LIE ALGEBRA COHOMOLOGY AND AFFINE ALGEBRAIC GROUPS

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1. Introduction

Let L be a finite-dimensional Lie algebra over a field F of characteristic 0. Let $\mathfrak{M}(L)$ denote the category of L-modules. The Lie algebra cohomology functor $H(L,\ )$ is a functor from $\mathfrak{M}(L)$ to the category of graded F-spaces, which we view as the sequence of injectively derived functors of the functor $H^0(L,\ )$, where, for every L-module V, the F-space $H^0(L, V)$ is the L-annihilated part V^L of V.

Replacing $\mathfrak{M}(L)$ with the category $\mathfrak{M}_f(L)$ of locally finite L-modules (sums of finite-dimensional L-modules), we obtain an analogous functor $H_f(L,\ )$. There is an evident natural transformation from $H_f(L,\ )$ to $H(L,\ )$, and we have $H_f^0(L, V) = V^L = H^0(L, V)$ for every locally finite L-module V.

Let A denote the radical of L. If V is an L-module then the action of L on V, together with the adjoint action of L on A, determines an L-module structure on $H(A, V)$, which factors through L/A. Taking the L-fixed part $H(A, V)^L$ for each V, we obtain a new functor $H(A,\ )^L$ from $\mathfrak{M}(L)$ to the category of graded F-spaces. The natural restriction map sends $H(L, V)$ into $H(A, V)^L$. Composing this with the natural map $H_f(L, V) \to H(L, V)$ for every locally finite L-module V, we obtain a natural transformation from $H_f(L,\ )$ to the functor $H(A,\ )^L$, restricted to $\mathfrak{M}_f(L)$. Our first main result is that this is actually a natural equivalence.

Let $\mathfrak{U}(L)$ denote the universal enveloping algebra of L. Endow $\mathfrak{U}(L)$ with the topology for which the two-sided ideals of finite codimension constitute a fundamental system of neighborhoods of 0. Then the continuous dual $\mathfrak{H}(L)$ of $\mathfrak{U}(L)$ is the Hopf algebra of representative functions on $\mathfrak{U}(L)$. In the case where F is algebraically closed, $\mathfrak{H}(L)$ is the algebra of polynomial functions of a connected pro-affine algebraic group P, whose elements are the F-algebra homomorphisms $\mathfrak{H}(L) \to F$. The category $\mathfrak{M}_f(L)$ is identifiable with the category of $\mathfrak{H}(L)$-comodules, and hence with the category of rational P-modules. Thus, the functor $H_f(L,\ )$ is naturally equivalent to the rational cohomology functor $H(P,\ )$.

Using the known structure theory of $\mathfrak{H}(L)$, we single out a naturally defined homomorphic image G of P, the *basic group* of L. This is an ordinary connected affine algebraic group whose Lie algebra contains L as an algebraically dense sub Lie algebra. In a sense made precise in §3, the rational cohomology of G is the same as that of P, and hence is given by the functor $H_f(L,\ )$ or $H(A,\ )^L$. Finally, we shall obtain a structural characterization of the basic group within the class of affine algebraic groups.

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For terminology and basic results concerning affine algebraic groups, as used here, we refer the reader to [5]. Thanks are due to M. E. Sweedler for his help in discussing the present material and, especially, for his clarification of the definition of the basic group.

2. Lie algebra cohomology

The equivalence of the functors $H_f(L, \quad)$ and $H(A, \quad)^L$ is obtained by combining an effaceability result for cohomology of solvable Lie algebras (due to J. L. Koszul and G. Leger) with a result on extendibility of representations (due to H. Zassenhaus). In a somewhat strengthened form, these results are exhibited, with proofs, in [2]. The following lemma is the formulation we need here.

LEMMA 2.1. *Let L be a finite-dimensional Lie algebra over a field of characteristic 0, and let A denote the radical of L. Let V be a locally finite L-module. Then there exists a locally finite L-module W containing V such that the canonical map $H^n(A, V) \to H^n(A, W)$ is the 0-map for every $n > 0$.*

Proof. Let S be any finite-dimensional L-submodule of V. By [2, Lemma 3], there is a finite-dimensional A-module S' containing S as an A-submodule such that the canonical map

$$H^n(A, S) \to H^n(A, S')$$

is the 0-map for every $n > 0$, and the representation of the commutator ideal $[L, A]$ on S' is nilpotent. Now [2, Lemma 4] applies to give the existence of a finite-dimensional L-module S^0 containing S' as an A-submodule and S as an L-submodule. Clearly, the canonical map $H^n(A, S) \to H^n(A, S^0)$ is the 0-map for every $n > 0$.

Now let us form the direct sum, Σ^0 say, of the family of S^0's, one for each finite-dimensional L-submodule S of V. This contains the direct sum, Σ say, of the family of the L-submodules S of V as an L-submodule. The injections $S \to V$ yield a surjective L-module homomorphism $\Sigma \to V$ in the natural way. Let Q denote the kernel of this homomorphism, and put $W = \Sigma^0/Q$. The injections $S^0 \to \Sigma^0$. followed by the canonical map $\Sigma^0 \to W$ are injective L-module homomorphisms $S^0 \to W$. The restrictions of these to the L-submodules S fit together to yield an injective L-module homomorphism $V \to W$, by means of which we identify V with its image in W.

Since L is finite-dimensional and V is locally finite, every element of $H^n(A, V)$ is the natural image of an element of $H^n(A, S)$, where S is some finite-dimensional L-submodule of V. Hence it is clear that W satisfies the requirements of Lemma 2.1.

THEOREM 2.2. *Let L be a finite-dimensional Lie algebra over a field F of characteristic 0, and let A be the radical of L. Let V be a locally finite L-module. Then the natural homomorphism*

$$H_f(L, V) \to H(L, V),$$

followed by the restriction homomorphism

$$H(L, V) \to H(A, V)^L,$$

is an isomorphism

$$H_f(L, V) \to H(A, V)^L,$$

whence there is a natural equivalence from the functor $H_f(L, \ \ )$ *to the restriction to* $\mathfrak{M}_f(L)$ *of the functor* $H(A, \ \ )^L$.

Proof. By Lemma 2.1, there is a locally finite L-module W containing V such that the canonical map $H^n(A, V) \to H^n(A, W)$ is the 0-map for every $n > 0$. By enlarging W, if necessary, we arrange that, furthermore, W is injective in the category $\mathfrak{M}_f(L)$. Now we show by induction on n that the natural homomorphism

$$H_f^n(L, V) \to H^n(A, V)^L$$

is an isomorphism. This is evident for $n = 0$.

Next, consider the case $n = 1$. We have the exact cohomology sequence

$$W^A \to (W/V)^A \to H^1(A, V) \to H^1(A, W).$$

By the choice of W, the last map here is the 0-map. All the maps are L-module homomorphisms, and L acts throughout via L/A. Since the modules are locally finite, they are therefore semisimple as L-modules. It follows that the induced sequence for the L-annihilated parts is still exact. On the other hand, we have a similar exact sequence of locally finite cohomology, as follows

$$W^L \to (W/V)^L \to H_f(L, V) \to H_f(L, W)$$

where the last term on the right is (0), because W is an injective of the category $\mathfrak{M}_f(L)$. Now these sequences fit together to make up the following commutative and exact diagram, in which the vertical map is the natural one:

$$\begin{array}{ccccc} & & H^1(A, V)^L & & \\ & \nearrow & \uparrow & \searrow & \\ W^L \to (W/V)^L & & & & (0) \\ & \searrow & & \nearrow & \\ & & H_f^1(L, V) & & \end{array}$$

From this, we see directly that the vertical map is an isomorphism, so that the case $n = 1$ is established.

Now suppose that $n > 1$ and that the result has been proved in the lower cases. Proceeding as above, we obtain the following commutative and exact diagram

$$\begin{array}{ccccccc} H^{n-1}(A, W)^L & \to & H^{n-1}(A, W/V)^L & \to & H^n(A, V)^L & \to & H^n(A, W)^L \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H_f^{n-1}(L, W) & \to & H_f^{n-1}(L, W/V) & \to & H_f^n(L, V) & \to & H_f^n(L, W). \end{array}$$

Since W is an injective of the category $\mathfrak{M}_f(L)$, the first and last term of the bottom row are both equal to (0). By the choice of W, the last map of the top row is the 0-map. By inductive hypothesis, the two vertical maps on the left are isomorphisms. Hence the diagram shows that the map

$$H_f^n(L, V) \to H^n(A, V)^L$$

is also an isomorphism. This completes the proof of Theorem 2.2.

3. The basic group

We continue the discussion of the end of the introduction, using the same notation. It will be convenient to view $\mathcal{H}(L)$ as a two-sided $\mathcal{U}(L)$-module. The left and right transforms $u \cdot f$ and $f \cdot u$ of an element f of $\mathcal{H}(L)$ by an element u of $\mathcal{U}(L)$ are given by $(u \cdot f)(x) = f(xu)$ and $(f \cdot u)(x) = f(ux)$. In each of these two ways, L acts on $\mathcal{H}(L)$ by F-algebra derivations. Let A again stand for the radical of L. The A-annihilated part $\mathcal{H}(L)^A$ of the left $\mathcal{U}(L)$-module $\mathcal{H}(L)$ is a sub Hopf algebra of $\mathcal{H}(L)$, which may be identified with $\mathcal{H}(L/A)$. Since A is an ideal of L, the left annihilated part $\mathcal{H}(L)^A$ actually coincides with the right annihilated part. Another important sub Hopf algebra of $\mathcal{H}(L)$ is the algebra C of the *trigonometric functions* (in the terminology of [3] and [4]). The elements of C are the representative functions associated with the semisimple representations of L that are trivial on the commutator ideal $[L, L]$. Quite generally, an element f of $\mathcal{H}(L)$ is associated with a semisimple representation of L if and only if the left $\mathcal{U}(L)$-module $\mathcal{U}(L) \cdot f$ is semisimple, or if and only if the right $\mathcal{U}(L)$-module $f \cdot \mathcal{U}(L)$ is semisimple. We say then that f is a *semisimple element* of $\mathcal{H}(L)$. The well-known fact that, over a field of characteristic 0, the tensor product of two finite-dimensional semisimple L-modules is still semisimple implies that the semisimple elements of $\mathcal{H}(L)$ constitute a subalgebra, and hence even a sub Hopf algebra of $\mathcal{H}(L)$. In fact, this sub Hopf algebra is $C(\mathcal{H}(L)^A) = C \otimes \mathcal{H}(L)^A$, as can be seen from Jacobson's structure theorem on fully reducible Lie algebras of linear endomorphisms.

The following result is contained in [3] and [4]. There is a right $\mathcal{U}(L)$-stable subalgebra B of $\mathcal{H}(L)$ satisfying the following conditions:

(1) B is finitely generated as an F-algebra.
(2) $\mathcal{H}(L) = C \otimes B$, and hence B separates the elements of L.
(3) The set B_s of all semisimple elements of B coincides with $\mathcal{H}(L)^A$.

Since B is finitely generated as an F-algebra, so is the smallest sub Hopf algebra of $\mathcal{H}(L)$ that contains B. We denote this sub Hopf algebra by R. It is known that B is unique up to left translations by elements of P, whence R is uniquely determined. The following characterization of R, for which I am indebted to M. E. Sweedler, is independent of the unicity result concerning B.

Let V be any non-zero right $\mathcal{U}(L)$-submodule of B, and let W be a $\mathcal{U}(L)$-submodule of V that is of minimal positive dimension. Then W is simple as a right $\mathcal{U}(L)$-module, whence $W \subset B_s = \mathcal{H}(L)^A$. Thus, in the category of right $\mathcal{U}(L)$-modules, B is an *essential extension* of $\mathcal{H}(L)^A$, in the sense that every non-zero submodule of B has a non-zero intersection with $\mathcal{H}(L)^A$. On the other hand, it is clear from property (2) of B that B is a direct right $\mathcal{U}(L)$-module summand of $\mathcal{H}(L)$. Hence B is an injective of the category of locally finite right $\mathcal{U}(L)$-modules. It is well known from the theory of Eckmann-Schopf [1] that the isomorphism class of an injective essential extension of a module is uniquely determined by that module. Hence the representative functions on $\mathcal{U}(L)$ that are associated with the right $\mathcal{U}(L)$-module B do not depend on the particular choice of B. These functions are precisely the elements of the smallest two-sidedly $\mathcal{U}(L)$-stable subspace of $\mathcal{H}(L)$ that contains B, i.e., they are the elements of $\mathcal{U}(L)\cdot B$. If η denotes the antipode of $\mathcal{H}(L)$, we have $R = (\mathcal{U}(L)\cdot B)\eta(\mathcal{U}(L)\cdot B)$. *Thus, R is the unique smallest sub Hopf algebra of $\mathcal{H}(L)$ containing an injective extension of the right $\mathcal{U}(L)$-module $\mathcal{H}(L)^A$* (every injective extension contains an injective essential extension).

In order to simplify our statements, we shall *assume from now on that F is algebraically closed.* We define the *basic group associated with L* as the affine algebraic group of all algebra homomorphisms $R \to F$. As is well known, $\mathcal{H}(L)$ is an integral domain, so that the corresponding pro-affine algebraic group P is connected. The basic group is the restriction image of P. It is a connected affine algebraic group in the usual sense. Let G denote the basic group, and let T denote the kernel of the restriction morphism $P \to G$.

Consider the action of T by left translations on $\mathcal{H}(L)$. Clearly, the sub Hopf algebra C is stable under the action of P. Viewed as functions on P, the elements of C are linear combinations of rational characters (group homomorphisms of P into the multiplicative group of F). In particular, it follows that the commutator group $[P, T]$ acts trivially on C. Since it acts trivially also on R, and so on B, and since $\mathcal{H}(L) = C \otimes B$, it must be trivial. Thus, T is contained in the center of P. Moreover, since T is determined by its action on C, we see that T is a *reductive* algebraic subgroup of P, in the sense that every rational representation of T is semisimple.

Now let V be any locally finite L-module. We may view V as a rational P-module. Since T is a reductive central algebraic subgroup of P, we have the canonical direct P-module decomposition $V = V^T + V_T$, where V^T is the T-fixed part of V and V_T is its unique T-module complement in V. The rational P-module V^T may be regarded as a rational G-module in the natural way. Using such a decomposition also for a rationally injective resolution of the rational P-module V, we find by familiar reasoning that the inflation map $H(G, V^T) \to H(P, V)$ is an isomorphism (note that $H(P, V) = H(P, V)^T = H(P, V^T)$, and that the T-fixed part of any rationally injective rational P-module is rationally injective as a G-module). On the other hand, if M is any rational G-module then we may regard M as a rational P-module, with $M = M^T$. In summary, we have the following result.

THEOREM 3.1. *Let L be a finite-dimensional Lie algebra over an algebraically closed field F of characteristic 0. Let G be the basic group associated with L, and* *let T be the kernel of the restriction map $P \to G$. On the category $\mathfrak{M}_f(L)$, we have natural isomorphisms $H_f(L, V)$ $H(G, V^T)$ establishing a natural equivalence between the functors $H_f(L, \ )$ and $H(G, \ ^T)$. Viewing rational G-modules as locally finite L-modules thereby gives a natural equivalence between the functor $H(G, \ )$ and the appropriate restriction of the functor $H_f(L, \ )$.*

Finally, we give a characterization of the basic group within the class of affine algebraic groups. The Lie algebra of our basic group G will be denoted $\mathfrak{L}(G)$. If K is any affine algebraic group and M is a sub Lie algebra of the Lie algebra $\mathfrak{L}(K)$ of K then M is called an *algebraic* sub Lie algebra if it is the Lie algebra of an algebraic subgroup of K, and M is said to be *algebraically dense* in $\mathfrak{L}(K)$ if the smallest algebraic sub Lie algebra of $\mathfrak{L}(K)$ that contains M coincides with $\mathfrak{L}(K)$.

THEOREM 3.2. *Let F, L, G be as in Theorem 3.1. Then L is algebraically dense in $\mathfrak{L}(G)$, and the intersection of the center of G with the radical of G is unipotent. If K is any connected affine algebraic group over F satisfying these conditions then there is a surjective morphism of affine algebraic groups $G \to K$ whose differential coincides with the identity map on L.*

Proof. It is clear from the definition of G that L may be identified with a sub Lie algebra of $\mathfrak{L}(G)$ and, as such, is algebraically dense in $\mathfrak{L}(G)$. Let Z denote the center of G, and let Z_s be the algebraic subgroup consisting of all the semisimple elements of Z. Then Z is the direct product of Z_s and an algebraic vector group $V = Z \cap G_u$, where G_u denotes the unipotent radical (not necessarily the full radical) of G. As before, let P be the pro-affine algebraic group of all F-algebra homomorphisms $\mathfrak{H}(L) \to F$, and let T denote the kernel of the restriction morphism $P \to G$. Let Z_s^0 be the inverse image of Z_s in P, so that the restriction morphism maps Z_s^0 onto Z_s and $T \subset Z_s^0$. Since Z_s and T are reductive, so is Z_s^0. Moreover, Z_s^0 is contained in the center of P, because $[Z_s^0, P]$ acts trivially on both B and C.

For any Hopf algebra E, let us denote the group of all algebra homomorphisms $E \to F$ by $\mathcal{G}(E)$. Let U be the kernel of the restriction morphism $Z_s^0 \to \mathcal{G}(\mathfrak{H}(L)^A)$, where A is the radical of L. Then U is a reductive central subgroup of P so that the U-fixed (right and left) part $\mathfrak{H}(L)^U$ is a direct (right and left) $\mathfrak{U}(L)$-module summand of $\mathfrak{H}(L)$. In particular, $\mathfrak{H}(L)^U$ is therefore an injective of the category of locally finite right or left $\mathfrak{U}(L)$-modules. Since it contains $\mathfrak{H}(L)^A$, we have from our above characterization of R that $R \subset \mathfrak{H}(L)^U$. This implies that the restriction morphism $Z_s \to \mathcal{G}(\mathfrak{H}(L)^A)$ is injective.

The Lie algebra of $\mathcal{G}(\mathfrak{H}(L)^A)$ is isomorphic with the semisimple Lie algebra L/A, as follows from the fact that a semisimple Lie algebra is universally algebraic. Hence the radical of G is contained in the kernel of the restriction morphism $G \to \mathcal{G}(\mathfrak{H}(L)^A)$. Since this restriction morphism is injective on Z_s, we conclude that the intersection of Z_s with the radical of G is trivial.

Therefore, the intersection of Z with the radical of G is precisely the vector group V, and thus is unipotent.

Now let K be any connected affine algebraic group over F such that L is algebraically dense in $\mathcal{L}(K)$ and the intersection of the center of K with the radical of K is unipotent. Let Q denote the Hopf algebra of polynomial functions of K. Since $L \subset \mathcal{L}(K)$, we may view Q as a left $\mathcal{U}(L)$-module in the natural fashion. This yields a Hopf algebra homomorphism $\rho : Q \to \mathcal{H}(L)$ such that, for q in Q and u in $\mathcal{U}(L)$, we have $\rho(q)(u) = c(u \cdot q)$, where c is the co-unit (augmentation) of $\mathcal{H}(L)$. The assumption that L is algebraically dense in $\mathcal{L}(K)$ implies that ρ is injective. Hence we may identify Q with a sub Hopf algebra of $\mathcal{H}(L)$, so that K becomes identified with the restriction image P_Q of P. From the assumption that L is algebraically dense in $\mathcal{L}(K)$ it follows also that the radical of K is the smallest algebraic subgroup of K whose Lie algebra contains the radical A of L. Let K' denote the factor group of K modulo its radical. Then we have $K' = \mathcal{G}(Q^A)$.

Let S be the group of all semisimple elements of the center of K. Our unipotency condition on K implies that the intersection of S with the radical of K is trivial. Hence the restriction to S of the canonical map $K \to K'$ is injective. Let t be any element of T. Then t is a semisimple element of the center of P, whence its restriction image t_Q belongs to S. The canonical image of t_Q in K' is the restriction image t_{Q^A}, which is the neutral element because $Q^A \subset \mathcal{H}(L)^A \subset R$. Hence we conclude that t_Q is the neutral element of K. Thus we have shown that the kernel T of the restriction morphism $P \to \mathcal{G}(R)$ is contained in the kernel of the restriction morphism $P \to \mathcal{G}(Q)$, so that it coincides with the kernel of the restriction morphism $P \to \mathcal{G}(QR)$. Since F is of characteristic 0, this implies that $QR = R$, i.e., that $Q \subset R$. Thus, K is the image of the restriction morphism $\mathcal{G}(R) \to \mathcal{G}(Q)$, and Theorem 3.2 is proved.

The structure of the basic group G can be analysed further. In particular, it can be seen from the construction of the subalgebra B of $\mathcal{H}(L)$, as carried out in [3] and [4], that *the dimension of the unipotent radical of G is equal to the dimension of the radical of L.* On the other hand, *the dimension of G is equal to that of L* (so that $L = \mathcal{L}(G)$) *if and only if the radical of L is nilpotent.*

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