SOME SUBGROUPS OF $\Omega(V)$ GENERATED BY GROUPS OF ROOT TYPE 1

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1. Introduction

The groups of the Lie type of Chevalley, Tits, Steinberg, Suzuki and Ree are generated by one-parameter subgroups, each isomorphic to the additive group of the field F in question. A standard notation for such a one-parameter subgroup is

$$\mathfrak{X}_r = \{x_r(t) \mid r \text{ is a root and } t \in F\}.$$

We say that H is a "group of root type" in G if H is a subgroup of a group G of Lie type such that $\alpha(H) = \mathfrak{X}_r$ for some root r and some automorphism α of G.

We will call a subgroup G_0 of G an "RT group" if G_0 is generated by groups of root type in G. In this paper we will classify certain RT subgroups of $\Omega(V)$, the commutator subgroup of a finite orthogonal group.

Several results on RT groups have already appeared. For example, Jack McLaughlin [4], [5] and Harriet Pollatsek [6] have studied RT subgroups of $SL(V)$, V finite dimensional over a finite field.

The recent work of John Thompson [11] on quadratic pairs is also related to the study of RT groups. Thompson defines a quadratic pair to be a finite non-trivial group G and an $\mathbf{F}_p G$ module M such that G acts faithfully and irreducibly on M and $G = \langle Q \rangle$ where

$$Q = \{g \in G - \{1\} \mid M(g-1)^2 = 0\}.$$

Thompson first proves that if $p \geq 5$ and (G, M) is a quadratic pair then there exist quadratic pairs (G_i, M_i) such that

(1) $M \cong M_1 \otimes \cdots \otimes M_n$
(2) G is a central product of the G_i's
(3) for all i, $G_i/Z(G_i)$ is simple.

Thompson's main result is a classification of the quadratic pairs (G, M) with $p \geq 5$ such that $G/Z(G)$ is simple. Specifically, he shows that $G/Z(G)$ must then be isomorphic to one of the following:

$$(*) \qquad \begin{array}{llll} A_n(q), & {}^2A_n(q), & B_n(q), & C_n(q), \\ D_n(q), & {}^2D_n(q), & {}^3D_4(q), & G_2(q), \\ F_4(q), & E_6(q), & {}^2E_6(q), & E_7(q), \end{array}$$

where $q = p^e$ for some $e > 0$.

Received November 19, 1971.

It turns out that many groups of root type have a natural representation in which the elements of the group are represented as linear maps with quadratic minimal polynomial. For example, the groups of root type in $SL(V)$ mentioned above, and the groups of "root type 1" (which we define below) in $\Omega(V)$ have such a representation.

In fact, for all the quadratic pairs (G, M) for $p \geq 5$, G is an RT group, and this fact arises in Thompson's proof. However, not all RT groups have representations as quadratic pairs. For example, $PSL_2(\mathbf{F}_p)$ is isomorphic to $\Omega(V)$ for V three dimensional over $\mathbf{F}_p$, and can be generated by two groups of "root type 2" (defined below) in $\Omega(V)$. (See [8] for a proof.) However, every group G of a quadratic pair (G, M) for $p \geq 5$ must contain $SL_2(\mathbf{F}_p)$ by Thompson's proof [11]. Thus $PSL_2(\mathbf{F}_p)$ is an RT group, but has no representation as a quadratic pair. Another example is $E_8(q)$ since it is a group of Lie type which does not appear on Thompson's list.

In this paper we classify the subgroups G of $\Omega(V)$ (for V finite dimensional over a finite field F of odd characteristic, $|F| = q$) which are transitive on the singular one dimensional subspaces of V and generated by groups of "root type 1". We find

(1) $G/Z(G) \cong {}^2A_n(q)$, unitary groups,
(2) $G = G/Z(G) \cong G_2(q)$, or
(3) $G = \Omega(V)$.

We recall $\Omega(V)/Z(\Omega(V)) = B_n(q)$, $D_n(q)$ or ${}^2D_n(q)$. Since (G, V) is a quadratic pair by the hypothesis of our theorem, G must be a central product of groups G_i such that $G_i/Z(G_i)$ is on Thompson's list (*). However it is not trivial to determine which of the groups on Thompson's list (*) satisfy our hypothesis. In fact, we do not use Thompson's methods or results at all. Instead, we rely on the geometry of V for a completely independent proof.

In a previous paper [8], the author has determined the subgroups G of $\Omega(V)$ which are transitive on the singular one-dimensional subspaces of V and generated by groups of root type, but not solely by groups of root type 1. These are:

(1) $G \cong A_5$, dim $V = 4$, index 1, $F = \mathbf{F}_3$,
(2) $G \cong G_{960}$, the semi-direct product of an elementary abelian group of order 16 by A_5, $F = \mathbf{F}_3$ and dim $V = 5$,
(3) $G = \Omega(V)$, dim $V = 3$ or 4, index $V = 1$.

(We remark that if the index of V is greater than 1, $\Omega(V)$ is generated solely by groups of root type 1.)

Thus this paper combined with [8] yields a determination of all RT subgroups of $\Omega(V)$ which are transitive on the one-dimensional singular subspaces of V.

2. Terminology and restatement of theorem

Let V be a finite dimensional vector space over a finite field of characteristic not 2. Let B be a symmetric bilinear form on V. B determines a quadratic form Q on V by $B(x, x) = 2Q(x)$. In addition, suppose there is no $x \neq 0$ in V such that $B(x, y) = 0$ for all y in V. Then we say B is nondegenerate on V. In this case, the group of linear transformations on V preserving B is called the orthogonal group (with respect to B), and is denoted $O(V)$. The commutator subgroup of $O(V)$ is denoted $\Omega(V)$.

If for all $x \in S \neq \{0\}$, $S \subseteq V$, $B(x, x) = 0$, we say S is singular. *We remark that the condition $S \neq \{0\}$ is non standard. Our "singular vectors" and "singular subspaces" are always non-zero.* If a vector u is non-zero and not singular, it is non-singular. This is standard. *Further, we use projective terminology. Thus, a one-dimensional subspace is a point and a two dimensional subspace is a line.* Let $\langle \ \rangle$ denote "subspace generated by". Thus $\langle x \rangle$ is the point generated by the vector x.

The set of vectors x such that $B(x, y) = 0$ for all $y \in Y \subseteq V$ is denoted $Y^{\perp}$. If $X \subseteq Y^{\perp}$ we say X is perpendicular to Y. Since B is bilinear, this is equivalent to saying $\langle X \rangle$ is perpendicular to $\langle Y \rangle$. Since B is symmetric, $X \subseteq Y^{\perp}$ implies $Y \subseteq X^{\perp}$.

Now let x be a singular vector (hence non-zero by definition), and let u be in $x^{\perp}$. Define a linear transformation $\rho_{x,u}$ as follows: for $z \in x^{\perp}$, $\rho_{x,u}$ sends z to $z + B(z, u)x$. This transformation preserves B on the $(n - 1)$-dimensional space $x^{\perp}$. (Note that in case $u \in \langle x \rangle$, $\rho_{x,u}$ acts as the identity on $x^{\perp}$.) By Witt's theorem (see, for example, Artin [1, p. 121]), every linear transformation which preserves B on a subspace of V can be extended to a member of $O(V)$. Tamagawa [10] shows that the extension ρ of $\rho_{x,u}$ to a member of $O(V)$ is unique. In fact, if y is a singular vector such that $B(y, u) = 0$ and $B(x, y) = 1$ (when $u \notin \langle x \rangle$, such vectors always exist) then ρ sends y to $y - Q(u)x - u$. We abuse notation by allowing $\rho_{x,u}$ to stand for its extension to a member of $O(V)$. $\rho_{x,u}$ is called a Siegel transformation.

Since, as a direct sonsequence of its definition,

$$\rho_{x,u}\,\rho_{x,v} = \rho_{x,u+v} \tag{1}$$

we see that the set

$$\Sigma = \{\rho_{x,ku} \mid x \text{ singular}, u \in x^{\perp}, u \notin \langle x \rangle;\ x, u \text{ fixed}, k \in F\}$$

is a group isomorphic to the additive group of F. If u is singular we say Σ is a group of root type 1. If u is non-singular we say Σ is a group of root type 2. The groups of root type in $\Omega(V)$ as defined above have this form. One can see this by using the explicit representations for the one-parameter groups of B_n and D_n given in [8], plus the paper of Steinberg [10] defining ${}^2D_n$. There will be no need, however, to refer to the Chevalley structure in the proof of our theorem. Instead, we use only the properties of Siegel transformations.

In order to have a group of root type 1, we must have a singular line $\langle x, u\rangle$. The dimension of a maximal singular subspace of V is called the index of V. Thus we must have index at least 2, and since the index of V is at most half the dimension (see Artin, [1, p. 143–144), the dimension of V is at least 4. In this case $\langle x\rangle^{\perp}$ is always spanned by singular vectors (again see Artin [1, p. 143–144]). Say $\langle x\rangle^{\perp} = \langle y_1, y_2, \cdots y_k\rangle$, where the y_i are singular. Then if $u \in \langle x\rangle^{\perp}$,

$$u = \textstyle\sum_{1\leq i\leq k} a_i y_i, \quad a_i \in F.$$

So

$$\rho_{x,u} = \rho_{x,\Sigma a_i y_i} = \textstyle\prod_{1\leq i\leq k} \rho_{x,a_i y_i}$$

by (1). Thus every Siegel transformation is a finite product of Siegel transformations $\rho_{x,y}$ with singular y. Hence the subgroup of $O(V)$ generated by all groups of root type 1 contains all $\rho_{x,u}$. Tamagawa [10] proves that the $\rho_{x,u}$ generate $\Omega(V)$ and that $\Omega(V)$ is transitive on the singular points of V.

Let $|F| = q$, dimension $V = t$ and index $V = \nu$. We restate our

THEOREM. *Let G be a subgroup of $\Omega(V)$ generated by groups of root type 1 and transitive of the singular points of V. Then either*

(1) $G \cong G_2(q)$, *simple groups discovered by Dickson in* 1901 [2],

(2) $G \cong SU(t/2, q^2)$, *groups of determinant one linear transformations preserving a hermitian form on a vector space of dimension $t/2$ over a field with q^2 elements* (*see, for example*, [3, p. 12]) *and* $t = 4m + 2$, $\nu = 2m$, $m \geq 1$ *or* $t = 4m$, $\nu = 2m$, $m > 1$, *or*

(3) $G = \Omega(V)$.

3. Correspondence between groups of root type 1 and singular lines; axis lines

We first show that groups of root type 1 are in one to one correspondence with the singular lines of V. It is trivial that a group of root type 1 determines a singular line. Let l be a singular line. Define Σl to be $\{\rho_{x,u} \mid l = \langle x, u\rangle\} \cup \{1\}$. In order to prove that Σl is a group of root type 1, we need only show that if $l = \langle x, u\rangle$, $\rho_{x,u} = \rho_{x_0,ku_0}$ for some fixed x_0, u_0 such that $l = \langle x_0, u_0\rangle$ and $k \in F^*$.

To do this, we observe that for any Siegel transformation $\rho_{x,u}$

$$\rho_{x,u+kx} = \rho_{x,u}, \quad k \in F, \tag{2}$$

and

$$\rho_{x,cu} = \rho_{cx,u}, \quad c \in F^*. \tag{3}$$

If, in addition, u is singular, then

$$\rho_{x,u} = \rho_{-u,x}. \tag{4}$$

One need only show that both sides of (2), (3), and (4) agree on $x^{\perp}$, since $\rho_{x,u}$ has a unique extension from its representation on $x^{\perp}$. For example, let us prove (4). If $z \in \langle x, u\rangle^{\perp}$, $\rho_{x,u}(z) = \rho_{-u,x}(z) = z$. There is a singular vector z_1 in $x^{\perp}$ such that $B(z_1, -u) = 1$. (Take some z' in $x^{\perp}$, z' not in $u^{\perp}$, then $z_1 = k(z' + tu)$ for some $t \in F$ and $k \in F^*$). Recall that when Tamagawa proves the uniqueness of the extension of $\rho_{s,t}$ from its action on $s^{\perp}$ to its action on V, he shows that if y is a singular vector such that $B(y, s) = 1$ and $B(y, t) = 0$, then $\rho_{s,t}$ sends y to $y - t - Q(t)s$. In this case our "t" is singular, so $Q(t) = 0$. Thus

$$\rho_{-u,x}(z_1) = z_1 - x = z_1 + B(z_1, u)x = \rho_{x,u}(z_1).$$

Since $x^{\perp} = \langle\langle x, u\rangle^{\perp}, z_1\rangle$, we are done.

Now let $x = ax_0 + bu_0$, $u = a'x_0 + b'u_0$ where $a, b, a', b' \in F$. If $b \neq 0$, we apply (2) to obtain $\rho_{ax_0+bu_0,a'x_0+b'u_0} = \rho_{ax_0+bu_0,tx_0}$ where $t = a' - b'ab^{-1}$. Since $l = \langle x, u\rangle$, $u \notin \langle x\rangle$ and $t \neq 0$. Hence we can apply (4) to get $\rho_{ax_0+bu_0,tx_0} = \rho_{-tx_0,ax_0+bu_0}$. If $b = 0$, one begins with this form. Now apply (2) and (3) for the desired result.

If $\Sigma l \subseteq G$ we say l is an axis line of G. Since our groups G are to be transitive on singular points, and since for $\tau \in O(V)$,

$$\rho^{\tau}_{x,u}(= \tau\rho_{x,u}\tau^{-1}) = \rho_{\tau(x),\tau(u)} \tag{5}$$

(proved by showing both sides agree on $\tau(x)^{\perp}$), each singular point lies on the same geometric configuration of axis lines for G. For example, suppose $\langle x, y\rangle$ and $\langle x, z\rangle$ are axis lines of G and $B(z, y) = k$, $k \in F$. Then for any $\tau \in G$, $\langle\tau(x), \tau(y)\rangle$ and $\langle\tau(x), \tau(z)\rangle$ are axis lines of G and $B(\tau(z), \tau(y)) = k$. In particular, we note that (5) implies that any element τ of a group G sends axis lines of G to axis lines of G.

4. Standard basis notation; The group G generated by Σl and Σl_0 where $l \cap l_0^{\perp} = \{0\}$

We use a standard basis notation:

$$V = \langle x_1, x_{-1}\rangle \oplus \langle x_2, x_{-2}\rangle \oplus \cdots \oplus \langle x_k, x_{-k}\rangle \oplus W$$

where the $x_i \in W^{\perp}$, W containr no singular vectors, $B(x_i, x_{-i}) = 1$, and all other products of the x_i are 0. W must have dimension 0, 1 or 2. Artin [1, p. 143–144] proves that V can always be represented in this way.

Witt's theorem implies that if $U_1, \cdots, U_t$ are non-zero vectors in V with the same multiplication table $(B(U_i, U_j))$ as some subset of a standard basis, the $U_1, \cdots, U_t$ can be extended to a standard basis. We use the notation $\{x_1, x_{-1}, \cdots, x_{-i}\}$ to denote a set of vectors having the same multiplication table as the corresponding vectors of a standard basis.

We begin by investigating the group G generated by Σl and Σl_0 where $l \cap l_0^{\perp} = \{0\}$.

The freedom allowed us by Witt's theorem for using the standard basis notation enables us to represent l by $\langle x_1, x_2\rangle$ and l_0 by $\langle x_{-1}, x_{-2}\rangle$. Then

$\rho_{x_{-1},kx_{-2}}$ sends

$$\langle x_1, x_2\rangle \quad \text{to} \quad \langle x_1 - kx_{-2}, x_2 + kx_{-1}\rangle.$$

Therefore Σl and Σl_0 generate a group G whose axis lines include

$$\langle x_{-1}, x_{-2}\rangle, \quad \text{and} \quad \langle x_1 - kx_{-2}, x_2 + kx_{-1}\rangle$$

for $k \in F$. Further, the sets of singular points

$$O_1 = \{\langle x_{-1}\rangle, \langle x_2 + kx_{-1}\rangle \mid k \in F\}$$

and

$$O_2 = \{\langle x_{-2}\rangle, \langle x_1 + kx_{-2}\rangle \mid k \in F\}$$

are fixed by Σl and Σl_0, hence by G. We show that the axis lines above are the only ones for G. Let $U = \langle l, l_0\rangle$. G fixes $U^{\perp}$ vectorwise since its generators fix $U^{\perp}$ vectorwise. Suppose there is an axis line l of G containing a vector y which does not lie in U, i.e. $y = w + v$ where $w \in U$, $v \in U^{\perp}$ and $v \neq 0$. Since $U^{\perp}$ is non-degenerate [1], there is an x in $U^{\perp}$ such that $B(x, v) \neq 0$. Choose $\langle z\rangle$ to be the point on l in $\langle x\rangle^{\perp}$. (We recall that $x^{\perp}$ has dimension $n - 1$, hence $\dim(x^{\perp} \cap W) = \dim W$ if $W \subseteq x^{\perp}$ or $(\dim W) - 1$ otherwise. In particular $x^{\perp} \cap$ (a line l) has dimension 1 or 2, i.e. for any point $\langle x\rangle$ there is always a (non-zero) point $\langle y\rangle$ on a given line l such that $y \in x^{\perp}$.)

Then $\rho_{z,y}$ does not fix x, contradicting $v \neq 0$. Thus all axis lines of G must lie in U. The only other singular lines in U are $\langle x_{-1}, x_2\rangle$ and $\langle x_1 - kx_2, x_{-2} + kx_{-1}\rangle$ $k \in F$. But no group of root type 1 with these axis lines fixes O_1 and O_2.

Since the axis lines of G do not intersect, there are $(q + 1)^2$ points of U on these axis lines. But this is the number of singular points in U. So each singular point of U is on one axis line of G. If we consider $V = U$ we have exhibited a subgroup of $\Omega(V)$ generated by two groups of root type 1, where each singular point is on exactly one axis line of G, but G is not transitive on the singular points of V.

In fact, as we show in section 7, G is isomorphic to $SU(2, q^2)$, the determinate one unitary group in 2 dimensions over a field of q^2 elements.

5. The lemma on the group generated by Σl_1 and Σl_2 where $l_1 \cap l_2^{\perp}$ is a point

The following lemma is a very important tool in the remainder of this paper.

LEMMA. *If l_1 and l_2 are axis lines for a group G_0 such that $l_1 \cap l_2 = \{0\}$ and the dimension of $l_1 \cap l_2^{\perp}$ is one, i.e., $l_1 \cap l_2^{\perp} = P$, a singular point, then P lies on two independent axis lines of G_0, namely l_1 and $\langle P, R\rangle$, where R is the point on l_2 in $l_1^{\perp}$.*

Proof. Using standard basis notation, let $l_1 = \langle x_1, x_2\rangle$, and let $P = \langle x_2\rangle$. Thus $x_1 \notin l_2^{\perp}$, but $x_2 \in l_2^{\perp}$. Let $\langle y\rangle$ be the point on l_2 in $x_1^{\perp}$. Since all of l_2 is perpendicular to x_2, $y \in l_1^{\perp}$. Thus $\langle y \rangle = R$, and we may name y "x_3". If $\langle x \rangle$ is a point on l_2 different from $\langle x_3 \rangle$, then $x \in \langle x_2, x_3 \rangle^{\perp}$ but $x \notin \langle x_1 \rangle^{\perp}$. Hence we may name $\langle x \rangle$ "$\langle x_{-1} \rangle$". But $\rho_{x_3, x_{-1}}$ sends $\langle x_2, x_1 \rangle$ to $\langle x_2, x_1 + x_3 \rangle$. We note that by (1), $\rho_{x_2, -kx_1} \rho_{x_2, k(x_1+x_3)} = \rho_{x_2, kx_3}$, $k \in F$. Thus $\langle x_2, x_3 \rangle = \langle P, R \rangle$ is an axis line of G_0 as claimed.

Remark. In general, (1) of Section 2 implies that if $\langle x, y \rangle$ and $\langle x, z \rangle$ are axis lines for G, so is $\langle x, y + kz \rangle$ for any $k \in F$. In particular, if $\langle x, y \rangle$ and $\langle x, y + z \rangle$ are axis lines, so is $\langle x, y + z - y \rangle = \langle x, z \rangle$.

6. Elimination of certain possibilities for dimension and index of V under the assumption of one axis line per singular point

In §6 and §7 we assume G satisfies the hypothesis of the theorem and in addition each singular point is on *exactly* one axis line. The lemma of §5 implies that under this assumption, if l_1 and l_2 are axis lines of G, $l_1 \cap l_2^{\perp} = l_1$ or $\langle 0 \rangle$. In particular, if l_1 is an axis line of G, and P is a point in $l_1^{\perp}$, then the axis line l of G containing P must lie in $l_1^{\perp}$. (Otherwise $l \cap l_1^{\perp} = P$.) Further, if R is a point not in $l_1^{\perp}$ and Q is the point on l_1 in $R^{\perp}$, then if the axis line l_2 containing R is $\langle R, T \rangle$, $T \notin Q^{\perp}$. (Otherwise $l_2 \cap l_1^{\perp} = Q$). This information can be extensively exploited.

Suppose $l_0 = \langle x_1, x_2 \rangle$ is an axis line for G. Since B is non-degenerate, there is a point $\langle x_{-1} \rangle$, $x_{-1} \notin x_1^{\perp}$, $x_{-1} \in x_2^{\perp}$. Let $l = \langle x_{-1}, y \rangle$ be the axis line containing $\langle x_{-1} \rangle$, and choose $\langle y \rangle$ to be the point on l in $x_1^{\perp}$. By the above argument, y must not lie in $x_2^{\perp}$. Hence we may name y "x_{-2}".

In §4, we saw that the group H generated by Σl and Σl_0 is not transitive on the singular points of $U = \langle l, l_0 \rangle$, but that every singular point of U is on an axis line of H. So if H is a subgroup of H_0, a group generated by groups of root type 1 in 4-dimensional space (index two), then either $H = H_0$ and H_0 is not transitive on singular points, or H is a proper subgroup of H_0 and there are axis lines of H_0 which are not axis lines of H. Thus H_0 would have singular points on more than one axis line. Hence for any group satisfying the hypothesis of this section, the dimension of V is greater than 4.

Now we proceed by induction. Suppose we have a system S_k of axis lines

$$\langle x_1, x_2 \rangle, \ \langle x_{-1}, x_{-2} \rangle, \ \cdots, \ \langle x_{2k-1}, x_{2k} \rangle, \ \langle x_{-(2k-1)}\, x_{-2k} \rangle.$$

Let $U = \langle S_k \rangle$. Let $\langle y \rangle$ be a singular point in $U^{\perp}$. Then the axis line l containing $\langle y \rangle$ is perpendicular to each line in S_k, so $l \subseteq U^{\perp}$. Using Witt's theorem, we name l "$\langle x_{2k+1}, x_{2k+2} \rangle$". Then by Witt's theorem, since B is non-degenerate, there is a point we may call $\langle x_{-(2k+2)} \rangle$ which is perpendicular to U and to $\langle x_{2k+1} \rangle$ but not to $\langle x_{2k+2} \rangle$. Then if $l_0 = \langle x_{-(2k+2)}, y \rangle$ is the axis line of G containing $x_{-(2k+2)}$, where $\langle y \rangle$ is chosen to be in $x_{2k+2}^{\perp}$, we see $\langle y \rangle$ must be in U and in $x_{-(2k+2)}^{\perp}$ and $\langle y \rangle$ must not be in $x_{2k+1}^{\perp}$. Hence we may name $\langle y \rangle$ "$\langle x_{-(2k+1)} \rangle$". Thus we obtain a system S_{k+1}. Since by Witt's theorem, every singular space is contained in a maximal singular space of dimension the index of V, we see that the index of V cannot be odd.

Now suppose we have even index and have a system S_k of axis lines as above. Suppose the dimension of V is $4k+1$ (i.e., the dimension of the W of standard basis notation is one). Let $W = \langle z\rangle$ where $Q(z) = g \in F^*$. Then

$$\langle x_1 - gx_{-1} + z\rangle = \langle y\rangle$$

is a singular point perpendicular to every axis line in S_k except $\langle x_1, x_2\rangle$ and $\langle x_{-1}, x_{-2}\rangle$. Hence the axis line l containing $\langle y\rangle$ must lie in $\langle x_1, x_2, x_{-1}, x_{-2}, z\rangle$. Let $\langle x\rangle$ be the point on l in $z^\perp$. Then $\langle x\rangle$ is in $\langle x_1, x_2, x_{-1}, x_{-2}\rangle$. But by §4, we know every singular point in this space is already on an axis line in the group generated by $\Sigma\langle x_1, x_2\rangle$ and $\Sigma\langle x_{-1}, x_{-2}\rangle$. Hence the dimension of V cannot be $4k+1$.

7. The existence of a group with each singular point on one axis line

The only remaining possibilities are $\dim V = 4k$, index $2k$, $k > 1$; or $\dim V = 4k+2$, index $2k$, $k \geq 1$. We show that in these cases a group G satisfying the hypothesis of the theorem and with each singular point on one axis line exists.

We have a system S_k of axis lines as above. Suppose $\dim V \geq 8$. Let $\langle x_2 + x_4, y\rangle$ be an axis line. Let $\langle y\rangle$ be the point on $\langle x_2 + x_4, y\rangle$ in $x_{-2}^\perp$. Then since $x_2 + x_4$ is in

$$\langle x_1, x_2\rangle^\perp, \ \langle x_3, x_4\rangle^\perp, \ \langle x_5, x_6\rangle^\perp, \ \cdots, \ \langle x_{-(2k-1)}, x_{-2k}\rangle^\perp$$

so is y, i.e. $y \in \langle x_1, x_2, x_3, x_4, W\rangle$ where W has dimension 0 or 2 and contains no singular vectors. Since y must be singular,

$$y \in \langle x_1, x_2, x_3, x_4\rangle.$$

Since $y \in x_{-2}^\perp$, $y \in \langle x_1, x_3, x_4\rangle$. Since

$$\langle x_2 + x_4, y\rangle \notin \langle x_{-1}, x_{-2}\rangle^\perp \quad \text{and} \quad x_{-1} \in \langle x_2 + x_4\rangle^\perp,$$

$x_{-1} \notin y^\perp$, i.e. y has an x_1 component. Similarly, y has an x_3 component. Hence, $\langle y\rangle = \langle x_1 + ax_3 + bx_4\rangle$ where $a \in F^*$ and $b \in F$.

Rewrite $ax_3 + bx_4$ as "x_3'". Then $\langle x_3, x_4\rangle = \langle x_3', x_4\rangle$ and if

$$x_{-4}' = -a^{-1}bx_{-3} + x_{-4} \ (a \neq 0) \quad \text{and} \quad x_{-3}' = a^{-1}x_{-3},$$

then $\langle x_{-3}, x_{-4}\rangle = \langle x_{-3}', x_{-4}'\rangle$ and the relationships among x_3, x_4, x_{-3} and x_{-4} are exactly those among x_3', x_4, x_{-3}' and x_{-4}'. So without loss of generality we may assume $\langle x_2 + x_4, x_1 + x_3\rangle$ is an axis line for G. But we cannot juggle things this way twice.

We may suppose $\langle x_2 + x_3, x_1 + ax_3 + bx_4\rangle$ is an axis line for G. (This time $b \neq 0$.) The product $\rho_{x_{-3}, sx_{-4}}\, \rho_{x_3, tx_4}\, \rho_{x_{-3}, kx_{-4}} = \tau$ where $s = -k/(1-tk)$ sends x_3 to $(1-tk)x_3$ and x_4 to $(1-tk)x_4$. (Assume $1 - tk \neq 0$.) Let

$1 - tk = r$ and apply τ to $\langle x_2 + x_4, x_1 + x_3 \rangle$ to obtain

$$\langle x_2 + rx_4, x_1 + rx_3 \rangle = l_1.$$

Similarly if $1 - tk = j$, τ sends

$$\langle x_2 + x_3, x_1 + ax_3 + bx_4 \rangle \quad \text{to} \quad \langle x_2 + jx_3, x_1 + ajx_3 + bjx_4 \rangle = l_2.$$

On l_1 there is a point

$$\langle x_2 + mx_1 + rx_4 + mrx_3 \rangle.$$

On l_2 there is a point

$$\langle x_2 + mx_1 + jx_3 + majx_3 + mbjx_4 \rangle.$$

Let $mbj = r$ and $j + maj = mr$. Eliminating m in these two equations we obtain $b + (r/j)a = (r/j)^2$. Thus if $b + Xa = X^2$ has a solution in F, then these two distinct axis lines, l_1 and l_2, contain a common point. Hence, $X^2 - aX - b$ is irreducible in F.

Now suppose $\dim V \geq 12$. As above, we get axis lines

$$\langle x_2 + x_6, x_1 + x_5 \rangle \quad \text{and} \quad \langle x_1 + a'x_5 + b'x_6, x_2 + x_5 \rangle$$

where $X^2 - a'X - b' = 0$ is irreducible in F. We show $a' = a$ and $b' = b$. First, note $\rho_{x_{-2}, x_{-1}}$ sends

$$\langle x_1 + x_3, x_2 + x_4 \rangle \quad \text{to} \quad \langle x_1 + x_{-2} + x_3, x_2 - x_{-1} + x_4 \rangle$$

and sends

$$\langle x_1 + ax_3 + bx_4, x_2 + x_3 \rangle \quad \text{to} \quad \langle x_1 + x_{-2} + ax_3 + bx_4, x_2 - x_{-1} + x_3 \rangle.$$

Then the product $\rho_{x_{-4}, x_{-3}} \rho_{x_{-5}, x_{-6}} \rho_{x_1 + x_{-2} + x_3, x_2 - x_{-1} + x_4}$ sends

$$\langle x_2 + x_6, x_1 + x_5 \rangle$$

to

$$l = \langle x_{-1} - x_4 + x_{-3} + x_6 + x_{-5}, -x_{-2} - x_3 - x_{-4} + x_5 - x_{-6} \rangle$$

and $\rho_{x_1 + x_{-2} + ax_3 + bx_4, x_2 - x_{-1} + x_3}$ sends

$$\langle x_2 + x_5, x_1 + a'x_5 + b'x_6 \rangle$$

to

$$\langle x_{-1} - x_3 + x_5, - x_{-2} - ax_3 - bx_4 + a'x_5 + b'x_6 \rangle = l_1.$$

Since $\langle x_{-1} - x_3 + x_5 \rangle$ is in $l^{\perp}$, the second vector of l_1 must also be in $l^{\perp}$. So $a = a'$ and $b = b'$.

Suppose $\dim V = 4n + 2$. Hence $\dim W = 2$ and $W = \langle w_1, w_2 \rangle$ and we may set $Q(w_1) = 1$, $Q(w_2) = g$ where $-g$ is a non-square and $B(w_1, w_2) = 0$ [1]. The singular point $\langle x_2 - x_{-2} + w_1 \rangle$ lies on an axis line l with a point $\langle y \rangle$. Say $\langle y \rangle$ is the point on l in $w_1^{\perp}$. Then

$$\langle y \rangle = \langle ex_2 + ex_{-2} + cx_1 + dx_{-1} + w_2 \rangle$$

where $c\, d \neq 0$ since $c^2 + c\, d = -g$, a non-square. Let y be that member of

$\langle y\rangle$ whose x_1 coefficient is 1, i.e.

$$y = ec^{-1}x_2 + ec^{-1}x_{-2} + x_1 + dc^{-1}x_{-1} + c^{-1}w_2 .$$

Let $e' = ec^{-1}$, $d' = dc^{-1}$, $w_2' = c^{-1}w_2$, and $Q(w_2') = c^{-2}g$. Let $c^{-2}g = g'$. Then if $\dim V \geq 10$, $e' = -a/2$ and $d' = b$. (Hence $-g' = a^2/4 + b$.)

Proof. The product $\rho_{x_{-3},-d'x_{-4}}\,\rho_{x_2-x_{-2}+w_1,x_1+d'x_{-1}+e'x_{-2}+e'x_2+w_2'}$ sends

$$\langle x_2 + x_4 , x_1 + x_3\rangle$$

to

$$\langle (1 + 2e')x_2 + x_1 + d'x_{-1} + e'w_1 + w_2' + x_4 - d'x_{-3} ,$$
$$x_1 + d'x_2 - d'x_{-2} + d'w_1 + x_3 + d'x_{-4}\rangle = l_1 .$$

Further $\rho_{x_{-1},d'x_{-2}}$ sends

$$\langle x_2 + x_3 , x_1 + ax_3 + bx_4\rangle$$

to

$$\langle x_2 + d'x_{-1} + x_3 , x_1 - d'x_{-2} + ax_3 + bx_4\rangle = l_2 .$$

But $\langle x_2 + d'x_{-1} + x_3\rangle$ is in $l_1^{\perp}$, so the second vector of l_2 must also be in $l_1^{\perp}$. This yields the desired equations.

Suppose now that either $\dim V \geq 8$, defining a and b, or if $\dim V = 6$ and $\langle x_2 - x_{-2} + w_1 , x_1 + d'x_{-1} + e'x_2 + e'x_{-2} + w_2'\rangle$ is an axis line, define a to be $-2e'$ and b to be d'. Then let α be a root of $X^2 - aX - b$. We embed V onto a vector space U of dimension $t/2$ (where $\dim V = t$) over $F(\alpha)$ by the following equations:

$$e_i = x_{2i} , \qquad \alpha e_i = x_{2i-1} ,$$
$$(\alpha - a/2)e_{-i} = gx_{-(2i-1)} \quad \text{where } -g = a^2/4 + b = (\alpha - a/2)^2$$
$$e_{-i} = x_{-2i} + (a/2)x_{-(2i-1)}$$
$$e_0 = w_1 , \qquad (\alpha - a/2)e_0 = w_2' \quad \text{when } \dim V = 4k + 2.$$

Then each of the following lines in V is a "point" in U (i.e. a one-dimensional subspace in the vector space U over $F(\alpha)$):

$\langle x_1 , x_2\rangle, \cdots , \langle x_{2k-1} , x_{2k}\rangle,$
$\langle x_{-1} , x_{-2}\rangle, \cdots , \langle x_{-(2k-1)} , x_{-2k}\rangle,$
$\langle x_1 + x_3 , x_2 + x_4\rangle, \cdots , \langle x_1 + x_{2k-1} , x_2 + x_{2k}\rangle,$
$\langle x_1 + ax_3 + bx_4 , x_2 + x_3\rangle, \cdots , \langle x_1 + ax_{2k-1} + bx_{2k} , x_2 + x_{2k-1}\rangle,$
$\langle x_2 - x_{-2} + w_1 , x_1 + bx_{-1} - (a/2)x_{-2} - (a/2)x_2 + w_2'\rangle.$

Call this system of lines "L".

For the remainder of this section some familiarity with unitary groups is needed and we refer the reader to Dieudonné [3].

Let B' be a map from $U \times U$ to $F(\alpha)$ with the following properties:

(i) $B'(x_1 + x_2 , y) = B'(x_1 , y) + B'(x_2 , y)$,
(ii) $B'(x, y_1 + y_2) = B'(x, y_1) + B'(x, y_2)$,

(iii) $B'(x, ay) = aB'(x, y)$ for $a \in F(\alpha)$,

(iv) $B'(ax, y) = \sigma(a)B'(x, y)$ for $a \in F(\alpha)$ and σ the automorphism of $F(\alpha)$ fixing F and taking α to its conjugate.

It follows that $B'(x, y) = \sigma(B'(y, x))$. B' is called a hermitian form. If, in addition, there is no $x \neq 0$ in U such that $B'(x, y) = 0$ for all y in U, say B' is non-degenerate.

Let B' be a hermitian form on U such that $B'(e_i, e_{-i}) = 1$ for $1 \leq i \leq k$ and $B'(e_0, e_0) = 2$ and let all other products of these basis vectors be zero. Then B' is non-degenerate. We call the group $U(U)$ of linear transformations on U fixing B' a unitary group. The subgroup of $U(U)$ consisting of determinant one transformations is denoted $SU(U)$.

We claim $B'(x, y) = B(x, y) + \beta^{-1}B(x, \beta y)$ where we abuse notation by allowing the same symbols for corresponding vectors in U and in V. We define $\beta = \alpha - a/2$. We repeat some relations defined earlier. Recall α is a zero of $X^2 - aX - b$.

$$\begin{aligned} e_i &= x_{2i}, & \alpha &= \beta + a/2, \\ \alpha e_i &= x_{2i-1}, & \beta^2 &= -g = b + a^2/4, \\ -\beta^{-1}e_{-i} &= x_{-(2i-1)}, & \alpha^2 &= a\alpha + b, \\ \alpha\beta^{-1}e_{-i} &= x_{-2i}, & \sigma(\alpha) &= -\beta + a/2, \\ e_0 &= w_1, & \sigma(\beta) &= -\beta, \\ \beta e_0 &= w_2', & \alpha\sigma(\alpha) &= -b. \end{aligned}$$

Let x be an arbitrary vector in V. Then

$$x = \left(\textstyle\sum_{i\neq 0, -2k \leq i \leq 2k} a_i x_i\right) + a_0 w_1 + a_0' w_2'$$

where the a_i, a_0, $a_0' \in F$. Thus in U,

$$x = \textstyle\sum_{1\leq i\leq k} \{(a_{2i} + \alpha a_{2i-1})e_i + (\alpha\beta^{-1}a_{-2i} - \beta^{-1}a_{-(2i-1)})e_{-i}\} + (\beta a_0' + a_0)e_0.$$

Let

$$y = \textstyle\sum_{1\leq i\leq k}\{(b_{2i} + \alpha b_{2i-1})e_i + (\alpha\beta^{-1}b_{-2i} - \beta^{-1}b_{-(2i-1)})e_{-i}\} + (\beta b_0' + b_0)e_0.$$

Then

$$\begin{aligned} B'(x, y) = {} & \textstyle\sum_{1\leq i\leq k} \{ (a_{2i} + \sigma(\alpha)a_{2i-1})(\alpha\beta^{-1}b_{-2i} - \beta^{-1}b_{-(2i-1)}) \\ & - \beta^{-1}(\sigma(\alpha)a_{-2i} - a_{-(2i-1)})(b_{2i} + \alpha b_{2i-1})\} + 2(a_0 - \beta a_0')(\beta b_0' + b_0) \\ = {} & \textstyle\sum_{1\leq i\leq k} \{a_{2i} b_{-2i}(1 + \frac{1}{2}a\beta^{-1}) + a_{2i-1} b_{-(2i-1)}(1 - \frac{1}{2}a\beta^{-1}) \\ & - b\beta^{-1}a_{2i-1} b_{-2i} - \beta^{-1}a_{2i} b_{-(2i-1)} + a_{-2i} b_{2i}(1 - \frac{1}{2}a\beta^{-1}) \\ & + a_{-(2i-1)} b_{2i-1}(1 + \frac{1}{2}a\beta^{-1}) + \beta^{-1}a_{-(2i-1)} b_{2i} + \beta^{-1}ba_{-2i} b_{2i-1}\} \\ & + 2(a_0 b_0 + ga_0' b_0') + 2\beta^{-1}(ga_0' b_0 - ga_0 b_0') \\ = {} & B(x, y) + \beta^{-1}(\textstyle\sum_{1\leq i\leq k} \{a_{-(2i-1)}(\frac{1}{2}ab_{2i-1} + b_{2i}) \\ & + a_{-2i}(-\frac{1}{2}ab_{2i} + bb_{2i-1}) + a_{2i}(\frac{1}{2}ab_{-2i} - b_{-(2i-1)}) \\ & + a_{2i-1}(-\frac{1}{2}ab_{-(2i-1)} - bb_{-2i})\} + 2g(a_0' b_0 - a_0 b_0'). \end{aligned}$$

We remark that in case $x = y$, (i.e., $a_i = b_i$ $(i \neq 0)$, $a_0 = b_0$, $a_0' = b_0'$), the above equation implies $B'(x, x) = B(x, x)$.

$$\begin{aligned}
\beta y &= \textstyle\sum_{1\leq i\leq k} \{(\beta b_{2i} + \beta\alpha b_{2i-1})c_i + (\alpha b_{-2i} - b_{-2i-1})e_{-i}\} + (-gb_0' + \beta b_0)c_0 \\
&= \textstyle\sum_{1\leq i\leq k} \{((\alpha - \frac{1}{2}a)b_{2i} + (a\alpha + b - \frac{1}{2}a\alpha)b_{2i-1})e_i \\
&\quad + ((b + \tfrac{1}{2}a\alpha)\beta^{-1}b_{-2i} - (\alpha - \tfrac{1}{2}a)\beta^{-1}b_{-(2i-1)})e_{-i}\} + (-gb_0' + \beta b_0)e_0 \\
&= \textstyle\sum \{(+b_{2i} + \frac{1}{2}ab_{2i-1})x_{2i-1} + (bb_{2i-1} - \frac{1}{2}ab_{2i})x_{2i} \\
&\quad + (\tfrac{1}{2}ab_{-2i} - b_{-(2i-t)})x_{-2i} + (-\tfrac{1}{2}ab_{-(2i-1)} - bb_{-2i})x_{-(2i-1)}\} \\
&\quad - gb_0' w_1 + b_0 w_2 .
\end{aligned}$$

$$\begin{aligned}
B(x, \beta y) = {} & \textstyle\sum_{1\leq i\leq k} \{a_{-(2i-1)}(b_{2i} + \frac{1}{2}ab_{2i-1}) + a_{-2i}(bb_{2i-1} - \frac{1}{2}ab_{2i}) \\
& + a_{2i}(\tfrac{1}{2}ab_{-2i} - b_{-(2i-1)}) + a_{2i-1}(-\tfrac{1}{2}ab_{-(2i-1)} - bb_{-2i})\} \\
& + 2a_0 b_0'(-g) + 2ga_0' b_0 .
\end{aligned}$$

We are done.

The lines in Q are points in U. Thus if $l \in L$, $l = \langle x, \beta x\rangle$ in V and a non-identity element in Σl can be written $\rho_{x,k\beta x}$, $k \in F^*$. If $B(z, x) = 0$, $\rho_{x,k\beta x}$ sends z to $z + kB(z, \beta x)x$. In this case $\beta B'(z, x) = B(z, \beta x)$ and since $B'(z, x) = \beta^{-1}B(z, \beta x) = \beta^{-1}t$ where $t \in F$, we see $B'(z, x) = -B'(x, z)$. Thus when $B(z, x) = 0$, $\rho_{x,k\beta x}$ sends z to $z - k\beta B'(x, z)x$. If $B(z, \beta x) = 0$, $\rho_{x,k\beta x} = \rho_{\beta x,-kx}$ sends z to $z - kB(x, z)\beta x$. In this case $B(x, z) = B'(x, z) \in F$. Since the vectors in $x^{\perp}$ and in $\beta x^{\perp}$ in V span V (over F) and since $\rho_{x,k\beta x}$ is a linear transformation in V, $\rho_{x,k\beta x}$ sends any z in V to $z - k\beta B'(x, z)x$.

If a transformation τ on U sends z to $z + \lambda B'(x, z)x$ where $\langle x\rangle$ is isotropic (i.e., $B'(x, x) = 0$ and $x \neq 0$) and $\sigma(\lambda) = -\lambda/0$, we say τ is a unitary transvection with center $\langle x\rangle$. Hence the Siegel transformations with axis lines in L act as unitary transvections on U. We remark that $\sigma(\lambda) = -\lambda$ if and only if $\lambda = k\beta$, $k \in F$. The set of all unitary transvections on U generates $SU(U)$ [3, pp. 43–47]. If $\tau \in J$, a subgroup of $U(U)$, we will say $\langle x\rangle$ is a center for J. Since $\tau^{\pi} = \pi\tau\pi^{-1}$ for $\pi \in U(U)$ is a unitary transvection with center $\langle\pi(x)\rangle$, elements of a group J send centers to centers. Hence a group which contains all the unitary transvections with a given center and which is also transitive on isotropic points will be $SU(U)$.

The groups of root type 1 with axis lines in L correspond to the groups of all unitary transvections with centers

$$\langle e_i\rangle, \quad \langle e_{-i}\rangle, \quad \langle e_1 + e_i\rangle, \quad \langle e_1 + \alpha e_i\rangle, \quad 1 \leq i \leq k$$

and

$$\langle e_1 - (1 + \tfrac{1}{2}a(\alpha - \tfrac{1}{2}a)^{-1})e_{-1} + e_0\rangle.$$

We wish to show that the group H generated by the groups of root type 1 with axis lines in L, when considered as a group of unitary transformations acting on U is $SU(U)$. Therefore, we have only to show that H is transitive on the isotropic points of U. This is done in the Appendix.

Since $B'(x, x) = B(x, x)$, the singular vectors of V correspond to the isotropic vectors of U. Then since $SU(U)$ is transitive on the isotropic vectors of U, for dim $U \geq 3$, (see the appendix), we see that H is transitive on the singular vectors and hence the singular points of V.

We remark that we have already proved that any group G satisfying the hypothesis of this section must contain H. But since H is transitive on the singular points of V, each singular point is on at least one axis line of H. Thus G can be no larger than H. We have only to show that each singular point is on no more than one axis line of H. Suppose each singular point is on more than one axis line. (Recall H is transitive on singular points, hence all singular points have the same incidence structure of axis lines containing them.) Then $\rho_{x_1,y} \in H$ where $y \notin \langle x_1, x_2\rangle$. Then by (1) we can write y with no x_2 component. But then $\rho_{x_1,y}$ fixes x_{-2} but not x_{-1}. So $\rho_{x_1,y}$ is not linear on U. So $\rho_{x_1,y}$ cannot be in H. So each singular point is on exactly one axis line in H.

Thus (for $t = \dim V \geq 6$; $t = 4m + 2$, $\nu = 2m$ or $t = 4m$ $\nu = 2m$) $H \cong SU(t/2, q^2)$ is a subgroup of $\Omega(V)$ transitive on the singular points of V and generated by groups of root type 1 in V such that each singular point of V is on exactly one axis line of H.

8. The system T_0

In Sections 8, 9 and 10 we assume that our group G satisfies the hypothesis of the theorem and in addition the space of axis lines containing a given singular point is spanned by two linearly independent axis lines perpendicular to other. For example, the space of axis lines containing $\langle x_1\rangle$ might be spanned by $\langle x_1, x_2\rangle$ and $\langle x_1, x_3\rangle$, but not by $\langle x_1, x_2\rangle$ and $\langle x_1, x_{-2}\rangle$.

We begin by choosing to call our first axis line "$\langle x_{-1}, x_3\rangle$". This is done so that our notation eventually conforms with Dickson's [2]. We shall show that with this assumption, there exists standard basis vectors x_1, x_2, x_{-2}, x_{-3} such that $\langle x_{-1}, x_2\rangle$, $\langle x_2, x_{-3}\rangle$ $\langle x_{-3}, x_1\rangle$, $\langle x_{-1}, x_{-2}\rangle$ and $\langle x_{-2}, x_3\rangle$ are also axis lines for G. We call the system of six axis lines thus obtained "T_0".

First we note that there must be a singular point P in $x_{-1}^{\perp}$ which is not in $x_3^{\perp}$. We may call P "$\langle x_{-3}\rangle$". We call the space spanned by the axis lines containing x_{-1} "Z". Then $x_{-3}^{\perp} \cap Z$ must have dimension 2 and include $\langle x_{-1}\rangle$. Say $x_{-3}^{\perp} \cap Z = \langle x_{-1}, y\rangle$. Thus, since every two-dimensional subspace of Z containing $\langle x_{-1}\rangle$ must be an axis line for G by (1), $\langle x_{-1}, y\rangle$ is an axis line. By the definition of y, $y \in \langle x_{-1}, x_3, x_{-3}\rangle^{\perp}$.

Let the space of axis lines containing $\langle x_{-3}\rangle$ be Z'. Then $\{x_3\}^{\perp} \cap Z'$ has two dimensions and we define

$$\langle x_3\rangle^{\perp} \cap Z' = \langle z, w\rangle.$$

Since $\langle x_{-1}\rangle^{\perp} \cap \langle z, w\rangle$ has dimension 1 or 2, we may assume without loss of generality that $\langle z\rangle \in x_{-1}^{\perp}$ (i.e. at least one point on $\langle z, w\rangle$ must be in $\langle x_{-1}\rangle^{\perp}$). Then ρ_{x_{-1},x_3} sends $\langle z, x_{-3}\rangle$ to $\langle z, x_{-3} + x_{-1}\rangle$. By (1), $\langle x_{-1}, z\rangle$ is an axis line. Thus $z = ax_{-1} + by + cx_3$ for some $a, b, c \in F$. But $z \in \langle x_{-3}\rangle^{\perp}$ since $\langle z, x_{-3}\rangle$ is an axis line. Hence $c = 0$ and $z \in \langle x_{-1}, y\rangle$.

If w is also in $\langle x_{-1}\rangle^{\perp}$, then $w \in \langle x_{-1}, y\rangle$ by the same reasoning. But this would imply $\langle z, w\rangle = \langle x_{-1}, y\rangle$. Then the space of axis lines containing $\langle x_{-3}\rangle$ is

$\langle x_{-3}, x_{-1}, y\rangle$, which implies that $\langle x_{-3}, x_{-1}\rangle$ is an axis line, contradicting the assumption that the space of axis lines containing $\langle x_{-1}\rangle$ be singular. Hence $w \notin \langle x_{-1}\rangle^{\perp}$, but $w \in \langle x_3, x_{-3}\rangle^{\perp}$. Thus we may call $\langle w\rangle$ "$\langle x_1\rangle$". This yields the axis line $\langle x_{-3}, x_1\rangle$. Let y' be the point on $\langle x_{-1}, y\rangle$ in $x_1^{\perp}$. Then

$$y' \in \langle x_3, x_{-1}, x_1, x_{-3}\rangle^{\perp}.$$

Hence we may name $\langle y'\rangle$ "$\langle x_2\rangle$". This yields the axis line $\langle x_{-1}, x_2\rangle$. Recall $z \in \langle x_{-1}, y\rangle = \langle x_{-1}, x_2\rangle$ and $z \in w^{\perp} = x_1^{\perp}$. Thus $\langle z\rangle = \langle x_2\rangle$ and this yields the line $\langle x_{-3}, x_2\rangle$.

Now suppose $\langle x_1, z'\rangle$ is an axis line where $z' \notin \langle x_1, x_{-3}\rangle$. Then z' can be chosen in

$$\langle x_1, x_{-1}, x_3, x_{-3}\rangle^{\perp}.$$

Thus ρ_{x_{-1},x_3} sends $\langle x_1, z'\rangle$ to $\langle x_1 - x_3, z'\rangle$. So $\langle x_3, z'\rangle$ is an axis line. If $z' \in x_2^{\perp}$, then ρ_{x_{-1},x_3} sends $\langle x_1, z'\rangle$ to $\langle x_1 - x_2, z'\rangle$. But then $\langle x_2, z'\rangle$ is an axis line and z' is on too many linearly independent axis lines. Thus $z' \notin x_2^{\perp}$. Hence we may name $\langle z'\rangle$ "$\langle x_{-2}\rangle$". Thus we obtain the axis, lines $\langle x_1, x_{-2}\rangle$ and $\langle x_{-2}, x_3\rangle$. This completes our system T_0.

9. Elimination of some possibilities under the assumption that the space of axis lines containing a given point is 3-dimensional and singular

Under the assumptions of Section 8 we shall eliminate all possibilities except $\dim V = 7$. We recall we have the following system T_0 of axis lines: $\langle x_{-1}, x_3\rangle$, $\langle x_{-1}, x_2\rangle$, $\langle x_{-3}, x_1\rangle$, $\langle x_{-3}, x_2\rangle$, $\langle x_1, x_{-2}\rangle$ and $\langle x_3, x_{-2}\rangle$.

Let $U_0 = \langle x_1, x_2, x_3, x_{-3}, x_{-2}, x_{-1}\rangle = \langle T_0\rangle$. Suppose $x \in U_0^{\perp}$ and is singular. Suppose $\langle x, y\rangle$ is an axis line. Suppose $y \notin U_0^{\perp}$. Without loss of generality, say y has an x_{-3} component. Then $\rho_{x_3,x_{-1}}$ sends $\langle x, y\rangle$ to $\langle x, y + t\rangle$ where $t \in \langle x_{-1}, x_3\rangle$, and $\rho_{x_3,x_{-2}}$ sends $\langle x, y\rangle$ to $\langle x, y + s\rangle$ where $s \in \langle x_3, x_{-2}\rangle$ $(s, t \notin \langle x_3\rangle)$. Since y has an x_{-3} component $y \notin \langle s, t\rangle$. Thus x is on too many linearly independent axis lines. Hence $y \in U_0^{\perp}$.

We thus build a system T_1 isomorphic to T_0 and in $T_0^{\perp}$. We can keep building T_i's until we run out of singular points in $T_j^{\perp}$, $j < i$. We see the index of V must be a multiple of 3.

Look at the axis line $\langle x_1 + x_1', y\rangle$ where $x_1 \in \langle T_0\rangle$ and $x_1' \in \langle T_1\rangle$. First, $y \in T_i^{\perp}$ for all $i > 1$. The perpendicular of all the T_i is our standard basis space W of dimension 0, 1 or 2 containing no singular vectors. So y must have a component in $\langle T_0, T_1\rangle$. Say y has an x_2 component. Then ρ_{x_{-2},x_3} sends y to $y + t$ where $t \in \langle x_{-2}, x_3\rangle$. Thus, by (1), $\langle x_1 + x_1', t\rangle$ is also an axis line. Further $\rho_{x'_{-1},x'_3}$ (from T_1) sends $\langle x_1 + x_1', t\rangle$ to $\langle x_1 + x_1' - x_3', t\rangle$, so t is on too many axis lines. Identical proofs work for any other components except x_1 and x_1'. Thus, $y \in \langle x_1, x_1', W\rangle$. But y is singular, so $y \in \langle x_1, x_1'\rangle$. Take the point on $\langle x_1 + x_1', y\rangle$ in $x_{-1}^{\perp}$. It must be $\langle x_1'\rangle$. But then $\langle x_1'\rangle$ is on too many axis lines. Therefore, the index of V is 3.

Thus the dimension of V is 6, 7, or 8. Suppose the dimension of V is 6.

Then $\langle x_{-1} + x_2, x_{-1}\rangle$ is an axis line and $\langle x_{-1} + x_2, y\rangle$ is an axis line where $y \notin \langle x_{-1}, x_2\rangle$ but $y \in \langle x_{-1}, x_2\rangle^{\perp}$. Hence

$$y = ax_{-1} + bx_{+2} + cx_3 + dx_{-3}, \quad a, b, c, d \in F.$$

Since y is singular, $cd = 0$. Suppose $d = 0$ and assume $\langle y\rangle$ is the point on $\langle x_{-1} + x_2, y\rangle$ in $x_1^{\perp}$. Then $y = bx_2 + cx_3$ where $c \neq 0$, because $c = 0$ implies $y \in \langle x_{-1}, x_2\rangle$. Since $\langle x_{-1}, x_2\rangle$ and $\langle x_{-1}, x_3\rangle$ are axis lines, (1) implies $\langle x_{-1}, bx_2 + cx_3\rangle$ is an axis lines. But then again by (1), $\langle x_2, bx_2 + cx_3\rangle$ is an axis line, contradicting the hypothesis that the space of axis lines containing $\langle x_2\rangle$ be singular. If $c = 0$, assume y is the point on $\langle x_{-1} + x_2, y\rangle$ in $x_{-2}^{\perp}$ for a similar contradiction. Thus dim V is 7 or 8.

We now show that each point of form $\langle x_1 - Q(w)x_{-1} + w\rangle$ with $w \in W \subseteq U_0^{\perp}$ is on an axis line with $\langle -ax_{-3} + bx_2\rangle$ where $a, b \in F^*$. But $\langle -ax_{-3} + bx_2, x_{-3}\rangle$ is an axis line in T_0. Hence $\langle -ax_{-3} + bx_2\rangle$ cannot be on axis lines with two distinct points of form $\langle x_1 - Q(w)x_{-1} + w\rangle$. Hence the number of ratios a/b in F^* determines the number of vectors $w \in W$. Thus dim $V = 7$. Now suppose $\langle y\rangle$ is the point on the axis line $\langle x_1 - Q(w)x_{-1} + w, y\rangle$ in $w^{\perp}$.

Let $y = a_1 x_1 + a_{-1} x_{-1} + a_2 x_2 + a_{-2} x_{-2} + a_3 x_3 + a_{-3} x_{-3} + w'$, where $w' \in W$ and $w' \in w^{\perp}$. Then $\rho_{x_2, kx_{-3}}$ fixes $\langle x_1 - Q(w)x_{-1} + w\rangle$ and translates y by $-a_{-2} kx_{-3} + ka_3 x_2$. By (1), $\langle x_1 - Q(w)x_{-1} + w\rangle$ must then be on an axis line with $\langle -a_{-2} x_{-3} + a_3 x_2\rangle$. Clearly either a_{-2} and a_3 are both zero or neither is zero, because $\langle x_{-3}\rangle$ and $\langle x_2\rangle$ are already on two axis lines in T_0. If they are both zero, then we show a_{-3} and a_2 are both non zero. For suppose a_2, a_{-2}, a_3 and a_{-3} are zero. Then $y = a_1 x_1 + a_{-1} x_{-1} + w'$. Then $a_1 a_{-1} = -Q(w')$ and $a_{-1} - a_1 Q(w) = 0$, so $a_1 a_{-1} = a_1^2 Q(w)$. Hence $-Q(w') = a_1^2 Q(w)$. But if $Q(w)$ is a square then $-Q(w')$ cannot be and vice-versa, or W would contain singular vectors. So suppose a_{-3} and a_2 are not both zero. Then using ρ_{x_{-2}, x_3} we see that $\langle x_1 - Q(w)x_{-1} + w\rangle$ is on an axis line with $\langle -a_2 x_3 + a_{-3} x_{-2}\rangle$. (Again we cannot have just one of a_{-3} and a_2 be zero.) Thus in all cases $\langle x_1 - Q(w)x_{-1} + w\rangle$ is on axis lines with $\langle -ax_{-3} + bx_2\rangle$ and $\langle bx_3 + ax_{-2}\rangle$ where a and $b \in F^*$. So dim $V = 7$ as claimed.

10. Existence of a group G with the space of axis lines containing a given point being 3-dimensional and singular

We have seen in Sections 8 and 9 that if a group exists satisfying the hypotheses of the theorem and such that the space of axis lines containing a given point is 3-dimensional and singular, then dim $V = 7$ and one can find a standard basis $x_1, x_{-1}, x_2, x_{-2}, x_3, x_{-3}, w$ by Witt's theorem such that the system T_0 of axis lines occurs, $Q(w) = 1$ and the axis lines

$$\langle x_1 - x_{-1} - w, x_2 - kx_{-3}\rangle \quad \text{and} \quad \langle x_1 - x_{-1} - w, x_{-2} + k^{-1}x_3\rangle$$

occur.

Now in our standard basis replace $-kx'_{-3}$ by x_{-3} and $k^{-1}x_3$ by $-x_3'$. This does not change T_0, (i.e., $\langle x_3, x_2\rangle = \langle x_3', x_2\rangle$ etc.). So we may assume that we had chosen basis vectors so that we have the system T_0 and the axis lines

$$\langle x_1 - x_{-1} - w, x_2 + x_{-3}\rangle \quad \text{and} \quad \langle x_1 - x_{-1} - w, x_{-2} - x_3\rangle.$$

Call the group generated by the groups of root type 1 with these axis lines G. We show that G is transitive on the singular points of V. Hence, each singular point is on at least 2 linearly independent axis lines of G.

We prove this in two parts: First, all singular points of V with a w component are in the same orbit under G, and secondly, any singular point without a w component is in an orbit with a singular point with a w component.

Recall that in Section 7 we proved that $\Sigma\langle x_{-1}, x_{-2}\rangle$ and $\Sigma\langle x_2, x_1\rangle$ generate a group isomorphic to $SU(2, q^2)$. In the appendix, we proved $SU(2, q^2) =$ "$SU(W_0)$" was transitive on the non-isotropic vectors of W_0 of a given length. These non-isotropic vectors correspond to the non-singular vectors in $\langle x_1, x_{-1}, x_2, x_{-2}\rangle =$ "W_0" since $B(x, x) = B'(x, x)$. Thus, the group generated by Σl and Σl_0 where $l \cap l_0^{\perp} = \{0\}$ is transitive on the non-singular vectors of $\langle l, l_0\rangle$ of a given length.

In particular, the group generated by $\Sigma\langle x_1, x_{-2}\rangle$ and $\Sigma\langle x_{-1}, x_2\rangle$ is transitive on the non-singular vectors of a given length in $\langle x_1, x_{-1}, x_2, x_{-2}\rangle$. Thus $x_1 - x_{-1}$ is in the same orbit O under G as $x_2 - x_{-2}$. Using $\Sigma\langle x_{+1}, x_{-3}\rangle$ and $\Sigma\langle x_{-1}, x_3\rangle$, we obtain $x_3 - x_{-3}$ in O as well. We show all vectors $y = \Sigma a_i x_i$ $(i = 1, 2, 3, -1, -2, -3)$ such that $Q(y) = -1$ are in O. This is equivalent to showing all singular points with a w component are in the same orbit under G, since such singular points can be expressed as $\langle\Sigma a_i x_i - w\rangle$.

If $a_1 a_{-1} \neq 0, -1$, then the group generated by

$$\Sigma\langle x_1, a_{-2} x_{-2} + a_{-3} x_{-3}\rangle \quad \text{and} \quad \Sigma\langle x_{-1}, a_2 x_2 + a_3 x_3\rangle$$

sends

$$x_1 - x_{-1} \quad \text{to} \quad a_1 x_1 + a_{-1} x_{-1} + a_{-2} x_{-2} + a_{-3} x_{-3} + a_2 x_2 + a_3 x_3.$$

If $a_1 a_{-1} = -1$, then $a_{-2} x_{-2} + a_{-3} x_{-3} \in \langle a_2 x_2 + a_3 x_3\rangle^{\perp}$ and the operation must be in two stages. First, use the group generated by

$$\Sigma\langle x_1, a_{-2} x_{-2} + a_{-3} x_{-3}\rangle$$

and

$$\Sigma\langle x_{-1}, x_2\rangle \text{ if } a_{-2} \neq 0 \quad \text{or} \quad \Sigma\langle x_{-1}, x_3\rangle \text{ if } a_{-3} \neq 0$$

(or skip this step if $a_{-2} = a_{-3} = 0$). This sends

$$x_1 - x_{-1} \quad \text{to} \quad a_1 x_1 + a_{-1} x_{-1} + a_{-2} x_{-2} + a_{-3} x_{-3}.$$

Then use the group generated by

$$\Sigma\langle x_{-1}, a_2 x_2 + a_3 x_3\rangle$$

and

$$\Sigma\langle x_1, x_{-2}\rangle \text{ if } a_2 \neq 0 \quad \text{or} \quad \Sigma\langle x_1, x_{-3}\rangle \text{ if } a_3 \neq 0.$$

This sends $a_1 x_1 + a_{-1} x_{-1}$ to $a_1 x_1 + a_{-1} x_{-1} + a_2 x_2 + a_3 x_3$ and fixes $a_{-2} x_{-2} + a_{-3} x_{-3}$. Thus, if $a_1 a_{-1} \neq 0$, we get our point. By symmetry we can do the same if $a_2 a_{-2} \neq 0$ (starting with $x_2 - x_{-2}$) or $a_3 a_{-3} \neq 0$ (starting with $x_3 - x_{-3}$). But not all of $a_1 a_{-1}$, $a_2 a_{-2}$, and $a_3 a_{-3}$ can be zero.

Thus we have all singular points in V with a w component. Now suppose $x \in w^{\perp}$, x singular. We show that the singular points on $w^{\perp}$ which are on axis lines with points not in $w^{\perp}$ span $w^{\perp}$. Then for every $x \in w^{\perp}$, there is a singular $y \notin x^{\perp}$, $y \in w^{\perp}$, y on an axis line l with a point not in $w^{\perp}$. But Σl sends x to $x + t$, $t \in l$, $t \notin \langle y \rangle$ and we are done.

Look at the points we know are on axis lines with points not in $w^{\perp}$: $\langle x_2 + x_{-3} \rangle$ and $\langle x_{-2} - x_3 \rangle$. Apply $\rho_{x_1, x_{-2}}$ to $x_2 + x_{-3}$ and get $x_2 + x_{-3} + x_1$. Apply ρ_{x_{-1}, x_3} and get $x_2 + x_{-3} + x_1 + x_{-1} - x_3$. Apply ρ_{x_{-1}, x_2} to $x_{-2} - x_3$ and get $x_{-2} - x_3 + x_{-1}$. Apply $\rho_{x_1, x_{-3}}$ and get $x_{-2} - x_3 - x_1 + x_{-1} - x_{-3}$. These six vectors span $w^{\perp}$.

The groups of root type 1 whose axis lines are in T_0 (denoted Q_{ijt} by Dickson) are contained in Dickson's group G_2 [2]. Further, since $\rho_{x_{-1}, w}\, \rho_{x_2, x_3}$ where $Q(w) = 1$ is in G_2 ($Y_{011} W_{231}$ in Dickson's notation) and sends $\langle x_1, cx_{-3} + dx_{-2} \rangle$ in T_0 to $\langle x_1 - x_{-1} - w, c(x_{-3} + x_2) + d(x_{-2} - x_3) \rangle$, we see $G \subseteq G_2$. Since G_2 is a proper subgroup of $\Omega(V)$, and since we show in the remainder of this paper that the singular points can be in no other configuration of axis lines without generating $\Omega(V)$, we see that each singular point is on exactly two linearly independent mutually perpendicular axis lines of G.

The only question remaining is: are there any elements in G_2 not in G? Clearly there are no groups of root type 1 in G_2 not in G. Therefore, suppose $\rho \in \Sigma \langle x, u \rangle \subseteq G$. Then ρ^{τ} is in G_2 for any $\tau \in G_2$. But then $\rho^{\tau} \in G$, for the set of all ρ^{τ} such that $\rho \in \Sigma \langle x, u \rangle$ and τ is fixed, is a group of root type 1 in G_2, hence in G. But if ρ is any element in G, $\rho = \rho_1 \cdots \rho_k$ where each ρ_i is an element of a group of root type 1 in G. Thus $\rho^{\tau} \in G$ for any $\rho \in G$ and $\tau \in G_2$. Hence G is normal in G_2. But G_2 is simple. So $G = G_2$.

11. Other possibilities: If the space of axis lines containing (x) is $\langle x \rangle \oplus U$, U must be non-degenerate

We show in this section and the next that if G satisfies the hypothesis of the theorem and has any axis line configuration other than (1) each singular point on exactly one axis line or (2) the space of axis lines containing a given point being 3-dimensional and singular, then $G = \Omega(V)$.

In these last two sections, we use the lemma of Section 5 and equation (1) of Section 2 extensively.

Let the space of axis lines containing $\langle x_1 \rangle$ be denoted $U \oplus \langle x_1 \rangle$ where $\oplus$ indicates that $\langle x_1 \rangle$ is linearly independent of U. In this section we show U must be non-degenerate. First, we shall show that U must be either singular or non-degenerate.

By Artin [1, p. 116], $U = Z \oplus Z'$ where Z is a maximal non-degenerate subspace of U, and $Z' \subseteq Z^{\perp}$ and Z' is singular.

Suppose Z is non-empty, (i.e., U is not singular). Then since H is spanned by singular vectors, there must be two linearly independent singular vectors

x and y in U with $B(x, y) \neq 0$. Thus, since Z is a maximal non-degenerate subspace of U, Z must have dimension at least 2. If Z has dimension exactly 2, then Z is spanned by singular vectors. (We know $U = \langle x, y\rangle \oplus \langle x, y\rangle'$ where $B(x, y) \neq 0$, x and y are singular and $\langle x, y\rangle'$ is a singular space in $\langle x, y\rangle^{\perp}$. It is not possible that also $U = W \oplus W'$ where W has dimension 2, W contains no singular vectors and $W' \subseteq W^{\perp}$, W' singular, because $W \oplus W'$ cannot be mapped to $\langle x, y\rangle \oplus \langle x, y\rangle'$ by a member of $O(V)$.)

Since any non-degenerate space of dimension greater than 2 is spanned by singular vectors, we see that in any case Z is spanned by singular vectors.

Let $\langle x_2\rangle$ be a singular point in Z. Let $\langle x_{-2}\rangle$ be a singular point in Z such that $B(x_2, x_{-2}) = 1$. Suppose $\{\langle x_2, t_i\rangle\}$ is a set of axis lines containing $\langle x_2\rangle$ such that the t_i span a space isomorphic to U. (Since G is transitive on singular points in V, such a set must exist.) Choose the $\langle t_i\rangle$ to be the points on $\langle x_2, t_i\rangle$ in $\langle x_{-2}\rangle^{\perp}$. Suppose all the t_i are also in $\langle x_1\rangle^{\perp}$. Then apply the lemma to $\langle x_2, t_i\rangle$ and $\langle x_1, x_{-2}\rangle$ to obtain the $\langle x_1, t_i\rangle$ as axis lines (unless $t_i \in \langle x_1\rangle$). But the t_i are all in $\langle x_2, x_{-2}\rangle^{\perp}$. Thus the t_i are all in $(\langle x_1\rangle \oplus U) \cap \langle x_2, x_{-2}\rangle^{\perp}$. But this has dimension one less than the dimension of U. Hence, some t_i is not in $x_1^{\perp}$. Without loss of generality, we may name this t_i "x_{-1}" and assume U has been chosen in $x_{-1}^{\perp}$ (i.e., translate whatever vectors had been chosen in U by a suitable multiple of x_1, until U is in $x_{-1}^{\perp}$).

Now assume Z' is also non-empty and $\langle x_3\rangle \in Z'$. Use the lemma with $\langle x_1, x_3\rangle$ and $\langle x_2, x_{-1}\rangle$ to see that $\langle x_3, x_2\rangle$ is an axis line. Similarly, if we had begun with x_{-2} or any other singular vector t in Z ($\subseteq \langle x_{-1}\rangle^{\perp}$), we would have gotten $\langle x_3, t + kx_1\rangle$ to be an axis line for some $k \in F$. Thus by (1), since x_3 is on an axis line with x_1, we see that x_3 is on axis lines with every singular point in $\langle x_1\rangle \oplus Z$.

Now look at the axis lines $\langle x_{-1}, s_i\rangle$ containing $\langle x_{-1}\rangle$. Suppose they are all in $\langle x_2\rangle^{\perp}$. Choose the s_i to be in $\langle x_1\rangle^{\perp}$. Use the lemma with $\langle x_{-1}, s_i\rangle$ and $\langle x_2, x_1\rangle$ to see the $\langle x_2, s_i\rangle$ are axis lines. Thus the s_i all lie in $\langle x_1, x_{-1}\rangle^{\perp} \cap$ (the space of axis lines containing $\langle x_2\rangle$). This has dimension one less than the dimension of U. Hence, some s_i, say s_0, must not be in $x_2^{\perp}$. Use the lemma with $\langle s_0, x_{-1}\rangle$ and $\langle x_3, x_2\rangle$ to see that x_{-1} is on an axis line with $ax_3 + bx_2$ where $a \in F^*$ and $b \in F$ and $s_0 \in \langle ax_3 + bx_2\rangle^{\perp}$. But $\langle x_{-1}, x_2\rangle$ is an axis line, so by (1), $\langle x_{-1}, x_3\rangle$ is an axis line. Thus $\langle x_3\rangle$ is on axis lines with all singular points in $\langle x_1\rangle \oplus Z$ and with $\langle x_{-1}\rangle$. But this is a non-degenerate space larger than Z, contradicting the maximality of Z. Hence, if Z is non-empty, Z' is empty, (i.e. U is either singular or non-degenerate).

We now suppose U is singular, (i.e., Z is empty), and the dimension of U is greater than 2. Since we have disposed of small singular U in previous section, suppose $\langle x_1, x_i\rangle$ ($2 \leq i \leq j$ where $j \geq 4$) are linearly independent axis lines containing $\langle x_1\rangle$. Using our standard basis notation, we look at the axis lines containing $\langle x_{-2}\rangle$. Let the space of axis lines containing $\langle x_{-2}\rangle$ be denoted $U_{\langle x_{-2}\rangle} \oplus \langle x_{-2}\rangle$ and choose $U_{\langle x_{-2}\rangle}$ to be in $x_2^{\perp}$. Let

$$U'_{\langle x_{-2}\rangle} = U_{\langle x_{-2}\rangle} \cap \langle x_1\rangle^{\perp}.$$

Then $U'_{\langle x_{-2}\rangle} \subseteq \langle x_1, x_2, x_{-2}\rangle^{\perp}$ and has dimension at least $j - 2$. Suppose $y_i \in U'_{\langle x_{-2}\rangle}$. Then using the lemma with $\langle x_{-2}, y_i\rangle$ and $\langle x_1, x_2\rangle$, we see that the $\langle y_i, x_1\rangle$ are axis lines. It is clear that $y_i \notin \langle x_1\rangle$, since $\langle x_{-2}, x_1\rangle$ is not an axis line i.e.

$$U'_{\langle x-2\rangle} \subseteq \langle x_1, x_2, x_3, \cdots x_j\rangle \cap \langle x_{-2}\rangle^{\perp} = \langle x_1, x_3, \cdots x_j\rangle.$$

But $\langle x_1\rangle \notin U'_{\langle x_{-2}\rangle}$. Thus $U'_{\langle x_{-2}\rangle} = \langle x_3, \cdots\cdots x_j\rangle$. (We recall that $U'_{\langle x_{-2}\rangle}$ must be at least this big.) Further if

$$U_{\langle x_{-2}\rangle} = U'_{\langle x_{-2}\rangle} \oplus \langle x\rangle,$$

$\langle x\rangle$ is in $\langle x_{-2}, x_2, x_i\rangle^{\perp}$, $3 \leq i \leq j$, but not in $\langle x_1\rangle^{\perp}$. So call $\langle x\rangle$ "$\langle x_{-1}\rangle$". Doing the same for x_{-3} instead of x_{-2} we see $\langle x_{-3}, x_i\rangle$ are axis lines for $i = 2$ or $4 \leq i \leq j$ and using the lemma with $\langle x_{-2}, x_{-1}\rangle$ and $\langle x_2, x_{-3}\rangle$, we see $\langle x_{-3}, x_{-1}\rangle$ is an axis line.

Now let $U_{\langle x_1+x_2\rangle}$ be in $\langle x_{-1}\rangle^{\perp}$ and look at

$$U'_{\langle x_1+x_2\rangle} = U_{\langle x_1+x_2\rangle} \cap \langle x_{-2}\rangle^{\perp}.$$

$U'_{\langle x_1+x_2\rangle}$ has dimension $j - 2$ since $x_2 \in U_{\langle x_1+x_2\rangle}$ but $x_2 \notin U'_{\langle x_1+x_2\rangle}$. Let

$$U''_{\langle x_1+x_2\rangle} = U'_{\langle x_1+x_2\rangle} \cap \langle x_{-3}\rangle^{\perp}.$$

This has dimension at least $j - 3$ which is ≥ 1 since $j \geq 4$. If $z \in U''_{\langle x_1+x_2\rangle}$ $(z \neq 0)$, then use the lemma with $\langle x_1 + x_2, z\rangle$ and $\langle x_{-1}, x_{-3}\rangle$ to see $\langle z, x_{-3}\rangle$ is an axis line. (Since $\langle x_{-3}, x_1 + x_2\rangle$ is not an axis line, $z \notin \langle x_{-3}\rangle$; since $z \in \langle x_1\rangle^{\perp}$, z has no x_{-1} component—thus $z \notin \langle x_{-1}, x_{-3}\rangle$.) Hence

$$z \in \langle x_4, \cdots, x_j\rangle = U_0.$$

But every vector in U_0 is on an axis line with x_{-2}. Hence z is on axis lines with x_{-2} and with $x_1 + x_2$, contradicting U singular.

12. If U is non-degenerate then $G = \Omega(V)$

We recall that the space of axis lines containing $\langle x\rangle$ is $U \oplus \langle x\rangle$. We have shown U must be non-degenerate. Suppose $\langle x_1, x_2\rangle$ is an axis line. Then $\langle x_2\rangle$ is on an axis line with some singular point $\langle y\rangle \notin x_1^{\perp}$. Call $\langle y\rangle$ "$\langle x_{-1}\rangle$". Let $U_{\langle x_1\rangle}$ (where the axis lines containing $\langle x_1\rangle$ span $U_{\langle x_1\rangle} \oplus \langle x_1\rangle$) be taken in $x_{-1}^{\perp}$. Then we may write

$$U_{\langle x_1\rangle} = \langle x_2, x_{-2}, x_3, x_{-3}, \cdots, x_k, x_{-k}, W\rangle$$

using standard basis notation. We remark that $k \geq 2$, since $U_{\langle x_1\rangle}$ must be spanned by singular vectors.

Now suppose $\langle x_2, z\rangle$ is an axis line where $z \notin \langle x_1, x_{-1}\rangle$, $z \in x_{-2}^{\perp}$. Thus $z = ax_1 + bx_{-1} + w$, where $w \in \langle x_1, x_{-1}, x_2, x_{-2}\rangle^{\perp}$, $a, b \in F$, and w may or may not be singular. Using (1), we see $\rho_{x_2,kw} \in G$ where $k \in F^*$. But $\rho_{x_2,kw}$ sends

$$\langle x_1, x_{-2}\rangle \quad \text{to} \quad \langle x_1, x_{-2} - kw - k^2Q(w)x_2\rangle.$$

Hence $x_{-2} - kw - k^2Q(w)x_2 \in U_{\langle x_1\rangle}$. Therefore, $w \in U_{\langle x_1\rangle}$. Thus

$$U_{\langle x_2\rangle} \cap \langle x_1, x_{-1}\rangle^{\perp} = U_{\langle x_1\rangle} \cap \langle x_2, x_{-2}\rangle^{\perp}.$$

Now suppose $\langle x_{-2}, x_{-1}\rangle$ is also an axis line. Then we can repeat the above argument used for $U_{\langle x_2\rangle}$ to see that

$$U_{\langle x_{-2}\rangle} \cap \langle x_1, x_{-1}\rangle^{\perp} = U_{\langle x_1\rangle} \cap \langle x_2, x_{-2}\rangle^{\perp}$$

and hence $U_{\langle x_2\rangle} = U_{\langle x_{-2}\rangle}$.

Therefore, we show that $\langle x_{-2}, x_{-1}\rangle$ must be an axis line. Let $\langle x_{-2}, y\rangle$ be an axis line such that $y \in \langle x_2, x_{-2}\rangle^{\perp}$ but $y \notin x_1^{\perp}$. Then

$$y = ax_1 + bx_{-1} + w \quad \text{where } b \neq 0 \text{ and } w \in \langle x_1, x_{-1}, x_2, x_{-2}\rangle^{\perp}.$$

So by (1), $\rho_{x_{-2},k(bx_{-1}+w)} \in G$ for all $k \in F^*$.

First suppose $Q(w) \neq 0$. Then note that $\rho_{x_{-2},k(bx_{-1}+w)}$ sends

$$\langle x_{-1}, x_2\rangle \quad \text{to} \quad \langle x_{-1}, x_2 - k(bx_{-1} + w) - k^2Q(w)x_{-2}\rangle.$$

Using (1) and (2), we see $\rho_{x_{-1},t(-kw-k^2Q(w)x_{-2})} \in G$ for all $t \in F^*$, and fixes x_{-2} and x_{-1} and sends w to $w - tkB(w, w)x_{-1}$. Hence $\rho_{x_{-2},k(bx_{-1}+w-tkB(w,w)x_{-1})} \in G$. Choose t such that $tkB(w, w) = b$. Then see $\rho_{x_2,kw} \in G$. Hence, by (1) we see $\langle x_{-1}, x_{-2}\rangle$ is an axis line.

Now suppose $Q(w) = 0$. Then $\rho_{x_{-2},k(bx_{-1}+w)}$ sends

$$\langle x_{-1}, x_2\rangle \quad \text{to} \quad \langle x_{-1}, x_2 - k(bx_{-1} + w)\rangle.$$

Using (1) and (2) we see $\rho_{x_{-1},-kw} \in G$ for $k \in F^*$. But $\rho_{x_{-1},-kw}$ sends

$$\langle x_{-2}, x_1\rangle \quad \text{to} \quad \langle x_{-2}, x_1 + kw\rangle,$$

so by (1), $\langle x_{-2}, w\rangle$ is an axis line, so again by (1), $\langle x_{-2}, x_{-1}\rangle$ is an axis line.

Let $T = U_{\langle x_2\rangle} \oplus \langle x_2, x_{-2}\rangle$. We show $\Omega(T) \subseteq G$. Let $\langle x, y\rangle$ be a singular line in T. If x_2 or x_{-2} is in $\langle x, y\rangle$, then $\langle x, y\rangle$ is an axis line for G. If $\langle x\rangle$ is in $x_2^{\perp}$ and $\langle y\rangle$ $(\neq \langle x\rangle)$ is in $x_{-2}^{\perp}$, then $\langle x_2, x\rangle$ and $\langle x_{-2}, y\rangle$ are axis lines and hence by the lemma, $\langle x, y\rangle$ is an axis line for G. So suppose

$$\langle x\rangle \subseteq \langle x_2, x_{-2}\rangle^{\perp}$$

and $\langle y\rangle$ $(\neq \langle x\rangle)$ is $\langle ax_2 + bx_{-2} + z\rangle$ where a and $b \in F^*$ and $z \in \langle x_2, x_{-2}\rangle^{\perp}$, and $Q(z) = -ab$. Then $\rho_{x_2,kz} \in G$ for $k \in F^*$, and sends

$$\langle x, x_{-2}\rangle \quad \text{to} \quad \langle x, x_{-2} - kz - k^2Q(z)x_2\rangle.$$

Choose $k = -b^{-1}$. Then

$$\langle x_{-2} - k_z - k^2Q(z)x_2\rangle = \langle x_{-2} + b^{-1}z + ab^{-1}x_2\rangle = \langle y\rangle.$$

Hence $\Omega(T) \subseteq G$. We wish to show $\Omega(T) = G$.

Since the axis lines of $\Omega(T)$ containing a singular point P in T are all the singular lines of T containing P by Tamagava [10] (i.e.

$$P \oplus U_P \cong U_{\langle x_2\rangle} \oplus \langle x_2\rangle$$

for any P in $T = U_{\langle x_2 \rangle} \oplus \langle x_2, x_{-2} \rangle)$, we see no point P in T can be on an axis line for G with a point outside T.

Suppose there is a singular point $\langle y \rangle \in T^\perp$. Let $\langle x \rangle$ be a singular point in T. Suppose $\langle x + y, s \rangle$ is an axis line. Suppose $s \in \langle x^\perp \cap T \rangle^\perp$ $(\subseteq x^\perp)$. Let $\langle x' \rangle$ be a singular point in T not in $x^\perp$. Let $\langle x', w \rangle$ $(\subseteq T)$ be an axis line containing x', where $\langle w \rangle$ is chosen to be that point on $\langle x', w \rangle$ in $\langle x \rangle^\perp$. (Hence $w \in x^\perp \cap T$.) Apply the lemma to $\langle x + y, s \rangle$ and $\langle x', w \rangle$. We see w must be on an axis line with some point on $\langle x + y, s \rangle$. But $w \in T$, and no point on $\langle x + y, s \rangle$ is in T, contradicting $s \in \langle x^\perp \cap T \rangle^\perp$.

Now suppose $s \in x^\perp$, but $s \notin \langle x^\perp \cap T \rangle^\perp$. Hence there is a singular point $\langle t \rangle$ in $\langle x^\perp \cap T \rangle$ (which is spanned by singular vectors) such that $s \notin t^\perp$. But $\langle x, t \rangle$ is an axis line. Use the lemma with $\langle x, t \rangle$ and $\langle x + y, s \rangle$ to see that $\langle x, x + y \rangle$ is an axis line, which is impossible, since $x \in T$ and $x + y \notin T$.

Hence $s \notin x^\perp$. But then there is a singular point $\langle t' \rangle$ in $T \cap \langle x, s \rangle^\perp$, which necessarily must be on an axis line with x. Apply the lemma to $\langle x, t' \rangle$ and $\langle x + y, s \rangle$ to see that $\langle t', x + y \rangle$ is an axis line, which is impossible since $t' \in T$ and $x + y \notin T$.

Hence there are no singular points in $T^\perp$. So suppose $y \in T^\perp$, $Q(y) \neq 0$. Then we have the axis line

$$\langle x_1 - Q(y)x_{-1} + y, s \rangle.$$

Choose s in $y^\perp$. If $T^\perp$ has only one dimension, we are done. So suppose $T^\perp$ has two dimensions (i.e., $T^\perp = W = \langle y, w \rangle$, a two-dimensional space with no singular points). If $U_{\langle x_1 - Q(y)x_{-1} + y \rangle}$ (taken to be in $\langle y \rangle^\perp$) has at least four dimensions, then $U_{\langle x_1 - Q(y)x_{-1} + y \rangle} \cap \langle w \rangle^\perp$ has at least three dimensions and hence contains a singular point which is in $\langle y, w \rangle = T^\perp$.

So assume the dimension of U is 2 or 3. If the dimension of U is 2, then *dim* $T = 4$ and $V = T \oplus W$ has dimension 6 and index 2. Now let

$$T = \langle x_1, x_{-1}, x_2, x_{-2} \rangle.$$

Let $W = \langle y, w \rangle$ where $Q(y) = 1$ and $\langle w \rangle = W \cap y^\perp$. Let $\langle x_1 - x_{-1} + y, s \rangle$ be an axis line. Take s in $y^\perp$. Then

$$s = cx_1 + cx_{-1} + ax_2 + bx_{-2} + w \quad \text{where } c^2 + ab = -Q(w),$$

so $ab \neq 0$. Apply $\rho_{x_2, k(x_1 + x_{-1})}$ to s to get

$$t = (c - bk)x_1 + (c - bk)_{-1} + (a + 2ck - k^2 b)x_2 + bx_{-2} + w.$$

If $c \neq 0$ choose $k = cb^{-1}$. Then $t = (a + c^2 b^{-1})x_2 + bx_{-2} + w$. Clearly, $t \notin \langle s \rangle$. If $c = 0$ choose $k = -b^{-1}$, get

$$t_0 = x_1 + x_{-1} + (a - b^{-1})x_2 + bx_{-2} + w.$$

So without loss of generality we may have assumed $c \neq 0$. Now apply $\rho_{x_{-2}, k(x_1 + x_{-1})}$ to s to get

$$t' = (c - ak)x_1 + (c - ak)x_{-t} + ax_2 + (b + 2ck - ak^2)x_{-2} + w.$$

Let $k = a^{-1}c$, so $t' = ax_2 + (b + c^2a^{-1})x_{-2} + w$. If $\langle x_1 + x_{-1} + y\rangle$ is not on too many axis lines, then $t' = ds + et$, d, $e \in F$. Hence, $1 = d + e$ (coeff of w), $dc = 0$ (coeff of x_1). Since $c \neq 0$, $d = 0$, and hence $e = 1$. Hence $t = t'$ and the coefficient of x_2 yields $a + c^2b^{-1} = a$, contradicting $c \neq 0$ and $ab \neq 0$.

Thus we have only one case left, dim $T = 5$. Let

$$T = \langle x_1, x_{-1}, x_2, x_{-2}, w\rangle \quad \text{where } w \in \langle x_1, x_{-1}, x_2, x_{-2}\rangle.$$

Let $T^\perp = \langle y, z\rangle$. Let $\langle y\rangle$ be the point on $T^\perp$ such that $Q(y) = -Q(w)$. Let $\langle x_1 - Q(y)x_{-1} + y, t\rangle$ be an axis line called l_1. Let the axis line

$$\langle x_1 + Q(y)x_{-1} + w, x_2\rangle = l_2 .$$

If $t \notin l_2^\perp$, then by the lemma, $\langle s, t\rangle$ is an axis line where $\langle s\rangle$ is the point on l_2 in $\langle t\rangle^\perp$. But then since $s \in T$, $t \in T$, causing a contradiction since t is on an axis line with a point outside T. Hence $l_1 \cap l_2^\perp = l_1$. If l_1 and l_2 have no points in common, the index of V is 4. If they have a point in common, some point of l_1 is in T, also causing a contradiction. This proves our theorem.

Appendix. $H = SU(U)$

Let α be a zero of $X^2 - aX - b$ which is irreducible over F, and let $\beta = \alpha - a/2$. Let σ be the automorphism of $F(\alpha)$ sending α to its conjugate and preserving F. Let B' be a hermitian form on the $2k$ (resp. $2k + 1$) dimensional vector space U over $F(\alpha)$ such that (1) $B'(ax, y) = \sigma(a)B'(x, y)$ for $a \in F(\alpha)$ and (2) $B'(e_i, e_{-j}) = 1$ for $1 \leq i \leq k$ (and resp $B'(e_0, e_0) = 2$) and other products of basis vectors are 0.

We show here that the group H generated by those transvections with centers

$$\langle e_i\rangle, \langle e_{-i}\rangle, \langle e_1 + e_i\rangle, \langle e_1 + \alpha e_i\rangle\ 1 \leq i \leq k \text{ and } \langle e_1 - (1 + \tfrac{1}{2}a(\alpha - \tfrac{1}{2}a)^{-1})e_{-1} + e_0\rangle$$

is transitive on the isotropic points of U. This is sufficient to show $H = SU(U)$ by [3, pp. 43-47].

Let dim $U = n$. We prove that H is transitive on the isotropic points first for $n = 2$ and 3, and then we use induction on n. For $n = 2$, $ae_1 + be_{-1}$ is isotropic only if $a = 0$ or $ba^{-1} = k\beta$ where $k \in F$. A unitary transvection centered at $\langle e_{-1}\rangle$ sends $\langle e_1\rangle$ to $\langle e_1 + k\beta e_{-1}\rangle$. Similarly a unitary transvection centered at $\langle e_1\rangle$ sends

$$\langle e_{-1}\rangle \quad \text{to} \quad \langle e_{-1} + k\beta e_1\rangle = \langle t\beta e_{-1} + e_1\rangle$$

for $t = k^{-1}\beta^{-2} \in F$. So all isotropic points are in the same orbit.

For $n = 3$ we first show that $SU(W_0)$ where W_0 has dimension 2 is transitive on non-isotropic vectors of a given length. Let $ae_1 + be_{-1}$ where a, $b \in F(\alpha)^*$ be an arbitrary non-isotropic vector in W_0. Then $a \neq k\beta b$ $(k \in F)$. Suppose τ in $U(W_0)$ fixes $ae_1 + be_{-1}$. We show τ can have arbitrary determinant.

Suppose τ sends e_1 to the isotropic vector $ce_1 + de_{-1}$. Then since

$$ce_1 + de_{-1} = (c - dab^{-1})e_1 + db^{-1}(ae_1 + be_{-1}),$$

we see $\det \tau = c - dab^{-1}$. Since

$$B'(\tau(e_1), ae_1 + be_{-1}) = B'(e_1, ae_1 + be_{-1}),$$

we obtain $b = \sigma(c)b + \sigma(d)a$. Since $ce_1 + de_{-1}$ is isotropic, either (1) $c = 0$, or (2) $d = k\beta c$ where $k \in F$. In case (1),

$$\det \tau = -da/b = -d/\sigma(d).$$

In case (2), $\det \tau = c(1 - k\beta a/b)$ and $b = \sigma(c)b(1 - k\beta a/b)$ yield

$$\det \tau = c/\sigma(c).$$

But the determinant y of any element in the unitary group is a unit, (i.e., $y\sigma(y) = 1$), and any unit can be written in this form. For let $\gamma = yd + \sigma(d)$. If $y \neq -\sigma(d)/d$, then $\gamma \neq 0$ and $y = \gamma/\sigma(\gamma)$. So a proper choice of c and d yields any determinant.

By Witt's theorem [4, p. 21], there is an element σ' in $U(W_0)$ which sends any non-singular vector to any other vector of the same length. If $\sigma'(x) = y$ ($B'(x, x) \neq 0$) and $\det \sigma' = s \neq 1$, then, by the above, there is a τ such that $\tau\sigma'(x) = y$ and $\tau\sigma' \in SU(W_0)$.

Now any isotropic point in U (dim 3) is either of form $\langle e_0 + x\rangle$ where x is a vector of length -2 in W_0, or else is already in W_0. We have seen that all the isotropic points in W_0 are centers for H since the unitary transvections centered at $\langle e_1\rangle$ and $\langle e_{-1}\rangle$ generate $SU(W_0)$. But

$$\langle e_1 - (1 + \tfrac{1}{2}a(\alpha - \tfrac{1}{2}a)^{-1})e_{-1} + e_0\rangle$$

is also a center for H. So we are done for U 3-dimensional.

We are now ready for the induction step. Suppose we have $SU(T)$ plus the unitary transvections centered at $\langle e_1 + e_k\rangle$, $\langle e_1 + \alpha e_k\rangle$, $\langle e_k\rangle$ and $\langle e_{-k}\rangle$ where the dimension of T is $2k - 1$ or $2k - 2$. Let τ_1 be a unitary transvection with center $\langle e_{-k}\rangle$; let τ_2 be a unitary transvection with center $\langle e_{-1}\rangle$; let τ_3 be a unitary transvection with center $\langle e_1 + \alpha e_k\rangle$. Then the product $\tau_3 \tau_2 \tau_1$ sends $e_1 + e_k$ to a vector $x + y$ such that

$$x \in \langle e_1, e_{-1}\rangle, \quad y \in \langle e_k, e_{-k}\rangle$$

and $B'(x, x) = b = -B'(y, y) \neq 0$. But by the argument above, $SU(T)$ is transitive on non-singular vectors of a given length when $\dim T \geq 2$. So we have all vectors $x' + y'$ such that $x' \in \langle e_k, e_{-k}\rangle^{\perp}$ (where $^{\perp}$ is with respect to B') $y' \in \langle e_k, e_{-k}\rangle$, $B'(x', x') = b = -B'(y', y')$. But then we have representatives of all centers of form

$$\langle x + y\rangle, \quad x \in \langle e_k, e_{-k}\rangle^{\perp}, \quad y \in \langle e_k, e_{-k}\rangle,$$

x and y non-isotropic.

We have only to obtain the centers of form

$$\langle x + y\rangle, \quad x \in \langle e_k, e_{-k}\rangle^{\perp}, \quad y \in \langle e_k, e_{-k}\rangle,$$

x and y isotropic. The trouble here is that $SU(W_0)$ is not transitive on isotropic vectors when $\dim W_0 = 2$.

If the dimension of U is ≥ 3, then $SU(U)$ is transitive on the isotropic vectors of U. For by Witt's theorem the group of unitary transformations on U, $U(U)$ is transitive on the isotropic vectors of U. Suppose $\tau \in U(U)$ and sends the isotropic vector y to z in U. Then one can find a $\tau' \in U(U)$ fixing vectorwise a 2-dimensional non-degenerate space containing z and with determinant $= (\det \tau)^{-1}$. Then $\tau'\tau$ sends y to z and has determinant 1. We know we have any center of form $\langle x + y\rangle$,x and y non-isotropic $x \in \langle e_1, e_{-1}\rangle$, $y \in \langle e_k, e_{-k}\rangle$. Say

$$x = ae_{-1} + \beta(a + b)e_1 \quad \text{and} \quad y = be_{-k} + \beta(b + a)e_k$$

where $a, b \in F(\alpha)^*$ and $ab^{-1} \notin F$. If τ sends z in U to

$$z - \beta B'(e_1 + e_k, z)(e_1 + e_k),$$

then τ sends $x + y$ to $ae_{-1} + be_{-k}$. For fixed a, be_{-k} runs through representatives of all but one of the orbits of singular vectors under $SU(W_0)$ where $W_0 = \langle e_k, e_{-k}\rangle$. This orbit is the one where $ab^{-1} \in F$. Let τ_1 send z to

$$z + g^{-1}\beta B'(\beta e_1 + e_{-1}, z)(\beta e_1 + e_{-1}).$$

Let τ_2 send z to $z + g^{-1}\beta B'(\beta e_k + e_{-k}, z)(\beta e_k + e_{-k})$. Then $\tau_2\tau_1$ sends $e_1 + e_k$ to

$$g^{-1}\beta e_{-1} + g^{-1}\beta e_{-k} \in \langle ae_{-1} + ae_{-k}\rangle.$$

So we are done.

Bibliography

1. E. Artin, *Geometric algebra*, Interscience, New York, 1964.
2. L. E. Dickson, *Theory of linear groups in an arbitrary field*, Trans. Amer. Math. Soc., vol. 2 (1901), pp. 363–394.
3. J. Dieudonne, *La Geometrie des Groupes Classiques*, Ergebnisse der Math., second edition, Springer Verlag, Berlin, 1963.
4. J. McLaughlin, *Some groups generated by transvections*, Arch. Math. (Basel), Vol. XVIII (1967), pp. 364–368.
5. ———, *Some subgroups of $SL_n(F_2)$*, Illinois J. Math., vol. 13 (1969), pp. 108–115.
6. H. Pollatsek, *Groups generated by transvections over the field of two elements*, J. Algebra, vol. 16 (1970), pp. 561–574.
7. R. Ree, *On some simple groups defined by C. Chevalley*, Trans. Amer. Math. Soc., vol. 84 (1957), pp. 392–400.
8. B. S. Stark, *Some subgroups of $\Omega(V)$ generated by groups of root type*, J. Algebra, to appear.
9. R. Steinberg, *Variations on a Theme of Chevalley*, Pacific J. Math., vol. 9 (1959), pp. 875–891.
10. T. Tamagawa, *On the structure of the orthogonal groups*, Amer. J. Math., vol. 80 (1958), 191–197.
11. J. Thompson, *Quadratic pairs*, to appear.

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