A NECESSARY AND SUFFICIENT CONDITION FOR THE RIEMANN HYPOTHESIS FOR RAMANUJAN'S ZETA FUNCTION

BY

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1. Introduction

The Ramanujan τ-function is defined as the *n-th* coefficient in the q-expansion of

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \qquad q = e^{2\pi i z} \quad \text{and} \quad \mathrm{Im}(z) > 0.$$

It is well known that the function $\Delta(z)$ spans the space of cusp forms of weight -12 associated with the unimodular group. $\Delta(z)$ is in fact an eigenfunction of the Hecke operators and as such its corresponding Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \tau(n) n^{-s}$$

has an Euler product

$$\varphi(s) = \prod_p (1 - \tau(p) p^{-s} + p^{11-2s})^{-1}.$$

In [4, p. 174], Hardy observed that the location of the nontrivial zeros of $\varphi(s)$ in the strip $11/2 \leq \mathrm{Re}(s) \leq 13/2$ gave rise to problems similar to those in the classical theory of the Riemann zeta function, i.e., Riemann Hypothesis, von Mangoldt formulae etc. Some of these questions were subsequently treated by various authors. Of particular interest is the paper by Goldstein [2] where the analogue of Merten's conjecture

$$\sum_{n \leq x} \mu(n) \ll x^{1/2+\varepsilon}$$

is established for Ramanujan's zeta function. A special case of Goldstein's main result is the

Theorem (3.6 in [2]). *A necessary and sufficient condition for Ramanujan's zeta function $\varphi(s)$ to have all its zeros on* $\mathrm{Re}(s) = 6$ *is that*

$$(1) \qquad \sum_{n \leq x} \mu_\varphi(n) \ll x^{6+\epsilon}$$

for all $\varepsilon > 0$, where $\mu_\varphi(n)$ is the Möbius function defined by expanding formally the product

$$(2) \qquad {}_x\prod_p (1 - \tau(p) p^{-s} + p^{11-2s}) = \sum_{n=1}^{\infty} \tau(n) n^{-s}.$$

In [2] Goldstein also suggested that the arithmetical function $\mu_\varphi(n)$ could be evaluated by means of the Selberg trace formula [7] in terms of ideal class numbers of certain imaginary quadratic fields and thus transform the sum in

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(1) into a sum involving class numbers. It was then hoped that the Riemann Hypothesis for $\varphi(s)$ could be settled by establishing a bound like (1) for the resulting sum of class numbers.

In the classical theory of the Riemann zeta function $\zeta(s)$ it is well known that a statement of the type

$$ {}_x\sum_{p\leq x} \log p - x \ll x^{1/2+\varepsilon} \tag{3}$$

for all $\varepsilon > 0$, is equivalent to the Riemann Hypothesis for $\zeta(s)$. Recently the author succeeded in formulating general prime number theorems for the coefficients of cusp forms of integral weight $-k$ associated with the full modular group. In this note we show that the Riemann Hypothesis for the Mellin transform of cusp forms that are eigenfunctions of the Hecke operator is equivalent to a prime number theorem with a sharp error term. The result we prove is the following

THEOREM 1. *Let $\varphi(s)$ be the Ramanujan zeta function. Then the following two statements are equivalent*:

(4) (Riemann Hypothesis)

$$\varphi(s) \neq 0 \quad \textit{for} \quad \mathrm{Re}(s) > 6.$$

$$\sum_{p\leq x} \tau(p) \ll x^6(\log x)^2. \tag{5}$$

Remarks. We have formulated Theorem 1 only for the space of cusp forms of weight -12 associated with the full modular group for various reasons. First, a general result can be obtained for cusp forms of arbitrary integral weight by the same method with only minor technical complications. Secondly, the space of cusp forms of weight -12 is of dimension 1 and hence the trace of the Hecke operators is identical with the Ramanujan τ-function. This will then lead to the possibility of obtaining a statement equivalent to (5) above involving simpler arithmetical functions. Lastly, we would like to mention that according to the results of Deligne [1], the statement (5) can be given an arithmetical interpretation in the sense that it implies a certain regularity in the distribution of the traces of the Frobenius endomorphism coming from an l-adic representation of the Galois group of a certain infinite Galois extension.

The plan of the paper is as follows. In §2 we recall some basic definitions and results about modular forms. Some of the results in [5] are restated in a form suitable to our needs. In §3 we derive some elementary estimates which make precise the idea that the main term in a prime number theorem is not affected by contributions which come from prime powers higher than the first. Such estimates had already occurred in a weak form in [5]. Here we shall make a more careful analysis of the situation which, as is well known, becomes rather complicated due to the fact that the true order of magnitude for $\tau(n)$ is not known. In §4 we use the Explicit Formula of [5] together with the results of §3 to prove Theorem 1. We then use the Selberg trace formula to evaluate the Ramanujan τ-function at a prime p and thus obtain a statement equivalent to the Riemann Hypothesis for $\varphi(s)$ but now involving ideal class numbers of imaginary quadratic fields. In §4 we also use a formula of Ramanujan which expresses the value of $\tau(p)$ in terms of the divisor function to obtain another statement equivalent to the Riemann Hypothesis for $\varphi(s)$.

2. Basic results

The notation will be that of [5]. The Mordell-Ramanujan identity for $\tau(n)$ is given by

$$\tau(p^k) = \tau(p)\tau(p^{k-1}) - p^{11}\tau(p^{k-2}) \tag{6}$$

for $k \geq 2$ and p a prime. The best estimate known for $\tau(n)$ is roughly

$$\tau(n) \ll n^{23/4+\epsilon}, \tag{7}$$

which can be obtained using A. Weil's estimates of Kloosterman sums [9]. We will also need the following result due to Rankin [7]:

$$\sum_{n\leq x} \tau^2(n) - \alpha x^{12} \ll x^{58/5}. \tag{8}$$

The Möbius function and the von Mangoldt functions for the Ramanujan τ-function are defined by formally expanding the Euler product

$$1/\varphi(s) = \prod_p (1 - \tau(p)p^{-s} + p^{11-s}) = \sum_{n=1}^{\infty} \mu_\varphi(n)n^{-s}$$

and convolving the two Dirichlet series

$$-\varphi'(s)/\varphi(s) = \sum_{n=1}^{\infty} \Lambda_\varphi(n)n^{-s},$$

respectively. The von Mangoldt function $\Lambda_\varphi(n)$ can also be defined by applying the differential operator $D = (d/ds)\log(\quad)$ to the Euler product of $\varphi(s)$. The advantage of this last definition lies in that it is readily seen that the function

$$\Lambda_\varphi(n) = \sum_{d|n} \mu_\varphi(n/d)\tau(d)\log d \tag{9}$$

has its support at the prime powers. Our estimates will be given in terms of the following summatory functions:

$$\Psi_\varphi(x) = \sum_{n\leq x} \Lambda_\varphi(n), \quad \vartheta_\varphi(x) = \sum_{p\leq x} \tau(p)\log p, \quad \pi_\varphi(x) = \sum_{p\leq x} \tau(p).$$

We recall from [5] the following results:

EXPLICIT FORMULA. *Let $\varphi(s)$ be the Ramanujan zeta function and $x \geq 2$. Then*

$$\Psi_\varphi(x) = -\varphi^{(3)}(0)/2\varphi^{(1)}(0) - \log(x-1) - \sum_{|\gamma|<T} x^\rho/\rho + R(x, T), \tag{10}$$

where the sum runs over the nontrivial zeros of $\varphi(s)$ in $11/2 < \mathrm{Re}(s) < 13/2$

and

$$R(x, T) \ll X^{13/2} (\log Tx) / T^2.$$

WEAK VON MANGOLDT FORMULA. *Let $N_\varphi(T)$ be the number of zeros of $\varphi(s)$ in the critical strip with ordinates $|\gamma| < T$. Then*

$$N_\varphi(T + 1) - N_\varphi(T) \ll \log T. \tag{11}$$

The Selberg trace formula [8] for the Hecke operators T_n acting on the space of cusp forms of dimension $-k$ is given by

$$\begin{aligned} \text{trace}\,(T_n) = & -\textstyle\sum^*_{|m|<(4n)^{1/2}} H(4n - m^2) F^{(k-2)}(\rho_n, \bar\rho_m) \\ & - \textstyle\sum_{d|n,\, d<n^{1/2}} d^{k-1} + \delta(n^{1/2})((k-1)/12 \\ & \cdot n^{(k-2)/2} - n^{(k-1)/2}) \\ & + 0 \quad \text{if } k > 2 \\ & \sigma(n) \quad \text{if } k = 2, \end{aligned} \tag{12}$$

where

$$F^{(k-2)}(\rho_m, \bar\rho_m) = (\rho_m^{k-1} - \bar\rho_m^{k-1})/\rho_m - \bar\rho_m \quad \text{with} \quad \rho_m = m + i(4n - m^2)^{1/2}/2$$

and $H(d)$ denotes the class number of the imaginary quadratic field $Q(\sqrt{(-d)})$, and the $(*)$ on the summation sign means that $H(d)$ is to be divided by the number of roots of unity in $Q(\sqrt{(-d)})$. $\delta(x) = 1$ when x is rational and zero otherwise.

Lastly we recall a formula of Ramanujan which expresses $\tau(n)$ in terms of simple arithmetical functions:

$$\begin{aligned} \tau(n) = & (65/756)\, \sigma_{11}(n) + (691/756)\, \sigma_5(n) \\ & - (691/3) \textstyle\sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n - m), \end{aligned} \tag{13}$$

where $\sigma_l(n)$ is the sum of the *l-th* powers of the divisors of n. (See [3, p. 55].)

3. Elementary computations

In this section we shall make quantitative our earlier observation that the distribution of powers of primes higher than the first do not affect the truth or falsity of the Riemann Hypothesis for $\varphi(s)$.

LEMMA 1. *Let $\Lambda_\varphi(n)$ be the von Mangoldt function associated with $\tau(n)$. Then*

$$\textstyle\sum_{n\le x} \Lambda_\varphi(n) - \sum_{p\le x} \tau(p) \log p \ll x^6 \log x.$$

Proof. We have already observed that $\Lambda_\varphi(n)$ vanishes outside the set of prime powers. From the definition of the Möbius function $\mu_\varphi(n)$ and the identity (9) it follows that $\Lambda_\varphi(p) = \tau(p) \log p$ for p a prime. Hence

$$\textstyle\sum_{n\le x} \Lambda_\varphi(n) - \sum_{p\le x} \tau(p) \log p = \sum_{\alpha\ge 2} \sum_{p^\alpha \le x} \Lambda_\varphi(p^\alpha). \tag{14}$$

To estimate the sum in the right hand side of (14) we use the formula

$$\Lambda_\varphi(p^\alpha) = (\tau(p^\alpha) - p^{11}\tau(p^{\alpha-2}))\log p, \quad \alpha \geq 2, \tag{15}$$

and the Mordell-Ramanujan identity (6) to obtain the following formulas:

$$\Lambda_\varphi(p^2) = (\tau^2(p) - 2p^{11})\log p, \quad \Lambda_\varphi(p^3) = (\tau^3(p) - 3p^{11}\tau(p))\log p,$$
$$\Lambda_\varphi(p^4) = (\tau^4(p) - 4p^{11}\tau^2(p) + 2p^{22})\log p.$$

The first formula is used to compute the term in the right hand side of (14) with $\alpha = 2$ as follows:

$$\sum_{p^2\leq x} \Lambda_\varphi(p^2) \ll \sum_{p\leq x^{1/2}} \tau^2(p)\log p + x^{11/2}\sum_{p\leq x^{1/2}}\log p.$$

If the second sum is estimated by means of (8) and the third sum by the Prime Number Theorem, we get

$$\sum_{p^2\leq x} \Lambda_\varphi(p^2) \ll x^6 (\log x). \tag{16}$$

To estimate the sum involving the cube powers we observe that (5) and (6) imply

$$\mathrm{Max}_{n\leq x^{1/3}} |\tau(n)| \ll x^{23/12+\epsilon/3} \quad \text{and} \quad \sum_{p\leq x^{1/3}} |\tau(p)| \ll x^{13/6}$$

respectively. Therefore

$$\begin{aligned}\sum_{p^3\leq x} \Lambda_\varphi(p) &\ll (\mathrm{Max}_{p\leq x^{1/3}} |\tau(p)|)\sum_{p\leq x^{1/3}} \tau^2(p)\log p \\ &\qquad + x^{11/3}\log x \sum_{p\leq x^{1/3}} |\tau(p)| \\ &\ll x^{23/12+\epsilon/3}x^4\log x + x^{11/3}x^{13/6}\log x \\ &\ll x^6\end{aligned} \tag{17}$$

provided that $\varepsilon < 1/4$. The estimation of the sum involving the fourth powers is done similarly by treating individually the terms in the sum

$$\sum_{p^4\leq x} \Lambda_\varphi(p^4) \ll \sum_{p^4\leq x} \tau^4(p)\log p + x^{11/4}\sum_{p^4\leq x}\tau^2(p)\log p + x^{11/2}\sum_{p^4\leq x}\log p$$

as we did for the squares and the cubes, thus obtaining

$$\sum_{p^4\leq x} \Lambda_\varphi(p^4) \ll x^6. \tag{18}$$

The terms in (14) with $\alpha \geq 5$ are treated as follows

$$\sum_{p^\alpha\leq x} \Lambda_\varphi(p^\alpha) \ll \sum_{p^\alpha\leq x} |\tau(p^\alpha)|\log p + \sum_{p^\alpha\leq x} p^{11}|\tau(p^{\alpha-2})|\log p.$$

By (6) and the Prime Number Theorem we have

$$\begin{aligned}\sum_{p^\alpha\leq x} |\tau(p^\alpha)|\log p &\ll (\mathrm{Max}_{p^\alpha\leq x} |\tau(p^\alpha)|)x^{1/\alpha} \\ &\ll x^{23/4+\epsilon+1/\alpha} \\ &\ll x^6.\end{aligned}$$

Similarly

$$\sum_{p^\alpha \le x} p^{11}\tau(p^{\alpha-2}) \log p \ll (\mathrm{Max}_{n \le x^{(\alpha-2)/\alpha}} | \tau(n)|) x^{11/\alpha+1/\alpha}$$
$$\ll x^{((\alpha-2)/\alpha)(23/4+\varepsilon)} x^{12/\alpha}$$
$$\ll x^6,$$

again provided that $\varepsilon < 1/20$ which can certainly be satisfied at the expense of the implied constants. Therefore we have

$$\sum_{p^\alpha \le x,\ \alpha \ge 5} \Lambda_\varphi(p^\alpha) \ll x^6. \tag{19}$$

Putting the estimates (16), (17), (18) and (19) together and observing that the highest power of a prime which can occur among the integers $\le x$ is not larger than 2 ($\log x$), we obtain

$$\sum_{p^\alpha \le x,\ \alpha \ge 2} \Lambda_\varphi(p^\alpha) \ll x^6 (\log x),$$

which can also be written as

$$\Psi_\varphi(x) - \vartheta_\varphi(x) \ll x^6 \log x.$$

This completes the proof of Lemma 1.

We will also need the following

LEMMA 2. *For Ramanujan's τ-function, the statements* (20) *and* (21) *below are equivalent*:

$$\pi_\varphi(x) = \sum_{p \le x} \tau(p) \ll x^6; \tag{20}$$

$$\vartheta_\varphi(x) = \sum_{p \le x} \tau(p) \log p \ll x^6 \log x. \tag{21}$$

The proof of Lemma 2 simply requires an elementary argument involving summation by parts and we leave it to the reader.

4. Proof of Theorem 1

First we prove that the Riemann Hypothesis for the Ramanujan zeta function implies the estimate (5). By Lemma 1 and Lemma 2 it is enough to show that

$$\Psi_\varphi(x) \ll x^6 (\log x)^2. \tag{22}$$

Substituting $T = x^{3/2}$ in the Explicit Formula (10) we have

$$\Psi_\varphi(x) \ll \sum_{|\gamma| \le x^{3/2}} | x^\rho/\bar{\rho} | + x^6 (\log x)^2.$$

Now, the Riemann Hypothesis for $\varphi(s)$ implies that $| x^\rho | = x^6$; therefore

$$\sum_{|\gamma| \le x^{3/2}} | x^\rho/\rho | \ll x^6 \sum_{|\gamma| \le 1} 1/| \rho | + x^6 \sum_{1 \le |\gamma| \le x^{3/2}} 1/| \gamma |$$
$$\ll x^6 + x^6 \sum_{n \le x^{3/2}} n^{-1} \sum_{n \le |\gamma| \le n+1} 1$$
$$\ll x^6 (\log x)^2,$$

where the last inequality is a consequence of (11). Hence (22).

To prove that the estimate (5) implies the Riemann Hypothesis for $\varphi(s)$ we again use Lemma 2 to deduce (22), and hence obtain that the integral

$$-\varphi'(s)/\varphi(s) = s \int_1^\infty (\sum_{n\leq x} \Lambda_\varphi(n))x^{-s-1}dx$$

represents a regular function for $\mathrm{Re}(s) > 6$. But this implies that $\varphi(s) \neq 0$ for $\mathrm{Re}(s) > 6$. This then completes the proof of Theorem 1.

We now use the Selberg trace formula to replace the sum in (5) of Theorem 1 by a sum involving class numbers of imaginary quadratic fields. In fact the Selberg trace formula (12) gives

$$\tau(p) = -\sum_{|m|<(4p)^{1/2}} H(4p - m^2)F^{(10)}(\rho_m, \bar{\rho}_m) - 1.$$

If this expression is substituted directly in (5) we would admittedly get a quite complicated condition implying the Riemann Hypothesis. Rather we observe the following simplifications:

$$F^{(10)}(\rho_m, \bar{\rho}_m) = (\rho_m^{11} - \rho_m^{11})/(\rho_m - \bar{\rho}_m) = \rho_m^{10} + \bar{\rho}_m^{10} + pF^{(8)}(\bar{\rho}_m, \bar{\rho}_m),$$

$$\bar{\rho}_m^{10} + \bar{\rho}_m^{10} = m^{10} - 18m^8p + 35m^6p - 50m^4p^3 + 25m^2p^4 - 2p^5 = M_{p,m}.$$

If we use the fact that the space of cusp forms of weight -10 contains only the cusp form which is identically zero, we get from the Selberg trace formula

$$0 = -\sum^*_{|m|<(4p)^{1/2}} H(4p - m^2)F^{(8)}(\rho_m, \bar{\rho}_m) - 1$$

that

$$\tau(p) = -\sum^*_{|m|<(4p)^{1/2}} H(4p - m^2)M_{p,m} - 1 + p.$$

The Riemann Hypothesis for the Ramanujan zeta function would then follow if we could show that

(G) $$\sum_{p\leq x} \sum_{|m|<(4p)^{1/2}} H(4p - m^2)M_{p,m} \ll x^6 (\log x)^A,$$

for some positive constant A. How one goes about establishing an estimate like (G) is not clear and will probably have to await the development of really deep tools in the analytic theory of numbers.

Another statement equivalent to (5) can be formulated by using the Ramanujan identity (13)

$$\tau(n) = 65\, \sigma_{11}(n)/756 + 691\, \sigma_5(n)/756 - (691/3) \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n - m)$$

which for a prime p can be transformed into the simple form

(23) $$\tau(p) = (65\, p^{11} + 691\, p^5)/756 + 1 - \sum_{ab+cd=p} (ac)^5$$

where the last sum runs over all quadruples of positive integers (a, b, c, d) such that $ab + cd = p$.

The representation (23) then implies that the inequality (5) could be de-deduced by the Prime Number Theorem and a rather sharp estimate for the sum

(24) $$\sum_{p\leq x} \sum_{ab+cd=p} (ac)^5.$$

Estimates for sums similar to (24) have already occurred in the literature (see Linnik [6]) but the results obtained are far from applicable to (24), and the simplification affected by (23) is of little help. For reasons which we hope to explain elsewhere, the condition (G) seems to be the most hopeful candidate for investigating the location of the nontrivial zeros of the Ramanujan zeta function $\varphi(s)$.

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