

ON A CLASS OF COMPLEX BANACH SPACES

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1. Introduction

A real (resp., complex) *Lindenstrauss space* is a real (resp., complex) Banach space V for which the dual V^* is isometric to an L^1 -space. The real Lindenstrauss spaces were introduced by Grothendieck [13], and were first investigated in depth by Lindenstrauss [20]. It has recently become apparent that one can often gain new insights into the structure of a Banach space V through the use of Choquet theory on the closed unit ball K of V^* . In particular, Lazar has proved in [19] that the real Lindenstrauss spaces V are characterized by a certain "simplex-like" condition on K . This theorem has played an important role in the subsequent development of the theory of real Lindenstrauss spaces (see [9], [10], [11], [24]).

In this paper we shall demonstrate that a complex analogue of Lazar's condition, that was proposed in [9], characterizes the complex Lindenstrauss spaces (see Theorem 4.3). We have reason to believe that this result will make the theory of complex Lindenstrauss spaces as accessible as that for the real spaces.

I am indebted to Professor Robert Phelps for a suggestion that considerably simplified the measure-theoretic arguments occurring in an earlier version of this paper. Specifically, he contributed the present definition for the measure $\omega(|\mu|)$, which he had arrived at in his investigations of complex function spaces.

Throughout this paper we shall use the following notation: If K is a compact Hausdorff space, then $C(K)$ is the Banach space of complex continuous functions on K with the uniform norm. If (X, \mathcal{S}, μ) is a measure space with $X \in \mathcal{S}$, $L^1(X, \mathcal{S}, \mu)$ is the Banach space of complex integrable functions on X with the L^1 -norm. If V is a complex Banach space, V^* is the Banach dual of V , i.e., the complex linear functions on V with the usual norm. If J is a subspace of V , we let J^0 denote the annihilator of J in V^* .

2. Complex L -spaces

We will say that a complex Banach space W is a C -space if it is isometric to $C(K)$ for some compact Hausdorff space K . We will say that W is an L -space if it is isometric to $L^1(X, \mathcal{S}, \mu)$ for some measure space (X, \mathcal{S}, μ) with $X \in \mathcal{S}$. The real analogue of the following result first appeared in [13]. For the complex case see [23, §1.13].

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LEMMA 2.1. *Let W be a complex Banach space. Then W is an L -space if and only if W^* is a C -space.*

If W is a C -space, then W^* is an L -space. The converse is false, i.e., the Lindenstrauss spaces constitute a wider class than do the C -spaces.

The real version of the following result is due to Goodner and Nachbin (see [7, p. 95], [12], [22]). The complex case is a result of Hasumi [14].

LEMMA 2.2. *Suppose that W is a complex Banach space. Then the following are equivalent:*

- (a) *W has the property that if $V_1 \subseteq V_2$ are complex Banach spaces, and $\theta_1 : V_1 \rightarrow W$ is a bounded linear map, then there is an extension $\theta_2 : V_2 \rightarrow W$ of θ_1 for which $\|\theta_2\| = \|\theta_1\|$.*
- (b) *W is isometric to $C(K)$, where K is a compact Stonean space.*

I am obligated to J. Lindenstrauss and L. Tzafriri for the following observation:

COROLLARY 2.3. *Suppose that W is an L -space and that π is a projection of W into itself with $\|\pi\| = 1$. Then $\pi(W)$ is an L -space.*

Proof. We have a natural isometry

$$\pi(W)^* \cong W^*/\pi(W)^0.$$

Letting π^* be the adjoint projection, it is a simple matter to verify that $u \rightarrow u + \ker \pi^*$ induces an isometry

$$W^*/\ker \pi^* \cong \pi^*(W^*).$$

From Lemma 2.1, W^* is isometric to $C(K)$ for a compact space K , and since $C(K)$ is a dual space, K is hyperstonean (see [8]), and W^* satisfies (a) of Lemma 2.2. Since $\|\pi^*\| \leq 1$, it follows that the same is true for $\pi^*(W^*)$, since one may first extend θ_1 to a map θ_2 of V_2 into W^* with the same norm, and then use $\pi^* \circ \theta_2$. Thus $\pi(W)^* \cong \pi^*(W^*)$ is a C -space, and from Lemma 2.1, $\pi(W)$ is an L -space.

Suppose that $W = L^1(X, \mathfrak{S}, \mu)$. The subspace of step functions is dense in W . In the proof of Theorem 4.3, we will need a systematic procedure for approximating general functions by step functions, which we will now explain.

Given $p \in W$, let

$$N(p) = \{x \in X : p(x) \neq 0\}$$

(this is determined to within a μ -null set). We say that a countable subset $\mathfrak{B} = \{B_1, B_2, \dots\}$ of \mathfrak{S} is a *partition* for p if

- (a) $0 < \mu(B_j) < \infty$ for all j ,
- (b) $B_j \cap B_k = \emptyset$ for $j \neq k$,
- (c) $N(p) \subseteq \cup B_j$.

Given a family $\mathfrak{C} = (C_1, C_2, \dots)$ satisfying $\mu(C_j) < \infty$ and (c), a simple induction will provide one with a family \mathfrak{B} satisfying (a)-(c). Thus given a sequence of functions p_1, p_2, \dots , the sets

$$C_{jk} = \{x \in X : |p_j(x)| \geq 1/k\}$$

will generate a common partition \mathfrak{B} for all of the functions p_1, p_2, \dots .

Given $p \in W$ and a partition $\mathfrak{B} = \{B_1, B_2, \dots\}$ for p , the *conditional expectation of p* with respect to \mathfrak{B} is the step function

$$E(p | \mathfrak{B}) = \sum_j \left[\mu(B_j)^{-1} \int_{B_j} p \, d\mu \right] \chi_{B_j},$$

where χ_B is the characteristic function

$$\begin{aligned} \chi_B(x) &= 1 & x \in B \\ &= 0 & x \notin B. \end{aligned}$$

$E(p | \mathfrak{B})$ is determined to within null sets, and the following relations hold for $p, q \in W$ and $c \in \mathbf{C}$ almost everywhere:

- (a) $E(p + q | \mathfrak{B}) = E(p | \mathfrak{B}) + E(q | \mathfrak{B})$,
- (b) $E(cp | \mathfrak{B}) = cE(p | \mathfrak{B})$,
- (c) $|E(p | \mathfrak{B})| \leq E(|p| | \mathfrak{B})$,
- (d) $\int E(p | \mathfrak{B}) \, d\mu = \int p \, d\mu$.

In particular from (c) and (d), $E(p | \mathfrak{B})$ is itself a member of W , with

$$(2.1) \quad \|E(p | \mathfrak{B})\| \leq \|p\|.$$

LEMMA 2.4. *Given $p_1, \dots, p_m \in W$ and $\varepsilon > 0$, there is a common partition \mathfrak{B} for p_1, \dots, p_m with*

$$\|p_k - E(p_k | \mathfrak{B})\| \leq \varepsilon, \quad k = 1, \dots, m.$$

Proof. If we let

$$C_{n,k} = \{x : 1/n \leq |p_k(x)| \leq n\},$$

then

$$\int_{X \setminus C_{n,k}} |p_k(x)| \, d\mu(x) \downarrow 0$$

as $n \rightarrow \infty$. From this it is apparent that there exists an integer n_0 and a Borel set $S_0 \in \mathfrak{S}$ such that for all k ,

- (a) $|p_k(x)| \leq n_0$ for all $x \in S_0$,
- (b) $\int_{X \setminus S_0} |p_k(x)| \, d\mu(x) \leq \varepsilon/3$,
- (c) $\mu(S_0) < \infty$.

Let $D \subseteq C$ be the closed disk of radius n_0 and center 0, and let $D_j, j = 1, 2, \dots, r$ be closed disks of diameter less than $\varepsilon/3\mu(S_0)$ with

$$D \subseteq \bigcup D_j.$$

Taking intersections and differences of the sets

$$S_{jk} = p_k^{-1}(D_j) \cap S_0, \quad j = 1, \dots, r; \quad k = 1, \dots, m,$$

we obtain disjoint B_1, \dots, B_s in \mathfrak{S} such that for all h, k ,

$$|p_k(x) - p_k(y)| < \varepsilon/3\mu(S_0) \quad \text{all } x, y \in B_h,$$

and $S_0 = \cup B_h$. We select disjoint B_{s+1}, B_{s+2}, \dots in \mathfrak{S} with

$$B_{s+h} \subseteq X \setminus S_0, \quad \mu(B_{s+h}) < \infty,$$

and for which

$$\cup_k N(p_k) \setminus S_0 \subseteq \cup_h B_{s+h}.$$

Deleting null-sets, $\mathfrak{B} = \{B_1, B_2, \dots\}$ is a partition for the p_k .

Given k , and $x \in B_h, h \leq s$, we have

$$\begin{aligned} |p_k(x) - E(p_k | \mathfrak{B})(x)| &= |p_k(x) - \mu(B_h)^{-1} \int_{B_h} p_k(y) d\mu(y)| \\ &= \mu(B_h)^{-1} \int_{B_h} [p_k(x) - p_k(y)] d\mu(y) \\ &\leq \varepsilon/3\mu(S_0), \end{aligned}$$

and if $x \in B_h, h > s$,

$$|p_k(x) - E(p_k | \mathfrak{B})(x)| \leq |p_k(x)| + \mu(B_h)^{-1} \int_{B_h} |p_k(y)| d\mu(y).$$

Thus

$$\|p_k - E(p_k | \mathfrak{B})\| \leq \sum_{h=1}^s \varepsilon \mu(B_h)/3\mu(S_0) + 2 \int_{X \setminus S_0} |p_k| d\mu \leq \varepsilon.$$

3. Complex measure fields

In this section we shall summarize the Bourbaki theory of measure fields. We will not need the sophisticated theory of "adequate" fields introduced in the second edition of *Intégration* [4], [5].

Let K be a compact Hausdorff space, and $M(K) = C(K)^*$ be the Banach space of complex regular Borel measures on K . We will use the notation:

$$M_1(K) = \{\mu \in M(K) : \|\mu\| \leq 1\},$$

$$M^+(K) = \{\mu \in M(K) : \mu \geq 0\},$$

$$P(K) = \{\mu \in M(K) : \mu \geq 0, \|\mu\| = 1\},$$

i.e., $P(K)$ consists of the probability measures on K . If $p \in K$, we let $\delta(p)$ be the unit mass at p .

Given $\mu \in M(K)$, let $|\mu|$ be the total variation of μ . We recall that for $f \in C^+(K)$,

$$|\mu|(f) = \sup \{|\mu(\varphi)| : |\varphi| \leq f, \varphi \in C(K)\}$$

(see [4, III, §1.7]). There exists a $|\mu|$ -essentially unique complex Borel

function φ on K such that $|\varphi(p)| = 1$ for all $p \in K$, and $\mu = \varphi|\nu|$ (see [15, p. 325]). The representation $\mu = \varphi|\nu|$ is called a *polar decomposition for μ* .

Suppose that T and K are compact Hausdorff spaces, and that ν is a measure on T . A *field* λ of complex K -measures on T is a map $\theta \rightarrow \lambda_\theta$ of T into $M(K)$. We define

$$\|\lambda\| = \sup \{\|\lambda_\theta\| : \theta \in T\}$$

and we will write $\lambda \geq 0$ if $\lambda_\theta \geq 0$ for all $\theta \in T$. Given a field λ , we define the field $|\lambda|$ "pointwise":

$$|\lambda|_\theta = |\lambda_\theta|.$$

Providing $M(K)$ with the weak* topology, λ is continuous (resp. Borel) if and only if $\theta \rightarrow \lambda_\theta(f)$ is continuous (resp. Borel) for each $f \in C(K)$. Given $f \in C^+(K)$,

$$|\lambda|_\theta(f) = \sup \{|\lambda_\theta(\varphi)| : \varphi \in C(K), |\varphi| \leq f\},$$

hence if λ is continuous, $\theta \rightarrow |\lambda|_\theta(f)$ is lower semi-continuous and thus Borel.

Let us assume $\|\lambda\| < \infty$. If λ is Borel, we define a measure

$$\beta = \int \lambda_\theta(f) d\nu(\theta)$$

on K by letting

$$(3.1) \quad \beta(f) = \int \lambda_\theta(f) d\nu(\theta) \quad \text{for } f \in C(K).$$

If λ is continuous and $f \in C^+(K)$, then given $\varphi \in C(K)$, $|\varphi| \leq f$,

$$|\beta(\varphi)| \leq \int |\lambda_\theta(\varphi)| d|\nu|(\theta) \leq \int |\lambda_\theta|_\theta(f) d|\nu|(\theta),$$

hence we have

$$(3.2) \quad \left| \int \lambda_\theta d\nu(\theta) \right| \leq \int |\lambda_\theta|_\theta d|\nu|(\theta).$$

For continuous λ one may use (3.1) to evaluate $\beta(f)$ for discontinuous f . We have:

LEMMA 3.1. *Suppose that $\lambda \geq 0$ is continuous and that f is a β -integrable complex function on K . Then*

- (a) f is λ_θ -integrable for ν -almost all θ ,
- (b) $\theta \rightarrow \lambda_\theta(f)$ is ν -measurable,
- (c) $\beta(f) = \int \lambda_\theta(f) d\nu(\theta)$.

Proof. See [5, §3.3, Th. 1] and [5, §3.1, Prop. 2].

"Image measures" are conveniently defined by means of measure fields. Suppose that m is a measure on K and that $\varphi : K \rightarrow T$ is a Borel map. Then it is evident that $p \rightarrow \delta(\varphi(p))$ is a Borel field. The *image measure*

$\varphi(m)$ is defined by

$$\varphi(m) = \int \delta(\varphi(p)) dm(p).$$

4. Complex measures on a ball

Throughout this section, T will denote the unit circle, and $d\alpha$ the unit Haar measure on T .

Let V be a complex Banach space, V^* the dual of V , and K the closed unit ball at the origin in V^* , with the weak* topology. K is a compact convex set. Each $\zeta \in T$ determines an affine weak* homeomorphism of K by $\sigma_\zeta(p) = \zeta p$. This in turn induces isomorphisms of $C(K)$ and $M(K)$ with the relevant structures via $\sigma_\zeta(f) = f \circ \sigma_\zeta^{-1}$ and $\sigma_\zeta(\mu) = \mu \circ \sigma_\zeta^{-1}$ (this coincides with the σ_ζ -image of μ).

We say that a function f on K is T -invariant (resp., T -homogeneous) if $f(\zeta p) = f(p)$ (resp., $f(\zeta p) = \zeta f(p)$) for all $\zeta \in T, p \in K$. Similarly, we say that a measure $\mu \in M(K)$ is T -invariant (resp., T -homogeneous) if $\sigma_\zeta \mu = \mu$ (resp., $\sigma_\zeta \mu = \zeta \mu$) for all $\zeta \in T$. We let $C_{\text{inv}}(K)$ (resp., $C_{\text{hom}}(K)$) and $M_{\text{inv}}(K)$ (resp., $M_{\text{hom}}(K)$) denote the corresponding linear spaces of functions and measures. If $f \in C(K)$, then the function

$$(\text{inv}_T f)(p) = \int f(\alpha p) d\alpha$$

is continuous and T -invariant. It is readily verified that inv_T is a norm-decreasing projection of $C(K)$ onto $C_{\text{inv}}(K)$. Similarly, if we let

$$(\text{hom}_T f)(p) = \int \alpha^{-1} f(\alpha p) d\alpha,$$

then hom_T is a norm-decreasing projection of $C(K)$ onto $C_{\text{hom}}(K)$. Taking the adjoints of these projections on $M(K)$,

$$\text{inv}_T \mu = \mu \circ \text{inv}_T, \quad \text{hom}_T \mu = \mu \circ \text{hom}_T,$$

we have norm-decreasing, weak* continuous projections of $M(K)$ onto $M_{\text{inv}}(K)$ and $M_{\text{hom}}(K)$, respectively.

Given $\mu \in M(K)$, the measure fields on T defined by

$$\alpha \rightarrow \sigma_\alpha \mu \quad \text{and} \quad \alpha \rightarrow \alpha^{-1} \sigma_\alpha \mu$$

are continuous. If $f \in C(K)$, then

$$\begin{aligned} (\text{inv}_T \mu)(f) &= \mu(\text{inv}_T f) \\ &= \iint f(\alpha p) d\alpha d\mu(p) \\ &= \iint f(\alpha p) d\mu(p) d\alpha \end{aligned}$$

$$= \int \sigma_\alpha(\mu)(f) d\alpha,$$

hence

$$\text{inv}_T \mu = \int \sigma_\alpha(\mu) d\alpha.$$

A similar argument shows that

$$\text{hom}_T \mu = \int \alpha^{-1} \sigma_\alpha(\mu) d\alpha.$$

In particular from (3.2),

$$(4.1) \quad |\text{hom}_T \mu| \leq \text{inv}_T |\mu|.$$

A simple calculation gives the important relation

$$(4.2) \quad \text{hom}_T \sigma_\alpha \mu = \alpha \text{hom}_T \mu.$$

Given $\mu \in M(K)$, let $\mu = \varphi |\mu|$ be a polar decomposition for μ (see §3). Since φ is Borel, it is evident that the same is true for the map $\omega : K \rightarrow K$ defined by

$$(4.3) \quad \omega(p) = \varphi(p)p.$$

If $f \in C(K)$ is T -homogeneous, then

$$\begin{aligned} \omega(|\mu|)(f) &= \int f(\varphi(p)p) d|\mu|(p) \\ &= \int \varphi(p)f(p) d|\mu|(p) \\ &= \mu(f), \end{aligned}$$

hence

$$(4.4) \quad \text{hom}_T \omega(|\mu|) = \text{hom}_T \mu.$$

On the other hand, if $f \in C(K)$ is T -invariant,

$$\omega(|\mu|)(f) = |\mu|(f),$$

hence

$$(4.5) \quad \text{inv}_T \omega(|\mu|) = \text{inv}_T |\mu|.$$

Regarding the elements of V as the weak* linear functions on V^* , the restriction map $v \rightarrow v|_K$ is a complex linear isometry of V into $C(K)$ (it can be shown that the image consists of the affine T -homogeneous functions in $C(K)$.) If $\mu \in M(K)$, $v \rightarrow \mu(v)$ is a bounded linear functional on V , and the *resultant* of μ is defined to be the unique point $r(\mu) \in V^*$ satisfying

$$r(\mu)(v) = \mu(v).$$

If $\mu \in P(K)$, we claim that $p = r(\mu)$ coincides with the "barycenter" of μ , i.e. if a is a real affine continuous function on K , then $\mu(a) = a(p)$. To

prove this, note that $a = \alpha + t$, where $\alpha \in \mathbf{R}$ and t is a real weak* continuous linear function on K (see [1, Cor. I.1.5]). On the other hand, $t = \operatorname{Re} v$ where $v \in V$ (let $v(p) = t(p) - it(ip)$). Thus

$$\mu(a) = \alpha + \mu(t) = \alpha + \mu(\operatorname{Re} v).$$

We have

$$\mu(\operatorname{Re} v) + i\mu(\operatorname{Im} v) = \mu(v) = v(p) = \operatorname{Re} v(p) + i \operatorname{Im} v(p),$$

hence

$$\mu(a) = \alpha + \operatorname{Re} v(p) = a(p).$$

It is readily verified that $r : M(K) \rightarrow V^*$ is a weak* continuous, norm-decreasing, linear surjection. Furthermore, we have the relations

$$r(\sigma_T \mu) = \zeta r(\mu), \quad r(\operatorname{hom}_T \mu) = r(\mu),$$

hence from (4.4),

$$r(\omega(|\mu|)) = r(\operatorname{hom}_T \omega(|\mu|)) = r(\mu).$$

We let $P(K)$ have the usual dilation order $<$, and we say that $\mu \in M(K)$ is maximal if $\mu = 0$, or $\mu \neq 0$ and the probability measure $|\mu|/\|\mu\|$ is $<$ -maximal. If $f \in C(K)$ is real, we recall that the upper envelope \hat{f} is defined by

$$\hat{f}(p) = \sup \{v(f) : v \in P(K), \quad r(v) = p\}.$$

If we let

$$B(f) = \{p \in K : \hat{f}(p) = f(p)\},$$

then $\mu \in M(K)$ is maximal if and only if

$$(4.6) \quad |\mu|(K \setminus B(f)) = 0$$

for all real $f \in C(K)$, and it suffices to verify (4.6) for all real convex $f \in C(K)$. We shall denote the maximal measures on K by $M^{\max}(K)$, and let $M^{\max}_{\operatorname{hom}}(K)$ be the corresponding subspace of maximal T -homogeneous measures.

LEMMA 4.1. *If $f \in C(K)$ is real and convex, then*

$$B(\operatorname{inv}_T f) \subseteq B(f).$$

Proof. If $p \notin B(f)$, then $\hat{f}(p) > f(p)$. Choose $\mu \in P(K)$ with $r(\mu) = p$ and $\mu(f) > f(p)$. For all $\alpha \in T$, $r(\sigma_\alpha \mu) = \alpha p$, and since f is convex,

$$(\sigma_\alpha \mu)(f) \geq f(\alpha p).$$

Noting that $\alpha \rightarrow \sigma_\alpha^{-1} f$ is norm-continuous, the function

$$\alpha \rightarrow (\sigma_\alpha \mu)(f) - f(\alpha p)$$

is continuous, non-negative, and strictly positive when $\alpha = 1$. Thus since $d\alpha$ has support T ,

$$0 < \int (\sigma_\alpha \mu)(f) - f(\alpha p) d\alpha$$

$$\begin{aligned}
&= \operatorname{inv}_T \mu(f) - \operatorname{inv}_T f(p) \\
&= \mu(\operatorname{inv}_T f) - \operatorname{inv}_T f(p) \\
&\leq (\operatorname{inv}_T f)^\wedge(p) - \operatorname{inv}_T f(p),
\end{aligned}$$

and $p \notin B(\operatorname{inv}_T f)$.

LEMMA 4.2. *If $\mu \in M(K)$ is maximal, then so are the measures $\operatorname{hom}_T \mu$ and $\omega(|\mu|)$.*

Proof. Given $f \in C(K)$ real and convex, let $F = \operatorname{inv}_T f$. It is easily seen that the set $K \setminus B(F)$ is invariant under the maps σ_α , $\alpha \in T$. Since

$$\alpha \rightarrow \sigma_\alpha(|\mu|)$$

is continuous, we have from Lemma 3.1,

$$\begin{aligned}
\operatorname{inv}_T |\mu|(K \setminus B(F)) &= \int \sigma_\alpha(|\mu|)(K \setminus B(F)) \, d\alpha \\
&= \int |\mu|(K \setminus B(F)) \, d\alpha \\
&= |\mu|(K \setminus B(F)) \\
&= 0
\end{aligned}$$

and from (4.1), $\operatorname{hom}_T \mu$ is maximal. On the other hand, from (4.5),

$$\omega(|\mu|)(K \setminus B(F)) = \operatorname{inv}_T |\mu|(K \setminus B(F)) = 0,$$

hence $\omega(|\mu|)$ is also maximal.

THEOREM 4.3. *Suppose that V is a complex Banach space with dual V^* , and let K be the closed unit ball of V^* . Then the following are equivalent:*

- (a) V is a Lindenstrauss space.
- (b) If ν_1 and ν_2 are maximal probability measures on K with $r(\nu_1) = r(\nu_2)$, then $\operatorname{hom}_T \nu_1 = \operatorname{hom}_T \nu_2$.

Proof. (a) \Rightarrow (b). Let us suppose that V is a Lindenstrauss space, and identify V^* with a complex L^1 space $L^1(X, \mathcal{S}, \mu)$ with $X \in \mathcal{S}$. Given maximal measures $\nu_1, \nu_2 \in P(K)$ with $r(\nu_1) = r(\nu_2)$, let

$$\nu = \frac{1}{2}(\nu_1 + \sigma_{-1} \nu_2).$$

Then ν is a maximal probability measure with $r(\nu) = 0$, and it suffices to prove that $\operatorname{hom}_T \nu = 0$.

We may select a net of atomic measures

$$\nu_\gamma = \sum_{k=1}^{n(\gamma)} c_{\gamma k} \delta(p_{\gamma k}), \quad \sum c_{\gamma k} = 1, \quad 0 \leq c_{\gamma k},$$

with

$$0 = r(\nu_\gamma) = \sum_{k=1}^{n(\gamma)} c_{\gamma k} p_{\gamma k}$$

(see [1, Prop. I.2.3]). Given $\varepsilon > 0$ and γ , we have from Lemma 2.4, a

common partition $\mathfrak{B}_\gamma^\varepsilon = \{B_1, B_2, \dots\}$ for $p_{\gamma_1}, \dots, p_{\gamma_n(\gamma)}$ such that for all k

$$\| p_{\gamma k} - E(p_{\gamma k} | \mathfrak{B}_\gamma^\varepsilon) \| < \varepsilon.$$

Letting $p_{\gamma k}^\varepsilon = E(p_{\gamma k} | \mathfrak{B}_\gamma^\varepsilon)$, the probability measures

$$\nu_\gamma^\varepsilon = \sum_k c_{\gamma k} \delta(p_{\gamma k}^\varepsilon)$$

also converge weak* to ν as $\gamma \rightarrow \infty$ and $\varepsilon \rightarrow 0$, and

$$(4.7) \quad r(\nu_\gamma^\varepsilon) = \sum_k E(p_{\gamma k} | \mathfrak{B}_\gamma^\varepsilon) = E(\sum_k c_{\gamma k} p_{\gamma k} | \mathfrak{B}_\gamma^\varepsilon) = 0.$$

It suffices to show that for each γ and $\varepsilon > 0$, there is a $\lambda_\gamma^\varepsilon \in P(K)$ with $\lambda_\gamma^\varepsilon \succ \nu_\gamma^\varepsilon$, and $\text{hom}_T \lambda_\gamma^\varepsilon = 0$, since if this is the case, let λ_δ be a weak* convergent subnet of $\lambda_\gamma^\varepsilon$. Letting $\lambda = \lim_\delta \lambda_\delta$, it is evident that $\lambda \succ \nu$, hence $\lambda = \nu$. Thus

$$\text{hom}_T \nu = \lim_\delta \text{hom}_T \lambda_\delta = 0.$$

We have

$$(4.8) \quad p_{\gamma k}^\varepsilon = \sum_{j=1}^\infty a_{jk} \chi_{B_j},$$

where from (2.1),

$$\sum_j |a_{jk}| \mu(B_j) = \| p_{\gamma k}^\varepsilon \| \leq \| p_{\gamma k} \| \leq 1.$$

Letting $a_{jk} = \zeta_{jk} |a_{jk}|$, $\zeta_{jk} \in T$, and $q_j = \chi_{B_j} \mu(B_j)^{-1}$, the probability measure

$$\lambda_\gamma^\varepsilon = \sum_j |a_{jk}| \mu(B_j) \delta(\zeta_{jk} q_j) + (1 - \| p_{\gamma k}^\varepsilon \|) \delta(0)$$

has resultant $p_{\gamma k}^\varepsilon$. It follows that

$$\lambda_\gamma^\varepsilon = \sum_k c_{\gamma k} \lambda_{\gamma k}^\varepsilon$$

is a dilation of ν_γ^ε . On the other hand, from (4.7),

$$\sum_k c_{\gamma k} p_{\gamma k}^\varepsilon = 0,$$

hence if we multiply by χ_{B_j} (see (4.8))

$$\sum_k c_{\gamma k} a_{jk} = 0.$$

We have from (4.2) with $\mu = \delta(q_j)$ and $\alpha = \zeta_{jk}$,

$$\begin{aligned} \text{hom}_T \lambda_\gamma^\varepsilon &= \sum_k c_{\gamma k} \text{hom}_T \lambda_{\gamma k}^\varepsilon \\ &= \sum_{j,k} c_{\gamma k} |a_{jk}| \mu(B_j) \text{hom}_T \delta(\zeta_{jk} q_j) \\ &= \sum_j [\sum_k c_{\gamma k} a_{jk}] \mu(B_j) \text{hom}_T \delta(q_j) \\ &= 0. \end{aligned}$$

(b) \Rightarrow (a). We define a map $H : K \rightarrow M_{\text{hom}}^{\text{max}}(K)$ by letting $H(p) = \text{hom}_T \nu$, where ν is any maximal measure in $P(K)$ with $r(\nu) = p$ (due to b , this is well defined.) We begin by showing that H extends to a linear isometry of V^* onto $M_{\text{hom}}^{\text{max}}(K)$.

H is affine since given $p, q \in K$ and $0 \leq \alpha \leq 1$, let $\mu, \nu \in P(K)$ be maximal

with resultants p, q respectively. Then $\alpha\mu + (1 - \alpha)\nu$ is maximal in $P(K)$, and it has resultant $\alpha p + (1 - \alpha)q$, hence

$$\begin{aligned} H(\alpha p + (1 - \alpha)q) &= \text{hom}_T(\alpha\mu + (1 - \alpha)\nu) \\ &= \alpha \text{hom}_T \mu + (1 - \alpha) \text{hom}_T \nu \\ &= \alpha H(p) + (1 - \alpha)H(q). \end{aligned}$$

H is T -homogeneous, since if $p \in K$ and $\alpha \in T$, let μ be a maximal probability measure with resultant p . Then $\sigma_\alpha(\mu)$ is maximal with resultant αp , hence from (4.2),

$$H(\alpha p) = \text{hom}_T \sigma_\alpha \mu = \alpha \text{hom}_T \mu = \alpha H(p).$$

Since H is both affine and T -homogeneous, it is complex linear.

If $\|p\| = 1$, let $\mu \in P(K)$ be maximal with $r(\mu) = p$. Then $H(p) = \text{hom}_T \mu$, or since hom_T is norm-decreasing, $\|H(p)\| \leq 1$. On the other hand, if we regard V as a subspace of $C(K)$, $H(p)$ is an extension of the linear function p to $C(K)$, hence

$$1 = \|p\| \leq \|H(p)\|.$$

It follows that H is an isometry.

Given $\nu \in M_{\text{hom}}^{\text{max}}(K)$, $\|\nu\| \leq 1$, choose $q \in E(K)$ (the extreme points of K) and let

$$\mu = \omega(|\nu|) + \frac{1}{2}(1 - \|\omega(|\nu|)\|)(\delta(q) + \delta(-q)).$$

From Lemma 4.2 and (4.4), μ is a maximal measure in $P(K)$ and

$$\text{hom}_T \mu = \text{hom}_T \nu = \nu.$$

Letting $r(\mu) = p$, we have $H(p) = \nu$, and we have proved that H is onto.

We next show that $M^{\text{max}}(K)$ is an L -space. Given $f \in C(K)$, let

$$M^f(K) = \{\mu \in M(K) : |\mu|(K \setminus B(f)) = 0\}.$$

We may define a projection e of $M(K)$ onto $M^f(K)$ by letting $e(\mu)$ be the restriction $\mu|B(f)$. Then

$$(1 - e)\mu = \mu|K \setminus B(f),$$

and we have

$$\|e\mu\| + \|(1 - e)\mu\| = \|\mu\|.$$

In the terminology of [6], $M^f(K)$ is the range of the " L -projection" e . It follows from [6] or [3, Prop. 1.13] that

$$M^{\text{max}}(K) = \bigcap_{f \in C(K)} M^f(K)$$

is the range of a real linear L -projection π . From the proof of the latter fact, it is apparent that π is also complex linear (one may instead use the fact that any real linear L -projection in a complex Banach space must be complex linear—see [16].) Since $\|\pi\| \leq 1$, we have from Corollary 2.3 that $M^{\text{max}}(K)$ is an L -space.

From Lemma 4.2, hom_T is a norm-decreasing projection of $M^{\max}(K)$ onto $M_{\text{hom}}^{\max}(K)$. Thus from Corollary 2.3, $M_{\text{hom}}^{\max}(K)$ is an L -space. Since H is an isometry of V^* onto the latter space, V^* is an L -space, and V is a Lindenstrauss space.

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