THE GENUS OF AN ORIENTABLE 3-MANIFOLD WITH CONNECTED BOUNDARY

BY

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The purpose of this paper is to relate several generalizations of the notion of the Heegaard genus of a closed 3-manifold to compact, orientable 3-manifolds with connected, nonempty boundary.

All spaces considered will be polyhedra and all maps will be piecewise linear. By a *solid torus of genus n* we mean a space homeomorphic to a regular neighborhood in $\mathbf{R}^3$ of a compact, connected graph with Euler characteristic $1 - n$. The Euler characteristic of any space X will be denoted $\chi(X)$. If D is a 2-cell, then $N(D)$ will denote a space homeomorphic to $D \times [-1, 1]$ where D corresponds to $D \times \{0\}$.

It is well known that any compact, orientable 3-manifold with nonempty connected boundary can be represented as $H \cup N(D_1) \cup \cdots \cup N(D_k)$ where H is a solid torus, D_i is a 2-cell for each i, $N(D_i) \cap N(D_j) = \emptyset$ if $i \neq j$ and $N(D_i) \cap H = \partial N(D_i) \cap \partial H$ corresponds to $\partial D_i \times [-1, 1]$ in $N(D_i)$. This will be called a *Heegaard splitting* (or *H-splitting*) for M, and $N(D_i)$ is called a *handle of index* 2. The *genus of the splitting* is the genus of H and the smallest possible genus of an H-splitting of M will be denoted $HG(M)$.

Downing [1] has shown that M may also be represented as $H_1 \cup H_2$ where H_1 and H_2 are solid tori of the same genus and $H_1 \cap H_2 = \partial H_1 \cap \partial H_2$. This may always be done so that $\partial H_j \cap \partial M$ is a disk with holes such that $\pi_1(\partial H_j \cap \partial M)$ injects naturally onto a free factor of $\pi_1(H_j)$ for $j = 1, 2$. In this case, we call this an *SD-splitting* of M and denote the minimal genus of such a splitting for M by $SD(M)$. If we require only that $\pi_1(\partial H_j \cap \partial M)$ injects naturally into $\pi_1(H_j)$, $j = 1, 2$, we call this a *D-splitting* and the minimal genus of any D-splitting for M is denoted $DG(M)$. If X is a subspace of Y, $N_Y(X)$ will denote a regular neighborhood of X in Y taken to be "small" with respect to all previously chosen objects in a given argument. The closure of any set A will be denoted $\mathrm{Cl}(A)$.

If F is a compact orientable surface of genus g with k boundary components, then $\chi(F) = 2 - 2g - k$ and $\pi_1(F)$ is free of rank $2g + k - 1$.

THEOREM 1. *Let M be a compact, orientable 3-manifold with connected nonempty boundary of genus k. Let $M = H_1 \cup H_2$ be an SD-splitting of M of genus n. Then M has an H-splitting of genus n.*

Proof. Let $K_i = \partial H_i \cap \partial M$ $(i = 1, 2)$. Then each K_i is a disk with k holes and $\mu_*(\pi_1(K_1))$ is a free factor of $\pi_1(H_1)$ where $\mu_* : \pi_1(K_1) \to \pi_1(H_1)$ is induced by inclusion. Now choose simple closed curves $\alpha_1, \cdots, \alpha_k$ in

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int (K_1) which meet only in the base point and which generate $\pi_1(K_1)$. This may be done so that the closure of each component of $K_1 - N_{K_1}(\bigcup_{i=1}^k (\alpha_i))$ is an annulus one of whose boundary components is a component of ∂K_1. Then [6] there exist properly embedded disks $D_1, \cdots, D_k$ in H_1 so that

$$\text{Cl}\ (H_1 - \textstyle\bigcup_{i=1}^k N_{H_1}(D_i))$$

is a solid torus of genus $(n - k)$, $D_i \cap \alpha_i$ is a point for each i, and $D_i \cap \alpha_j = \emptyset$ if $i \neq j$. Then, by an isotopy if necessary, $D_j \cap K_1 = \partial D_j \cap K_1$ may be taken to be a single simple arc properly embedded in K_1.

For $j = 1, \cdots, k$, let $\beta_j = \text{Cl}\ (\partial D_j - K_1)$. Then β_j is a simple arc in $\partial D_j \cap \partial H_2$. Now we find pairwise disjoint, properly embedded disks $D_{k+1}, \cdots, D_n$ in H_1 so that $\text{Cl}\ (H_1 - \bigcup_{i=1}^n N_{H_1}(D_i))$ is a 3-cell. Since

$$\text{Cl}\ (K_1 - \textstyle\bigcup_{i=1}^n N_{H_1}(D_i))$$

is a disk, we may assume $D_j \cap K_1 = \emptyset$ for $j = k + 1, \cdots, n$.

Now $H_2 \cup (\bigcup_{i=1}^n N_{H_1}(D_i)) \approx H_2 \cup (\bigcup_{i=k+1}^n N_{H_1}(D_i))$ is a solid torus of genus n with $(n - k)$ handles of index 2 attached and $\text{Cl}\ (H_1 - \bigcup_{i=1}^n N_{H_1}(D_i))$ is a 3-cell meeting this in a 2-cell on their common boundary. Hence,

$$M \approx H_2 \cup (\textstyle\bigcup_{i=1}^n N_{H_1}(D_i)) \approx H_2 \cup (\bigcup_{i=k+1}^n N_{H_1}(D_i)). \qquad \square$$

COROLLARY. *If M is a compact, orientable 3-manifold with connected, non-empty boundary, then $HG(M) \leq SD(M)$.*

THEOREM 2. *Let M be a compact, orientable 3-manifold with connected non-empty boundary of genus k. Suppose $M = H \cup N(D_1) \cup \cdots \cup N(D_{n-k})$ is an H-splitting for M of genus n. Then M has a D-splitting of genus n.*

Proof. If $n - k = 0$, the result is trivial, so assume $n - k \geq 1$. Let

$$S = \text{Cl}\ (\partial H - \textstyle\bigcup_{i=1}^{n-k} N(D_i)).$$

Then S is an orientable surface of genus k with $2(n - k)$ boundary components, say $\alpha_1, \beta_1, \cdots, \alpha_{n-k}, \beta_{n-k}$ where $\alpha_i \cup \beta_i \subset \partial N(D_i)$ for $i = 1, \cdots, n - k$.

Now we choose simple, properly embedded, pairwise disjoint arcs $\gamma_1, \cdots, \gamma_n$ in S so that each γ_i joins some α_j to β_j and $T' = \text{Cl}\ (S - \bigcup_{i=1}^n N_S(\gamma_i))$ is connected. Now $\chi(S) = 2 - 2n$ and $\chi(T') = 2 - 2n + n = 2 - n$. This may be done so that T' has n boundary components and is a surface of genus 0. Now, as indicated in Figure 1, choose properly embedded, pairwise disjoint arcs $\delta_1, \cdots, \delta_{n-k-1}$ in T' so that each δ_i joins some γ_j to γ_r $(j \neq r)$ and $T = \text{Cl}\ (T' - \bigcup_{i=1}^{n-k-1} N_{T'}(\delta_i))$ is connected. Then T is a disk with k holes and the inclusion induced homomorphism $\mu* : \pi_1(T) \to \pi_1(S)$ is an injection.

Now we assume that the inclusion induced homomorphism $\nu* : \pi_1(S) \to \pi_1(H)$ is an injection. Then $\nu*\mu* : \pi_1(T) \to \pi_1(H)$ is an injection. Let

$$H_1 = (\textstyle\bigcup_{i=1}^{n-k} N(D_i)) \cup (\bigcup_{i=1}^n N_H(\gamma_i)) \cup (\bigcup_{i=1}^{n-k-1} N_H(\delta_i))$$

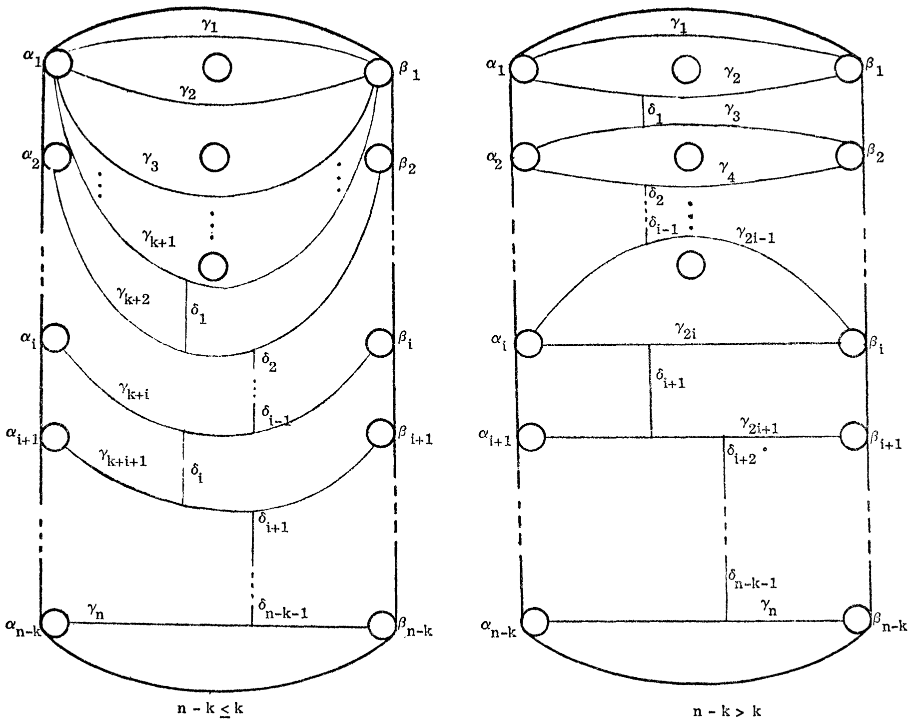

FIGURE 1.

where

$$[(\bigcup_{i=1}^{n} N_H(\gamma_i)) \cup (\bigcup_{i=1}^{n-k-1} N_H(\delta_i))] \cap S = (\bigcup_{i=1}^{n} N_S(\gamma_i)) \cup (\bigcup_{i=1}^{n-k-1} N_{T'}(\delta_i)).$$

Let $H_2 = \mathrm{Cl}\,(H - H_1)$. Then H_1 and H_2 are solid tori of genus n and $M = H_1 \cup H_2$.

Since the pair $(H_2, H_2 \cap \partial M)$ is homeomorphic to (H, T). we have that $\pi_1(H_2 \cap \partial M)$ injects into $\pi_1(H_2)$. Now

$$H_1 \cap \partial M = (\bigcup_{i=1}^{n-k} (D_i \times \{-1, 1\})) \cup (\bigcup_{i=1}^{n} N_S(\gamma_i)) \cup (\bigcup_{i=1}^{n-k-1} N_{T'}(\delta_i))$$

is connected, has $k + 1$ boundary components and $\chi(H_1 \cap \partial M) = 2 - (k + 1)$. Hence, $H_1 \cap \partial M$ is a disk with k holes. By the construction of H_1 we also have that the inclusion induced homomorphism $\pi_1(H_1 \cap \partial M) \to \pi_1(H_1)$ is injective. Hence, M has a D-splitting of genus n.

If $\nu* : \pi_1(S) \to \pi_1(H)$ is not injective, we find by Dehn's lemma [5] and the loop theorem [4] a simple closed curve J in S that does not contract in S but bounds a disk E in H. Cutting along E, either we separate M into manifolds M_1 and M_2 with H-splittings of genuses n_1, n_2 (both >0) so that $n_1 + n_2 = n$ or we remove a handle of index 1 from M to get a manifold M_1 with an H-splitting of genus $n - 1$.

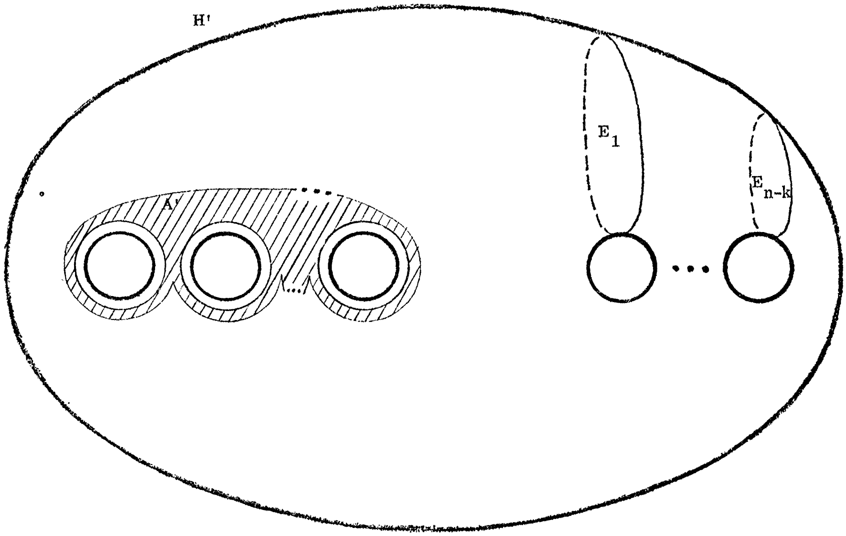

FIGURE 2.

Now by [2], if $H_1 \cup H_2$ is a D-splitting for M_i, any disk or pair of disks in ∂M_i can by an isotopy be assumed to meet $H_j \cap \partial M_i$ in a disk for $j = 1, 2$. Hence, by induction on n and the fact that the theorem is trivial if $n = 1$, we are finished. □

COROLLARY. *If M is a compact, orientable 3-manifold with connected, nonempty boundary, then $DG(M) \leq HG(M)$.*

We now give a partial converse to Theorem 1.

PROPOSITION 3. *Let M be a compact, orientable 3-manifold with connected, nonempty boundary of genus k. Let $M = H \cup N(D_1) \cup \cdots \cup N(D_{n-k})$ be an H-splitting for M of genus n. Suppose K is a surface of genus 0 with $k + 1$ boundary components in $\partial H - \bigcup N(D_i)$. Further assume that the inclusion induced map $\pi_1(K) \to \pi_1(H)$ is an injection onto a free factor of $\pi_1(H)$ and that*

$$\partial H - (K \cup N(D_1) \cup \cdots \cup N(D_{n-k}))$$

is connected. Then M has an SD-splitting of genus n.

Proof. Let H' be a solid torus of genus n as in Figure 2. For each $i = 1, \cdots, n - k$, let J_i be a simple closed curve in $N(D_i) \cap H$ so that $N(D_i) \cap H = N_{\partial H}(J_i)$. Then there is a homeomorphism $h : \partial H' - \text{Int } A' \to \partial H - \text{Int } K$ such that $h(\partial E_i) = J_i$ for $i = 1, \cdots, n - k$.

Let $M' = H \cup_h H'$. Then this gives an SD-splitting of M' of genus n. However, H' collapses to $(\partial H' - \text{Int } A') \cup E_1 \cup \cdots \cup E_{n-k}$ and so M' collapses to $H \cup E_1 \cup \cdots \cup E_{n-k}$. Hence M' is homeomorphic to M. □

COROLLARY. *Let $M = H \cup N(D)$ where H is a solid torus of genus 2 and*

∂M is connected. Suppose K is a simple closed curve in $\partial H - N(D)$ which represents a primitive element for $\pi_1(H)$. Then M has an SD-splitting of genus 2.

Proof. If $\partial H - (N(D) \cup K)$ is not connected, then K and one component of $\partial N(D) \cap \partial H$ cobound an annulus. Hence, $\partial N(D) \cap \partial H$ represents a primitive element of $\pi_1(H)$ and we may choose a new curve K' which represents a complementary primitive element. Therefore we may assume that $\partial H - (N(D) \cup K)$ is connected and Proposition 3 may be applied.

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