COMPARISON THEOREMS FOR ELLIPTIC AND PARABOLIC INEQUALITIES

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A recent comparison theorem of Kurt Kreith [2] will be extended to quasilinear elliptic and parabolic differential inequalities of second order. Accordingly, Kreith's theorem is generalized in three directions at once. Our proof is extremely easy, reducing the proposition to the Hopf maximum principle (in the elliptic case [3, p. 67]) or the Friedman theorem (in the parabolic case [3, p. 174]). Our hypotheses are *explicit*, unlike Kreith's indirect hypotheses that "the boundary problems are sufficiently regular so that certain resolvents can be represented as integral operators."

Let L be the elliptic differential operator defined by

$$L(x, u) = \sum_{i,j=1}^{n} D_i[a_{ij}(x)D_j u] + \sum_{i=1}^{n} b_i(x)D_i u,$$
$$x = (x_1, \cdots, x_n), \quad D_i = \partial/\partial x_i, \quad i = 1, \cdots, n$$

for x in a bounded domain $G \subset R^n$. It is assumed that the coefficients a_{ij} and b_i are real-valued continuous functions on $\bar{G}$, and that the matrix (a_{ij}) is symmetric and uniformly positive definite in G (uniform ellipticity condition).

Let I be a real interval containing zero and let H be a domain in R^n. The differential inequalities to be compared are

$$L(x, u) \geq c(x, u, \nabla u)u, \qquad L(x, v) \leq c^*(x, v, \nabla v)v \tag{1}$$

where $\nabla u = (D_1 u, \cdots, D_n u)$ and c and c^* are continuous functions in $\bar{G} \times I \times H$. The respective solutions u and v are to satisfy the following homogeneous boundary conditions on ∂G:

$$\nu \cdot \nabla u + \sigma(x)u = 0, \qquad \nu \cdot \nabla v + \tau(x)v = 0, \tag{2}$$

where ν denotes the external unit normal to ∂G, and the directional derivatives are regarded as limits from within G. The functions σ and τ are supposed to be piecewise continuous on ∂G with values in $(-\infty, +\infty]$. As usual, the notation $\sigma(x^0) = +\infty$ denotes the boundary condition $u(x^0) = 0$.

THEOREM 1. *Let G be a bounded domain in R^n whose boundary has continuous curvature at every point. Let u and v satisfy the uniformly elliptic inequalities* (1) *in G and the boundary conditions* (2) *on ∂G, and suppose that $u(x)$ is positive throughout G. If*

(i) $c^*(x, v(x), \nabla v(x)) \leq c(x, u(x), \nabla u(x)), x \in G,$

(ii) $-\infty < \tau(x) \leq \sigma(x) \leq +\infty, x \in \partial G,$

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then $v(x)$ cannot be positive throughout $\bar{G}$ unless $v(x)$ is a constant multiple of $u(x)$.

Proof. If $v(x) > 0$ throughout $\bar{G}$, it follows from (1) that

$$vL(x, u) - uL(x, v) \geq [c(x, u, \nabla u) - c^*(x, v, \nabla v)]uv,$$

and hence that the function $w = u/v$ satisfies the differential inequality

$$\text{(3)} \quad \textstyle\sum_{i,j} D_i(a_{ij} v^2 D_j w) + \sum_i b_i v^2 D_i w + [c^*(x, v, \nabla v) - c(x, u, \nabla u)]v^2 w \geq 0.$$

On account of hypothesis (i), the Hopf maximum principle [3, p. 67] shows that w is either identically constant in $\bar{G}$ or $\nu \cdot \nabla w(x^0) > 0$ at any maximum point x^0 of w on ∂G. However, (2) and hypothesis (ii) imply the inequality

$$\nu \cdot \nabla w = [\tau(x) - \sigma(x)]w \leq 0 \quad \text{on} \quad \partial G,$$

providing a contradiction unless u/v is constant throughout $\bar{G}$.

An inequality similar to (3) has been used by Protter and Weinberger [4] to establish lower bounds for eigenvalues.

COROLLARY. *Let G be as in Theorem 1. Suppose that u is a positive solution of the first inequality (1) in G and that v satisfies the uniformly elliptic linear differential equation $L(x, v) = c^*(x)v$ in G. If*

(i) $c^*(x) \leq c(x, u(x), \nabla u(x)), x \in G$,

(ii) *u and v satisfy the boundary conditions (2), where $\tau(x) < \infty$ and*

$$-\infty < \tau(x) \leq \sigma(x) \leq +\infty, \quad x \in \partial G,$$

then either $v(x)$ changes sign in G or $v(x)$ is a constant multiple of $u(x)$ in $\bar{G}$.

Proof. If $v(x) > 0$ in G, also $v(x)$ cannot have a zero $x^0 \in \partial G$, for if $v(x^0) = 0$, then $(\nu \cdot \nabla v)(x^0) < 0$ [6, Lemma on p. 246], contradicting the second boundary condition (2). Theorem 1 shows, therefore, that $v(x)$ cannot be positive throughout G unless $v(x)$ is a constant multiple of $u(x)$. In the present linear case, $v(x)$ cannot be negative throughout G either since $-v(x)$ satisfies the same differential equation as $v(x)$. Since $v(x)$ cannot have a zero minimum (or maximum) in G by the lemma quoted above [6, p. 246], it must either change sign in G or be a constant multiple of $u(x)$.

Remark. Under the additional hypothesis that u (as well as v) satisfies a uniformly elliptic linear differential equation, Kreith [2] obtains the strong conclusion of the Corollary without the assumption $\tau(x) < \infty$. However, the proof of Theorem 1 of [2] is not clear since the hypotheses of Theorem 2.6 of his reference 4 are not fulfilled.

Theorem 1 can easily be extended to uniformly *parabolic* operators of the form

$$L(x, u) = \textstyle\sum_{i,j=1}^{n-1} D_i[a_{ij}(x)D_j u] + \sum_{i=1}^{n-1} b_i(x)D_i u - D_n u$$

where now the $(n - 1)$-square matrix (a_{ij}) is symmetric and uniformly positive definite in G (uniform parabolicity condition).

For $y \in G$, the notation G_y will be used to denote the set of all points $z \in G$ such that z can be connected to y by a path consisting only of line segments contained in hyperplanes $x_n =$ constant and segments parallel to the positive x_n-axis.

THEOREM 2. *If the hypotheses of Theorem* 1 *hold with "elliptic" replaced by "parabolic", and if* $u(x^0) = 0$ *at every point* $x^0 \in \partial G$ *at which* $\nu(x^0)$ *is parallel to the* x_n*-axis, then* $v(x)$ *cannot be positive throughout* $\bar{G}$ *unless* $v(x)$ *is a constant multiple of* $u(x)$ *in some subdomain* G_y *of* G.

Proof. The Friedman maximum principle [3, p. 174] for parabolic differential inequalities is now used instead of the Hopf maximum principle. If $v(x) > 0$ throughout $\bar{G}$, then $w = u/v$ satisfies the uniformly parabolic inequality (the analogue of (3))

$$(1/v^2)\sum_{i,j=1}^{n-1} D_i(a_{ij}v^2D_jw) + \sum_{i=1}^{n-1} b_iD_iw - D_nw \\ + [c^*(x, v, \nabla v) - c(x, u, \nabla u)]w \geq 0.$$

Hence either (i) w attains its maximum in a subdomain of type G_y of G, or (ii) w attains its maximum M at a boundary point x^0 and $w(x) < M$ for all $x \in G$. Now, $\nu(x^0)$ cannot be parallel to the x_n axis, or else $w(x^0) = u(x^0)/v(x^0) = 0$ by hypothesis, and the maximum of w would not be at x^0. The Friedman maximum principle then shows in alternative (ii) that $\nu \cdot \nabla w(x^0) > 0$, giving a contradiction as in the proof of Theorem 1.

It is also possible to extend Theorem 2 to more general parabolic operators of the form

$$L(x, u) = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + \sum_{i=1}^{n} b_iD_iu + cu$$

when the matrix (a_{ij}), $i, j = 1, \cdots, n$, is only positive semidefinite in G, and the coefficients a_{ij}, b_i, and c are allowed to be functions of x, u, and ∇u. For example, Fichera's maximum principle [1] could be applied to produce an analogue of Theorem 2.

We remark that a comparison theorem for two inequalities

$$L(x, u) \geq 0, \qquad L^*(x, v) \leq 0$$

in which *all* the coefficients can differ (not just c and c^*) has been obtained by the second author [5, p. 199]. However, this theorem does *not* contain Theorem 1 above since hypothesis (i) of Theorem 1 does not imply the hypothesis

$$\int_G [c(x) - c^*(x) - g(x)]u^2(x)\,dx \geq 0$$

in [5] (specialized to (1)). Here $g(x)$ is a positive continuous function such that $\det Q(x) > 0$ in G, where

$$Q(x) = \begin{pmatrix} a(x) & -b^T(x) \\ -b(x) & g(x) \end{pmatrix}, \qquad a = (a_{ij}), \quad b = (b_i)$$

Actually, the comparison theorem in [5] is false under the weaker hypothesis

$$\int_G [c(x) - c^*(x)]u^2(x)\,dx \geq 0. \tag{4}$$

Likewise Theorem 1 above is false (for domains G with piecewise C^1 boundaries) when the pointwise inequality in assumption (i) is replaced by the integral inequality (4).

A counterexample is provided in the case that $n = 1$, G is the interval $(0, \pi)$ and the inequalities (1) are the linear ordinary differential equations

$$u'' - 2u' + 2u = 0, \qquad v'' - 2v' + (x + 2 - k)v = 0 \tag{5}$$

where $k = \pi - 1$. Equations (5) have the solutions

$$u(x) = e^x \sin x, \qquad v(x) = e^x f(x),$$

for any solution f of

$$f'' + (x + 2 - \pi)f = 0. \tag{6}$$

We note that $u(x)$ is positive throughout $(0, \pi)$, $u(0) = u(\pi) = 0$, and (4) is satisfied since

$$\begin{aligned}\int_0^\pi (x - k)e^{2x} \sin^2 x\,dx &= \left[\frac{e^{2x}}{16}\left(4x - 2 - 2x \sin 2x - 2x \cos 2x + \sin 2x\right)\right.\\ &\qquad \left. - \frac{ke^{2x}}{8}(2 - \sin 2x - \cos 2x)\right]_0^\pi\\ &= \frac{e^{2\pi}}{8}(\pi - 1 - k) + \frac{k+1}{8} > 0.\end{aligned}$$

However, we assert that (6) has a positive solution on $[0, \pi]$ given by

$$f(x) = g(z) = 2g_1(z) + g_2(z),$$

where $z = x + 2 - \pi$, $2 - \pi \leq z \leq 2$, and

$$g_1(z) = 1 + \sum_{n=1}^\infty \frac{(-1)^n z^{3n}}{2\cdot 3 \cdots (3n - 1)(3n)} \tag{7}$$

$$g_2(z) = z + \sum_{n=1}^\infty \frac{(-1)^n z^{3n+1}}{3\cdot 4 \cdots (3n)(3n + 1)} \tag{8}$$

It is easily proved that $g_1(z)$ is decreasing in $(-\infty, 2]$ and hence $g_1(z) > g_1(2) > -0.02$ in this interval. Since

$$g_2(z) > z - z^4/12 \geq 2/3 \quad \text{on} \quad 1 \leq z \leq 2,$$

it follows that $g(z) = 2g_1(z) + g_2(z) > 0$ on $1 \leq z \leq 2$. The alternating series (7) and (8) show that $g(z) > 0$ in $0 \leq z < 1$. Since all terms in

(7) or in (8) have the same signs for $z < 0$, it follows that

$$g_1(z) > g_1(0) = 1, \qquad g_2(z) > g_1(2 - \pi) > -1.38$$

on $2 - \pi \leq z \leq 0$. Then $g(z) > 0$ on this interval also, and hence $g(z) > 0$ on the whole interval $[2 - \pi, 2]$. This proves that the second equation (5) has a positive solution $v(x) = e^x g(x + 2 - \pi)$ on $0 \leq x \leq \pi$.

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