the Euler summation formula to the evaluation of $\sum_{k=1}^{\infty} p_k^{-s}$. We obtain

$$\sum_{k=1}^{\infty} p_k^{-s} = \int_1^{\infty} p_u^{-s} du + \int_1^{\infty} \frac{d}{du}((p_u)^{-s})(u - [u] - \tfrac{1}{2}) du + \tfrac{1}{2} p_1^{-s}, \qquad \operatorname{Re}(s) > 1,$$

where we have put

$$p_u = H^{-1}(u), \quad u \ge 1.$$

Make the substitution $u = H(y)$, and let $c > 2$ be the real number such that

$1 = H(c)$. Then

$$\begin{aligned} &\textstyle\sum_{k=1}^{\infty} p_k^{-s} \\ (12) \quad &= \int_c^\infty y^{-s} H'(y)\,dy + \int_c^\infty (-sy^{-s-1})(H(y) - [H(y)] - \tfrac{1}{2})\,dy \\ &\qquad + \tfrac{1}{2}p_1^{-s}, \\ &= \int_c^\infty y^{-s} H(y)\,dy + \Phi_2(s), \end{aligned}$$

where $\Phi_2(s)$ is analytic for $\mathrm{Re}(s) > 0$.

When $s > 1$, we may evaluate the integral $\int_c^\infty y^{-s}H'(y)\,dy$ termwise, obtaining

$$\begin{aligned} \int_c^\infty y^{-s}H'(y)\,dy &= \int_c^\infty \frac{y^{-s}}{\log y}\,dy - \sum_{m=1}^{\infty} 2^{-m} \int_c^\infty y^{-s-l+\sigma_m} \frac{\cos(r_m \log y)}{\log y}\,dy \\ &= \int_{\log c}^\infty v^{-1} e^{-(s-1)v}\,dv - \sum_{m=1}^{\infty} 2^{-m} \int_{\log c}^\infty e^{-(s\sigma_m)v} \frac{\cos(r_m v)}{v}\,dv \end{aligned}$$

where we have made the substitution $y = e^v$. Thus,

$$\begin{aligned} &\int_c^\infty y^{-s}H'(y)\,dy \\ &= -\log(s-1) + \Phi_3(s) - \textstyle\sum_{m=1}^{\infty} 2^{-m}\,\mathrm{Re}\left\{\displaystyle\int_{\log c}^\infty v^{-1}\right. \\ (13) \quad &\qquad \left. \cdot \exp(-(s - \sigma_m - ir_m)v)\,dv\right\} \\ &= -\log(s-1) + \Phi_3(s) + \textstyle\sum_{m=1}^{\infty} 2^{-m-1} \log((s-\sigma_m)^2 + r_m^2) + \Phi_4(s), \end{aligned}$$

where $\Phi_3(s)$ and $\Phi_4(s)$ are entire functions of s. Combining equations (11), (12), and (13), we have for $s > 1$,

$$\log \zeta_p(s) = -\log(s-1) + \textstyle\sum_{m=1}^{\infty} 2^{-m-1} \log((s-\sigma_m)^2 + r_m^2) + \Phi_5(s),$$

where $\Phi_5(s)$ is analytic for $\mathrm{Re}(s) > \frac{1}{2}$. Hence, for $\mathrm{Re}(s) > 1$,

$$\zeta_p(s) = (s-1)^{-1} \textstyle\prod_{m=1}^{\infty} ((s-\sigma_m)^2 + r_m^2)^{2^{-m-1}} \exp(\Phi_5(s)).$$

But this representation of $\zeta_p(s)$ shows that it cannot be continued across the line $\mathrm{Re}(s) = 1$. This proves Theorem 4.

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