ON GROUPS WITH A QUATERNION SYLOW 2-SUBGROUP

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A theorem of Brauer and Suzuki states:

Let G be a group with a generalized quaternion Sylow 2-subgroup S. Then the center of $G/O_{2'}(G)$ is of order 2.

The case in which $|S| > 8$ has been proved by the theory of ordinary characters (e.g., in Chapter 12 of [2]). The published proofs of the case in which $|S| = 8$ require the theory of blocks of characters (e.g. [1, pages 321–324]). In this paper, we prove the latter case without using blocks.

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We shall adapt the proof for the case in which $|S| > 8$, as given in Chapter 12 of [2]. Hence we adopt some of the notation of [2] and add some further notation.

Assume that G is a counterexample to the theorem of minimal order. Since we assume the case in which $|S| > 8$, we will suppose that $|S| = 8$. Let x be an element of order four in S. Let $X = \langle x \rangle$, $T = \langle x^2 \rangle$, $C^* = C_G(X)$, $N^* = N_G(X)$, and $H^* = O_{2'}(C^*)$.

Let A^* be the subset $C^* - TH^*$ of C^*. Let B be the set of all conjugates of elements of A^* in G.

Denote the principal characters of C^* and G by 1_{C^*} and 1_G.

(1) (a) $N^* = SH^*$ *and* $C^* = XH^* = X \times H^*$;
(b) A^* *is disjoint from its conjugates in* G *and* $N^* = N_G(A^*)$.

Proof. These results are analogues of Lemmas 12.1.2 and 12.1.3 of [2]. Note that $XH^* = X \times H^*$ because H^* centralizes X and intersects it in the identity group.

By (1a), $TH^* \lhd C^*$ and $|C^*/TH^*| = 2$.
Let λ be the unique linear character of C^* with kernel TH^*. Let $\tilde{1}_{C^*}$ and $\tilde{\lambda}$ be the characters of N^* induced by 1_{C^*} and λ, and let $\zeta = \tilde{1}_{C^*} - \tilde{\lambda}$. Let η be the generalized character of G induced by ζ.

(2) (a) $(\zeta, \zeta)_N = 4$;
(b) $\zeta(1) = 0$ *and* $\zeta(y) = 0$ *for every* $y \in N^* - A^*$;
(c) *there exist distinct nonprincipal irreducible characters* χ_1, χ_2, χ_3

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of G and signs $\varepsilon_i = \pm 1$ $(i = 1, 2, 3)$ *such that*

$$\eta = 1_G + \sum_{1 \le i \le 3} \varepsilon_i \chi_i .$$

Proof. These results are analogues of Lemmas 12.1.4 and 12.1.5 of [2]. Note that $\tilde{\lambda}$ is not an irreducible character of N^*, but is the sum of two distinct linear characters of N^*.

(3) (a) $\eta(y) = 1 + \sum \varepsilon_i \chi_i(y) = 0$ *if* $y \in G - B$;
(b) $\zeta(y) = 4$, *if* $y \in A^*$;
(c) $\eta(y) = 1 + \sum \varepsilon_i \chi_i(y) = 4$ *if* $y \in B$;
(d) *for every involution u of G,*

$$1 + \sum \varepsilon_i (\chi_i(u))^2 / \chi_i(1) = 0.$$

Note. Here and in later results, in a summation involving an index i, we will take i to run over the values 1, 2, 3.

Proof. (a) This is obvious from (2b), since η is induced by ζ.
(b) By the definition of an induced character,

$$\zeta(y) = \tilde{1}_{C^*}(y) - \tilde{\lambda}(y) = 2(1_{C^*}(y) - \lambda(y)) = 4.$$

(c) We can assume that $y \in A^*$. By (b) and (1b), $\eta(y) = \zeta(y) = 4$.
(d) By the definitions of A^* and B, every element of B is of even order. Hence no element of B is the product of two involutions of G. (This is Lemma 12.1.7 of [2], and its proof does not require any restriction on the order of S.) This yields (d), which is the analogue of Lemma 12.1.8 of [2].

(4) (a) *For* $i = 1, 2, 3$, x^2 *does not lie in the kernel of* χ_i;
(b) *all the elements of order four in G are conjugate.*

Proof. Let $K = O_{2'}(G)$. Since G is a minimal counterexample to the theorem for the case in which $|S| = 8$, $|Z(G/K)| \ne 2$.

Now, $O_{2'}(G/K) = K/K = 1$, and S is isomorphic to a Sylow 2-subgroup of G/K. If $K \ne 1$, then $|G/K| < |G|$ and, consequently,

$$2 = |Z(G/K)/O_{2'}(G/K)| = |Z(G/K)|.$$

Hence $K = 1$, that is, G is "core-free" in the sense of Brauer [1]. So, $|Z(G)| = |Z(G/K)| \ne 2$. Therefore, G satisfies the hypothesis for Brauer's proof, and, as Brauer shows (pages 321–322 of [1]), (a) and (b) are easy to obtain. (If (b) fails, then $N_G(P)/C_G(P)$ is a 2-group for every P of S. So then G has a normal 2-complement and $|Z(G)| = |Z(G/K)| = |Z(S)| = 2$, by a theorem of Frobenius [2, page 253]. If (a) fails, for some i, let G^* be the kernel of χ_i. Then

$$G^* \subset G \quad \text{and} \quad O_{2'}(G^*) \subseteq O_{2'}(G) = 1.$$

By (b), either $\langle x^2 \rangle$ or S is a Sylow 2-subgroup of G^*. In either case, we find that $\langle x^2 \rangle = Z(G^*)$ and then that $\langle x^2 \rangle = Z(G)$).

We introduce some further notation. Let

$$x_i = \chi_i(1), \quad y_i = \chi_i(x^2), \quad z_i = x_i - y_i, \quad \text{for } i = 1, 2, 3.$$

Since x^2 has order two, the numbers x_i, y_i, z_i are rational integers.

(5) (a) $z_i > 0$ *for* $i = 1, 2, 3$;
(b) $1 + \sum \varepsilon_i x_i = \sum \varepsilon_i z_i = \sum \varepsilon_i(z_i^2/x_i) = 0$.

Proof. (a) This follows from (4a).

(b) From the definitions of A^* and B, we note that neither of them contains the identity element or an involution. Hence, by (3a),

$$1 + \sum \varepsilon_i x_i = 1 + \sum \varepsilon_i \chi_i(1) = 0 = 1 + \sum \varepsilon_i \chi_i(x^2) = 1 + \sum \varepsilon_i y_i.$$

Therefore, $\sum \varepsilon_i z_i = 0$.

For each i,

$$y_i^2/x_i = (z_i - x_i)^2/x_i = (z_i^2/x_i) - 2z_i + x_i$$

Thus (3d) yields

$$0 = 1 + \sum \varepsilon_i(y_i^2/x_i) = 1 + \sum \varepsilon_i(z_i^2/x_i) - 2\sum \varepsilon_i z_i + \sum \varepsilon_i x_i$$
$$= \sum \varepsilon_i(z_i^2/x_i).$$

(6) *For every generalized character* χ *of* G,

$$(\chi, \eta)_G = (1/|H^*|) \textstyle\sum_{u \in H^*} \chi(xu).$$

Proof. By (3b), $\zeta(y) = 4$ for every $y \in A^*$. Since η is induced from ζ, the Frobenius Reciprocity Theorem and (2b) yield

$$\begin{aligned}(\chi, \eta)_G &= (\chi|_{N^*}, \zeta)_{N^*} \\ &= (1/|N^*|) \textstyle\sum_{y \in A^*} 4\chi(y) \\ &= (1/|N^*|) \textstyle\sum_{u \in H^*}(4\chi(xu) + 4\chi(x^{-1}u)).\end{aligned}$$

Now, $|N^*| = 8\,|H^*|$. Take $y \in S - \langle x \rangle$. Then y normalizes H^* and $x^y = x^{-1}$. Hence y maps the set xH^* onto the set $x^{-1}H^*$ by conjugation. Thus

$$\textstyle\sum_{u \in H^*} \chi(x^{-1}u) = \sum_{u \in H^*} \chi((x^{-1}u)^y) = \sum_{u \in H^*} \chi(xu);$$

$$(\chi, \eta)_G = (1/8\,|H^*|) \textstyle\sum_{u \in H^*} 8\chi(xu) = (1/|H^*|) \sum_{u \in H^*} \chi(xu).$$

(7) *For* $i = 1, 2, 3$,
(a) *the values of* χ_i *are rational integers*;
(b) $\chi_i(xu) = \chi_i(xu^{-1})$ *for every* $u \in H^*$.

Proof. Recall that the values of the characters of G are algebraic integers in the cyclotomic field, K, of the $|G|$-*th* roots of unity.

(a) Suppose that $1 \le i \le 3$ and the values of χ_i are not rational integers. We may assume that $i = 1$. Then some value of χ_1 is irrational. Since K

is a normal extension of the rational field, there exists an automorphism α of K that moves some value of χ_1. Since α permutes the irreducible characters of G by the definition

$$\chi^\alpha(y) = (\chi(y))^\alpha, \quad y \in G,$$

χ_1^α is an irreducible character of G distinct from χ_1. As η is rational-valued,

$$(\chi_1^\alpha, \eta)_G = (\chi_1, \eta)_G = \varepsilon_1.$$

By (2c), χ_1^α is χ_j for some j such that $\varepsilon_j = \varepsilon_1$. We may and will assume that $j = 2$.

Since $\chi_1(x^2)$ is rational,

$$y_2 = \chi_2(x^2) = (\chi_1(x^2))^\alpha = \chi_1(x^2) = y_1.$$

Similarly, $x_2 = x_1$. Thus $z_2 = z_1$. By (5b),

$$0 = 2\varepsilon_1 z_1 + \varepsilon_3 z_3 = 2\varepsilon_1(z_1^2/x_1) + \varepsilon_3(z_3^2/x_3).$$

Hence $z_3 = -2\varepsilon_1 \varepsilon_3 z_1$ and so

$$0 = 2\varepsilon_1(z_1^2/x_1) + 4\varepsilon_3(z_1^2/x_3).$$

By (5a), $z_1 \neq 0$, so $0 = (2\varepsilon_1/x_1) + (4\varepsilon_3/x_3)$ and

$$\varepsilon_3 x_3 = -2\varepsilon_1 x_1.$$

But, by (5b), $1 + 2\varepsilon_1 x_1 + \varepsilon_3 x_3 = 0$, a contradiction.

(b) By (1a), $xu = ux$ for every $u \in H^*$. Let β be an automorphism of K that fixes a primitive fourth root of unity and takes every root of unity of odd order into its inverse. Then, by (a),

$$\chi_i(xu) = (\chi_i(xu))^\beta = \chi_i(xu^{-1}).$$

(8) *Suppose* $1 \leq i \leq 3$. *Then* $\chi_i(x)$ *and* $(\chi_i^2, \eta)_G$ *are odd.*

Proof. Let $\chi = \chi_i$. By (6),

$$\varepsilon_i = (\chi, \eta)_G = (1/|H^*|) \textstyle\sum_{u \in H^*} \chi(xu).$$

Let I be a subset of $H^* - \{1\}$ that contains precisely one element from each pair $\{u, u^{-1}\}$ of elements of $H^* - \{1\}$. By (7), χ is integer-valued and

$$\chi(x) + 2 \textstyle\sum_{u \in I} \chi(xu) = \chi(x) + \sum_{u \in I} (\chi(xu) + \chi(xu^{-1}))$$
$$= \textstyle\sum_{u \in H^*} \chi(xu) = |H^*| \varepsilon_i.$$

Since $|H^*|$ and ε_i are odd, $\chi(x)$ is odd.

Applying (6) and (7) again, we have

$$|H^*| (\chi^2, \eta)_G \equiv \textstyle\sum_{u \in H^*} \chi^2(xu) \equiv \sum_{u \in H^*} \chi(xu) \equiv \varepsilon_i |H^*| \equiv 1 \pmod{2}.$$

So $(\chi^2, \eta)_G$ is odd.

(9) *Suppose* $1 \leq i \leq 3$. *Then* $\chi_i(x) = \varepsilon_i$.

Proof. Suppose $\chi_i(x) \neq \varepsilon_i$. Let $\chi = \chi_i$. By (7a), the values of χ are rational integers. By (8), $\chi(x)$ is odd. Hence $|\chi(x)| > 1$ or $\chi(x) = -\varepsilon_i$ In either case, $\chi(x)^2 > \varepsilon_i \chi(x)$. Similarly, $\chi(y)^2 \geq \varepsilon_i \chi(y)$ for all $y \in G$. Hence, by (6),

$$\begin{aligned}(\chi^2, \eta)_G &= (1/|H^*|) \textstyle\sum_{u \in H^*} \chi(xu)^2 \\ &> (1/|H^*| \textstyle\sum_{u \in H^*} \varepsilon_i \chi(xu) \\ &= \varepsilon_i(\chi, \eta)_G = 1.\end{aligned}$$

By (8), $(\chi^2, \eta)_G \geq 3$.

Now by (4a), x^2 is not in the kernel of χ; since x^2 is in the derived group of S, $\chi(1) > 1$. By (3), $\eta(y) = 4$ if $y \in B$ and $\eta(y) = 0$ if $y \in G - B$. As $(\chi, 1_G)_G = 0$, an argument like that of the previous paragraph yields

$$\begin{aligned}\textstyle\sum_{y \in G-B} \chi(y)^2 > \textstyle\sum_{y \in G-B} (-\varepsilon_i)\chi(y) &= \textstyle\sum_{y \in B} \varepsilon_i \chi(y) \\ &= (\tfrac{1}{4}) \textstyle\sum_{y \in G} \varepsilon_i \chi(y)\eta(y^{-1}) = (|G|/4)(\varepsilon_i \chi, \eta)_G = (|G|/4).\end{aligned}$$

Hence

$$\begin{aligned}|G| = \textstyle\sum_{y \in G-B} \chi(y)^2 + \textstyle\sum_{y \in B} \chi(y)^2 &> (|G|/4) + (\tfrac{1}{4}) \textstyle\sum_{y \in G} \chi(y)^2\eta(y^{-1}) \\ &= (|G|/4) + (|G|/4)(\chi^2, \eta)_G \geq |G|,\end{aligned}$$

a contradiction.

(10) *Suppose* $1 \leq i \leq 3$. *Then*
(a) z_i *is divisible by* 4;
(b) x_i *is odd*;
(c) *if* z_i *is divisible by* 8, *then* $x_i - \varepsilon_i$ *is divisible by* 4.

Proof. Let $\chi = \chi_i$. Let 1_S be the principal character of S and let ψ be the unique irreducible character of S of degree two. Then

$$\psi(1) = 2, \quad \psi(x^2) = -2, \quad \text{and} \quad \psi(y) = 0 \quad \text{for all } y \in S - \langle x^2 \rangle.$$

By (4b) and (9), $\chi(y) = \varepsilon_i$ for all $y \in S - \langle x^2 \rangle$. Since $(\chi|_S, \psi)_S$ is an integer,

$$\begin{aligned}0 \equiv 8(\chi|_S, \psi)_S \equiv \textstyle\sum_{y \in S} \chi(y)\psi(y^{-1}) &\equiv 2\chi(1) - 2\chi(x^2) \\ &\equiv 2x_i - 2y_i \equiv 2z_i \pmod{8}.\end{aligned}$$

This proves (a). Similarly,

$$\begin{aligned}0 \equiv 8(\chi|_S, 1_S)_S \equiv \textstyle\sum_{y \in S} \chi(y) &\equiv x_i + y_i + 6\varepsilon_i \\ &\equiv 2x_i - z_i - 2\varepsilon_i \equiv 2(x_i - \varepsilon_i) - z_i \pmod{8}.\end{aligned}$$

By (a), 4 divides z_i and therefore divides $2(x_i - \varepsilon_i)$. So $x_i - \varepsilon_i$ is even, and x_i is odd. Finally, suppose 8 divides z_i. Then 8 divides $2(x_i - \varepsilon_i)$, which yields (c).

(11) *There exists* i *such that* $1 \leq i \leq 3$, z_i *is divisible by* 8, *and* $x_i + \varepsilon_i$ *is divisible by* 4.

Proof. Let 2^k be the highest power of 2 that divides every z_i. By (10), $k \geq 2$. Let $w_i = z_i/2^k$ for $i = 1, 2, 3$. Then some w_i is odd, say, w_3. By (5b),

$$\sum \varepsilon_i(z_i^2/x_i) = 0 = \sum \varepsilon_i(w_i^2/x_i),$$

so

$$0 = \varepsilon_1 x_2 x_3 w_1^2 + \varepsilon_2 x_1 x_3 w_2^2 + \varepsilon_3 x_1 x_2 w_3^2.$$

By (10), each x_i is odd. As 0 is not a sum of three odd numbers, w_i is even for some i, say, for $i = 1$. It follows that w_2 is odd. Since $z_1 = 2^k w_1$ and $k \geq 2$, z_1 is divisible by 8.

From the above equation,

$$0 \equiv -\varepsilon_1 x_2 x_3 w_1^2 \equiv \varepsilon_2 x_1 x_3 w_2^2 + \varepsilon_3 x_1 x_2 w_3^2 \equiv \varepsilon_2 x_1 x_3 + \varepsilon_3 x_1 x_2$$
$$\equiv \varepsilon_2 \varepsilon_3 x_1(\varepsilon_3 x_3 + \varepsilon_2 x_2) \quad (\text{modulo } 4).$$

Therefore 4 divides $\varepsilon_3 x_3 + \varepsilon_2 x_2$. By (5), $1 + \sum \varepsilon_i x_i = 0$. Hence

$$0 \equiv \varepsilon_3 x_3 + \varepsilon_2 x_2 \equiv -(1 + \varepsilon_1 x_1) \equiv -\varepsilon_1(x_1 + \varepsilon_1) \quad (\text{modulo } 4),$$

which yields that 4 divides $x_1 + \varepsilon_1$.

Since (10) and (11) contradict one another, this completes the proof of the theorem for the case in which $|S| = 8$.

References

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