SEQUENCE MIXING AND α-MIXING

BY

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1. Introduction

Let $(\Omega, \mathfrak{A}, m)$ be a probability space and let τ be a bimeasurable, invertible transformation mapping Ω onto Ω. All sets discussed throughout will be assumed to be elements of $\mathfrak{A}$. τ is measure-preserving if $m(\tau A) = m(A)$ for all A, it is ergodic if

$$\lim\ (1/n) \sum_{j=0}^{n-1} m(\tau^j A \cap B) = m(A)m(B) \quad \text{for all } A \text{ and } B,$$

it is weak mixing if

$$\lim\ (1/n) \sum_{j=0}^{n-1} \mid m(\tau^j A \cap B) - m(A)m(B) \mid = 0 \quad \text{for all } A \text{ and } B,$$

and strong mixing if

$$\lim m(\tau^n A \cap B) = m(A)m(B) \quad \text{for all } A \text{ and } B.$$

Since weak mixing already implies that $\lim m(\tau^n A \cap B) = m(A)m(B)$, except possibly along a sequence of asymptotic density zero (which may depend on A and B), it might be supposed that there is no room between weak mixing and strong mixing. At a symposium on ergodic theory held at Tulane University in October 1961, one of the authors proposed the notion of sequence mixing. τ is sequence mixing if for every A with $m(A) > 0$ and every infinite sequence of integers $\{k_n\}$ we have $m(\bigcup \tau^{k_n} A) = 1$. It is trivial to verify that strong mixing implies sequence mixing but for a number of years it remained an open question whether the converse holds. Recently Friedman and Ornstein, [2] showed that this is not the case. They define a transformation τ to be α-mixing for $\alpha \in (0, 1)$ if

$$\text{liminf}_n\, m(\tau^n A \cap B) \geq \alpha m(A)m(B) \quad \text{for all } A \text{ and } B,$$

and show that for every $\alpha \in (0, 1)$ there exist transformations which are α-mixing but not $(\alpha + \varepsilon)$-mixing for any $\varepsilon > 0$. Thus we may suppose that for every $\alpha \in (0, 1)$ there exists an α-mixing transformation and sets A and B with $m(A) > 0$, $m(B) > 0$ and $\text{liminf}_n\, m(\tau^n A \cap B) = \alpha m(A)m(B)$.

In this paper we construct a transformation which is sequence mixing but not α-mixing for any $\alpha \in (0, 1)$. It follows from the lemma below that α-mixing implies sequence mixing and it follows from [1] that sequence mixing implies weak mixing. Therefore α-mixing is strictly between weak and strong mixing. Also Friedman [3] gives an example of a weak mixing transformation T such that for some set A with $0 < m(A) < 1$ we have

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$\limsup_n m(T^n A \cap A) = m(A)$. It is easy to see that such a transformation cannot be sequence mixing. Thus sequence mixing is strictly between weak mixing and α-mixing.

2. A sequence mixing transformation

We begin with a

LEMMA. *τ is sequence mixing if and only if for $m(A) > 0$, $m(B) > 0$ we have*

$$\liminf_n m(\tau^n A \cap B) > 0.$$

Note that the lemma shows at once that α-mixing implies sequence mixing. To prove the lemma, suppose there exists A with $m(A) > 0$, and an infinite sequence of integers $k_1, k_2, \cdots$ such that $m(\bigcup_n \tau^{k_n} A) < 1$. Let $B = (\bigcup \tau^{k_n} A)^C$ to obtain

$$\liminf_n m\{\tau^n A \cap B\} = 0.$$

Conversely pick a sequence $\{k_n\}$ so that $\sum_{n=1}^{\infty} m\{\tau^{k_n} A \cap B\} < m(B)$. Then clearly $m\{\bigcup \tau^{k_n} A\} < 1$. This proves the lemma.

Now let I be the unit interval, $\mathcal{B}$ the Borel sets of I, and let m be the Lebesgue measure. Let $(\Omega, \mathcal{A}, \mu)$ be the measure space obtained by taking infinitely many copies of I and endowing it with the usual product field and product measure. We shall call $A \in \mathcal{A}$ a cylinder of dimension N if

$$A = A_1 \times A_2 \times \cdots \times A_N \times I \times I \times \cdots$$

for some integer $N > 0$, where $A_i \in \mathcal{B}$ for $i = 1, \cdots, N$. Let $\mathcal{A}_0$ be the algebra of sets consisting of finite unions of cylinders, so that $\mathcal{A}$ is the smallest σ-algebra of sets containing $\mathcal{A}_0$. Then for $A \in \mathcal{A}$ and $\varepsilon > 0$ it is possible to find $A_0 \in \mathcal{A}_0$ with $\mu(A \triangle A_0) < \varepsilon$, where $A \triangle A_0$ is the symmetric difference of A and A_0.

To define our transformation we begin by choosing $\alpha \in (0, 1)$ and τ to be α-mixing on I, with the property that there exist $A \subset I$, $B \subset I$ with $m(A) > 0$, $m(B) > 0$ and

$$\liminf_n m(\tau^n A \cap B) = \alpha m(A) m(B).$$

Now if $x = (x_1, x_2, \cdots) \in \Omega$ we define $T(x) = (\tau(x_1), \tau(x_2), \cdots)$. Clearly T maps cylinders onto cylinder and since τ preserves m it follows that T preserves μ on cylinders, and from this it follows easily that T is measure preserving for μ.

Now let A and B be subsets of I with the above property that

$$\liminf_n m\{\tau^n A \cap B\} = \alpha m(A) m(B).$$

For integers $k \geq 1$ define $A^{(k)}$ (similarly $B^{(k)}$) by

$$A^{(k)} = A \times \cdots \times A \times I \times \cdots \quad (k \text{ factors of } A).$$

Then

$$T^n A^{(k)} \cap B^{(k)} = (\tau^n A \cap B) \times \cdots \times (\tau^n A \cap B) \times I \times \cdots$$
$$(k \text{ factors of } \tau^n A \cap B)$$

and it is easily verified that

$$\mu(T^n A^{(k)} \cap B^{(k)}) = [m(\tau^n A \cap B)]^k.$$

Hence

$$\liminf_n \mu(T^n A^{(k)} \cap B^{(k)}) = [\alpha m(A) m(B)]^k.$$

Thus we see that T cannot be α-mixing for any $\alpha \in (0, 1)$.

Next we show that T is sequence mixing. Choose A and B with $\mu(A)$ and $\mu(B)$ positive. From the lemma it will be sufficient to show that

$$\liminf_n \mu(T^n A \cap B) > 0.$$

Let $\{\beta_i, i = 1, 2, \cdots\}$ be a sequence such that $0 < \beta_i < 1$, all i and such that

$$\prod_{i=1}^{n} \beta_i = \gamma > \tfrac{3}{4}.$$

Let $C_1 \in \mathcal{A}_0$ be such $\mu(C_1 \Delta A) \leq (1 - \beta_1)\mu(A)$, and suppose that C_1 is a cylinder set of dimension N_1. Then $\mu(C_1 \cap A) \geq \beta_1 \mu(A)$. Let $A_1 = C_1 \cap A$ and find $C_2 \in \mathcal{A}_0$ with

$$\mu(C_2 \triangle A_1) \leq (1 - \beta_2)\mu(A_1)$$

and so that $C_2 \subset C_1$. Again note that $\mu(C_2 \cap A_1) \geq \beta_2 \mu(A_1) \geq \beta_1 \beta_2 \mu(A)$.

Proceeding inductively we define C_k and A_{k-1} so that $C_k \subset C_{k-1}$, with C_k a cylinder set of dimension N_k, and such that $\mu(C_k \triangle A_{k-1}) \leq (1 - \beta_k)\mu(A_{k-1})$, which in turn implies that

$$\mu(C_k \cap A_{k-1}) \geq \prod_{i=0}^{k} \beta_i \mu(A).$$

Then define

$$A_k = C_k \cap A_{k-1}.$$

Let

$$C_\infty = \bigcap_{k=1}^{\infty} C_k \quad \text{and} \quad A_\infty = \bigcap_{k=1}^{\infty} A_k.$$

Now

$$\liminf_n \mu(C_k) = \liminf_n \mu(A_{k-1}) \geq \gamma\mu(A).$$

Hence $\mu(C_\infty) = \mu(A_\infty) \geq \gamma\mu(A)$. Clearly $C_\infty \subset A$ except for a null set.

We shall now describe a specific possible construction of the C_k which we shall use. For each k, let $C_k = \bigcup_i C_{k,i}$ where i is in a finite index set and such that

$$C_{k,i} = E_1^{(k,i)} \times E_2^{(k,i)} \times \cdots \times E_{N_k}^{(k,i)} \times I \times I \times \cdots.$$

Then $C_{k,i} = G_{k,i} \cap H_{k,i}$ where

$$G_{,li} = E_1^{(k,i)} \times \cdots \times E_{N_1}^{(k,i)} \times I \times I \times \cdots,$$

and

$$H_{k,i} = I \times \cdots \times I \times E_{N_1+1}^{(k,i)} \times \cdots \times E_{N_k}^{(k,i)} \times I \times I \times \cdots$$
$$(N_1 \text{ initial factors of } I).$$

In this construction we can choose the $G_{k,i}$ so that for fixed k and arbitrary i and j we have $G_{k,i} = G_{k,j}$ or $G_{k,i} \cap G_{k,i} = \emptyset$. This can always be accomplished by breaking up each $C_{k,i}$ into several pieces when necessary. Since $C_k \subset C_1$ for all k we have $\bigcup_i G_{k,i} \subset C_1$. Now define

$$J_k = [i \mid \mu\{\bigcup_{G_{k,j}=G_{k,i}} H_{k,j}\} \geq 2\gamma - 1].$$

Then

$$\begin{aligned}\gamma\mu(C_1) \leq \mu(C_k) &= \mu(\bigcup_i C_{k,i}) = \mu(\bigcup_i G_{k,i} \cap H_{k,i}) \\ &= \mu(\bigcup_{i \in J_k} H_{k,i})) + \mu(\bigcup_{i \in J_k} (G_{k,i} \cap H_{k,i})) \\ &\leq \mu(\bigcup_{i \in J_k} G_{k,i}) + (2\gamma - 1)\mu(\bigcup_{i \in J_k} G_{k,i}).\end{aligned}$$

Thus

$$\gamma\mu(C_1) \leq \mu(\bigcup_{i \in J_k} G_{k,i}) + (2\gamma - 1)(\mu(C_1) - \mu(\bigcup_{i \in J_k} G_{k,i})).$$

By a simple rearrangement we have

$$\tfrac{1}{2}\mu(C_1) \leq \mu(\bigcup_{i \in J_k} G_{k,i}).$$

Now we can make precisely the same constructions for the set B as we have done for A and we shall denote all sets constructed by superscripts A and B respectively. Moreover we can arrange matters so that the cylinders comprising C_k^A and C_k^B have the same dimensions for each k.

Now

$$(\bigcup_{i \in J_k^A} T^n C_{k,i}^A) \cap (\bigcup_{j \in J_k^B} C_{k,j}^B) \subset T^n C_k^A \cap C_k^B .$$

Fix $i \in J_k^A$ and $j \in J_k^B$. Then

$$\begin{aligned}\mu[(\bigcup_{G_{k,s}^A = G_{k,i}^A} T^n C_{k,s}^A) \cap (\bigcup_{G_{k,s}^B = G_{k,j}^B} C_{k,s}^B)] \\ &= \mu[\bigcup_s T^k (G_{k,s}^A \cap H_{k,s}^A) \cap \bigcup_s (G_{k,s}^B \cap H_{k,s}^B)] \\ &= \mu[T^n G_{k,i}^A \cap G_{k,j}^B]\mu[(\bigcup_s T^n H_{k,s}^A) \cap (\bigcup_s H_{k,s}^B)].\end{aligned}$$

Since $i \in J_k^A$ and $j \in J_k^B$ we have

$$\begin{aligned}\mu[\bigcup_s T^n H_{k,s}^A) \cap (\bigcup H_{k,s}^B)] &\geq \mu[(\bigcup_s T^n H_{k,s}^A)] + \mu[(\bigcup_s H_{k,s}^B)] - 1 \\ &\geq (2\gamma - 1) + (2\gamma - 1) - 1 = 4\gamma - 3 > 0.\end{aligned}$$

Hence

$$\begin{aligned}\liminf_n \mu[(\bigcup_s T^n C_{k,s}^A) \cap (\bigcup_s C_{k,s}^B)] &\geq (4\gamma - 3) \liminf_n \mu(T^n G_{k,i}^A \cap G_{k,j}^B] \\ &= (4\gamma - 3) \liminf_n \textstyle\prod_{r=1}^{N_1} m[\tau^n E_r^{(A,k,i)} \cap E_r^{(B,k,j)}].\end{aligned}$$

Recalling that m is α-mixing with respect to τ we have that the last expression exceeds

$$\alpha^{N_1}(4\gamma - 3) \textstyle\prod_{r=1}^{N_1} m(E_r^{(A,k,i)})m(E_r^{(B,k,j)}) = \alpha^{N_1}(4\gamma - 3)\mu[G_{k,j}^B].$$

Now recall that for distinct $i \in J_h^A$ the $G_{k,i}^A$ are disjoint. Similarly for distinct j in J_k^B. Hence we obtain

$$\liminf_n \mu[(\bigcup_{i \in J_k^A} T^n C_{k,i}^A) \cap (\bigcup_{j \in J_k^B} C_{k,j}^B)] \geq \alpha^{N_1}(4\gamma - 3)\tfrac{1}{4}\mu(C_1^A)\mu(C_1^B) = p > 0$$

or

$$\liminf_n \mu[(T^n C_k^A) \cap C_k^B] \geq p.$$

Now C_k^A and C_k^B decreases in k to C_∞^A and C_∞^B respectively so that

$$\liminf_n \mu[T^n C_\infty^A \cap C_\infty^B] \geq p > 0.$$

As noted above $C_\infty^A \subset A$ and $C_\infty^B \subset B$ except for null sets, so that

$$\liminf_n \mu(T^n A \cap B) > 0.$$

Since A and B were arbitrary sets of positive measure, we have that T is sequence mixing.

References

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