PARABOLIC POTENTIALS WITH SUPPORT ON A HALF-SPACE

BY
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1. Introduction

We study the class of parabolic potentials $\mathcal{L}_\alpha^p$ introduced by Jones [4]. These spaces arise in the study of the heat equation; they are analogous to Sobolev spaces of fractional order.

We direct our attention to the problem of deciding whether the restriction of a function in $\mathcal{L}_\alpha^p$ to a half-space necessarily agrees with a function in $\mathcal{L}_\alpha^p$ supported on that half-space. In the case of Sobolev spaces the result is well known; one method of answering this question appears in Strichartz [7, §3]. Essentially the same approach is used here, but the presence of the time variable raises a number of complications.

For $1 < p < \infty$, it is possible to describe $\mathcal{L}_\alpha^p$ in terms of Sobolev spaces on R. This is done, for example, in [2]. Such a characterization could also be used here to give a somewhat shorter proof of the main theorem. However, the techniques used here produce additional insight.

2. Definitions and basic properties

Definition. A function f is in $\mathcal{L}_\alpha^p(R^{n+1})$ if $\hat{f} = (1 + |x|^2 + it)^{-\alpha/2}\hat{\phi}$ for some $\phi \in L^p(R^{n+1})$. Here $x \in R^n$, $t \in R$, and $\wedge$ denotes the Fourier transform in R^{n+1}. The norm of f is $\| f \|_{p,\alpha} = \| \phi \|_p$.

Definition.

$$H_\alpha(x, t) = t^{(\alpha-n)/2-1} \exp\{-x^2/4t\}, \quad t > 0$$
$$= 0, \qquad t \leq 0$$

Sampson [5] proves that if $f \in \mathcal{L}_\alpha^p$, $0 < \alpha < n + 2$, then $f = H_\alpha * g$ for some $g \in L^p$ with $\| g \|_p \leq c_{p,\alpha} \| f \|_{p,\alpha}$.

The following functional is useful in examining these spaces:

$$S_\alpha f(x, t) =$$

$$\left\{\int_0^\infty \left[\int_{|y|<1} \int_0^1 | f(x - ry, t - r^2 s) - f(x, y) | \, ds \, dy\right]^2 r^{-1-2\alpha} \, dr\right\}^{1/2}.$$

Theorem 2.2 of [1] states that for $0 < \alpha < 1$ and $1 < p < \infty$, $f \in \mathcal{L}_\alpha^p$ if and only if both $f \in L^p$ and $S_\alpha f \in L^p$, and that $\| f \|_{p,\alpha} \approx \| f \|_p + \| S_\alpha f \|_p$. Since

$$S_\alpha(fg) \leq \| f \|_\infty S_\alpha g + | g | S_\alpha f,$$

this characterization of $\mathcal{L}_\alpha^p$ is especially useful when products of functions are involved.

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Note that an equivalent functional is obtained if the y-integration is performed over the region $-1 \leq y_i \leq 1$, $i = 1, \cdots, n$; for our purposes, this will be more convenient.

3. Main theorem

Let $f \in \mathcal{L}_\alpha^p(R^{n+1})$, where $0 < \alpha < 1/p < 1$. Let $\zeta \in R^n \sim \{0\}$, and let

$$\begin{aligned} g(x, t) &= f(x, t), \quad t > x\cdot\zeta \\ &= 0, \qquad\quad t \leq x\cdot\zeta. \end{aligned}$$

Then $g \in \mathcal{L}_\alpha^p(R^{n+1})$, with $\| g \|_{p,\alpha} \leq c_{p,\alpha} \| f \|_{p,\alpha}$.

4. Some mixed-norm Sobolev inequalities

LEMMA 1. *Let $f \in \mathcal{L}_\alpha^p(R^{n+1})$, where $0 < \alpha < 1/p < 1$. Let $1/u = 1/p - \alpha/2$. Then for a.e. $x \in R^n$, $f(x, \cdot) \in L^u(R)$ and $\int \| f(x, \cdot) \|_u^p \, dx \leq c_{p,\alpha} \| f \|_{p\alpha}^p$.*

LEMMA 2: *Let $f \in \mathcal{L}_\alpha^p(R^{n+2})$, where $n \geq 0$ and $0 < \alpha < 1/p < 1$. Denote points in R^{n+1} as (x, ζ, t), where $x \in R^n$, $\zeta \in R$, and $t \in R$. Let $1/v = 1/p - \alpha$. Then for a.e. $(x, t) \in R^{n+1}$,*

$$f(x, \cdot, t) \in L^v(R) \quad \textit{and} \quad \iint \| f(x, \cdot, t) \|_v^p \, dx \, dt \leq c_{p,\alpha} \| f \|_{p,\alpha}^p.$$

Proof of Lemma 1. Let $f = H_\alpha * g$, $g \in L^p(R^{n+1})$, $\| g \|_p \leq c_{p,\alpha} \| f \|_{p,\alpha}$. Then

$$f(x, t) = \int_0^\infty s^{(\alpha-n)/2-1} \, ds \int_{R^n} \exp\{-|y|^2/4s\} g(x - y, t - s) \, dy.$$

Now

$$\begin{aligned} &\left| \int_{R^n} \exp\{-|y|^2/4s\} g(x - y) \, dy \right| \\ &\qquad \leq \textstyle\sum_{j=1}^\infty \displaystyle\int_{(j-1)s \leq |y|^2 \leq js} \exp\{-|y|^2/4s\} \, |g(x - y, t - s)| \, dy \\ &\qquad \leq \textstyle\sum_{j=1}^\infty \exp\{-(j - 1)/4\} \displaystyle\int_{|y|^2 \leq js} |g(x - y, t - s)| \, dy \\ &\qquad \leq \textstyle\sum_{j=1}^\infty \exp\{-(j - 1)/4\} c_n (js)^{n/2} M_1 g(x, t - s) \\ &\qquad = A s^{n/2} M_1 g(x, t - s), \end{aligned}$$

where $A = \sum_{j=1}^\infty c_n j^{n/2} \exp\{-(j - 1)/4\}$ and M_1 denotes a partial maximal function defined by

$$M_1 g(x, t) = \sup_{r>0} \frac{1}{m\{y : |y| \leq r\}} \int_{|y| \leq r} |g(x - y, t)| \, dx.$$

Thus

$$|f(x, t)| \leq A \int_0^\infty s^{\alpha/2-1} M_1 g(x, t - s) \, ds.$$

By the standard fractional integration theorem and the L^p-boundedness of the maximal function,

$$\| f(x, \cdot) \|_u \leq c_{p,\alpha} \| M_1 g(x, \cdot) \|_p$$

and

$$\int \| f(x, \cdot) \|_u^p \, dx \leq c_{p,\alpha} \int\int M_1 g(x, t)^p \, dt \, dx \leq c_{p,\alpha} \| g \|_p^p \leq c_{p,\alpha} \| f \|_{p,\alpha}^p .$$

The fractional integration theorem is proved in Hardy, Littlewood, and Polya [3, Theorem 383] and in Zygmund [8]. Stein [6, Chapter 1] contains a discussion of the maximal function.

Proof of Lemma 2. Again we set $f = H_\alpha * g$; this time

$$f(x, \zeta, t) = \int_{-\infty}^{\infty} d\eta \int_0^\infty s^{(\alpha-n-1)/2-1} ds \int_{R^n} \exp \{-|y|^2 + \eta^2)/4s\} \cdot g(x - y, \zeta - \eta, t - s) \, dy.$$

Just as in Lemma 1, the first integral is bounded by

$$As^{n/2} \exp \{- \eta^2/4s\} M_1 g(x, \zeta - \eta, t - s).$$

Now we bound the s-integral:

$$\begin{aligned} A \int_0^\infty & s^{(\alpha-1)/2-1} \exp \{-\eta^2/4s\} M_1 g(x, \zeta - \eta, t - s) \, ds \\ &= A \textstyle\sum_{j=-\infty}^{\infty} \displaystyle\int_{2^j\eta^2}^{2^{j+1}\eta^2} s^{(\alpha-1)/2-1} \exp \{-\eta^2/4s\} M_1 g(x, \zeta - \eta, t - s) \, ds \\ &\leq A \textstyle\sum_{j=-\infty}^{\infty} (2^j\eta^2)^{(\alpha-1)/2-1} \exp \{-1/4 \cdot 2^{j+1}\} \\ &\qquad \cdot \displaystyle\int_0^{2^{j+1}\eta^2} M_1 g(x, \zeta - \eta, t - s) \, ds \\ &\leq A \textstyle\sum_{j=-\infty}^{\infty} (2^j\eta^2)^{(\alpha-1)/2-1} \exp \{-2^{-j-3}\} \cdot 2^{j+1}\eta^2 M_3 M_1 g(x, \zeta - \eta, t) \\ &= AB |\eta|^{\alpha-1} M_3 M_1 g(x, \zeta - \eta, t), \end{aligned}$$

where $B = \sum_{j=-\infty}^{\infty} 2^{1+(\alpha-1)c/2} \exp \{- 2^{-j-3}\}$ and M_3 denotes another partial maximal function. Thus

$$|f(x, \zeta, t)| \leq AB \int_{-\infty}^{\infty} |\eta|^{\alpha-1} M_3 M_1 g(x, \zeta - \eta, t) \, d\eta;$$

the desired conclusion follows as in Lemma 1.

5. Proof of main theorem

Let

$$\begin{aligned} \chi(x, t) &= 1, \quad t > x \cdot \zeta \\ &= 0, \quad t \leq x \cdot \zeta. \end{aligned}$$

Then we must prove that $f \to \chi f$ defines a continuous mapping from $\mathfrak{L}_\alpha^p$ into itself for $0 < \alpha < 1/p < 1$.

Clearly $\| \chi f \|_p \le \| f \|_p \le \| f \|_{p,\alpha}$; it remains only to show $\| S_\alpha(\chi f) \|_p \le c \| f \|_{p,\alpha}$. Now

$$S_\alpha(\chi f) \le \| \chi \|_\infty S_\alpha f + | f | S_\alpha \chi.$$

As $\| \chi \|_\infty = 1$ and $S_\alpha f \in L^p$, we must show $| f | S_\alpha \chi \in L^p$.

Rotate coordinates in R^n so that ζ becomes $(0, \cdots, 0, | \zeta |)$. Then χ and consequently $S_\alpha \chi$ are independent of $x_1, \cdots, x_{n-1}$; to simplify notation we assume $n = 1$ and $\zeta = \lambda > 0$.

In the next section we show $S_\alpha \chi(x, t) \le c_\alpha(\lambda^\alpha | t - \lambda x |^{-\alpha} + | t - \lambda x |^{-\alpha/2})$.

By Lemma 1, we have for $\phi \in L^{2/\alpha}(R)$,

$$\int \int | \phi(t) f(x, t) |^p \, dt \, dx \le \int \| \phi \|_{2/\alpha}^p \| f(x, \cdot) \|_u^p \, dx$$

$$\le c_{p,\alpha} \| \phi \|_{2/\alpha}^p \| f \|_{p,\alpha}^p$$

since $\alpha/2 + 1/u = 1/p$. Using the same technique as in Strichartz [7, Theorem 3.6], it follows that also

$$\int \int | \phi(t) f(x, t) |^p \, dt \, dx \le M^p c_{p,\alpha} \| f \|_{p,\alpha}^p$$

provided only that

$$m\{t : | \phi(t) | > \eta\} \le (M\eta^{-1})^{2/\alpha}.$$

Since $m\{t : | t - \lambda x |^{-\alpha/2} > \eta\} = 2\eta^{-2/\alpha}$, $| t - \lambda x |^{-\alpha/2} | f | \in L^p$ with norm bounded by $c_{p,\alpha} \| f \|_{p,\alpha}$.

Using Lemma 2 and the same technique, we also have that $\lambda^\alpha | t - \lambda x |^{-\alpha} | f | \in L^p$.

It is interesting to note that the estimate thus obtained for $\| \chi f \|_{p,\alpha}$ is independent of λ.

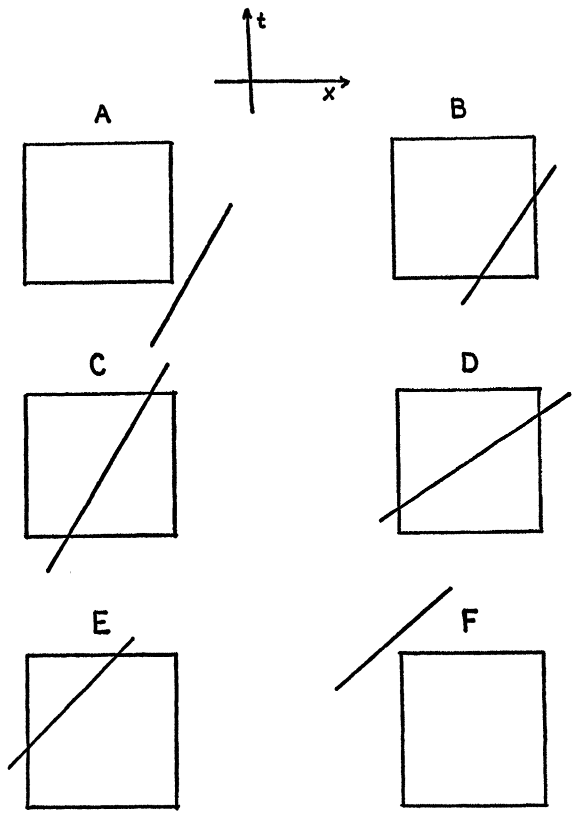
6. Estimates for $S_\alpha \chi$

Let

$$I = \int_{-1}^{1} \int_0^1 | \chi(x - ry, t - r^2 s) - \chi(x, t) | \, dy \, ds,$$

where χ is the characteristic function of $\{ (x, t) : t > \lambda x \}$. Then $S_\alpha \chi^2 = \int_0^\infty I^2 r^{-1-2\alpha} \, dr$. Note that I is simply the measure of the set of points (y, s) in $[-1, 1] \times [0, 1]$ for which (x, t) and $(x - ry, t - r^2 s)$ lie on opposite sides of the line $t = \lambda x$.

As (y, s) ranges over $[-1, 1] \times [0, 1]$, the points $(x - ry, t - r^2 s)$ sweep out a rectangle R with vertices at $(x - r, t)$, $(x + r, t)$, $(x + r, t - r^2)$, and $(x - r, t - r^2)$. Ignoring the cases in which the line $t = \lambda x$ passes through a vertex, there are six possible configurations.



We see that in cases A and F, $I = 0$. In case B, I is the area of a triangle. In cases C and D, I is the area of a trapezoid. In case E, I is either the area of a triangle or the area of its complement in R, depending on the sign of $t - \lambda x$.

First we consider the possible cases when $t - \lambda x > 0$. For small r, case A occurs and $I = 0$.

Let r increase. We enter case B when

$$t - r^2 = \lambda(x + r) \quad \text{or} \quad r = \tfrac{1}{2}(-\lambda + \sqrt{\lambda^2 + 4(t - \lambda x)}) = r_0 .$$

During case B, the line crosses the bottom of R when $t - r^2 = \lambda(x - ry)$, i.e.,

$$y = [r^2 - (t - \lambda x)]/\lambda r = y_0$$

and the right-hand side of R when

$$t - r^2 s = \lambda(x + r), \quad \text{i.e.,} \quad s = (t - \lambda x + \lambda r)r^{-2} = s_0 .$$

Thus we have

$$I = \tfrac{1}{2}(y_0 + 1)(1 - s_0) = [r^2 + \lambda r - (t - \lambda x)]/2\lambda r^3.$$

As r increases, the upper right-hand corner of R crosses the line when $t = \lambda(x + r)$, i.e., $r = \lambda^{-1}(t - \lambda x) = r_1$, and the bottom left-hand corner crosses the line when $t - r^2 = \lambda(x - r)$, i.e.,

$$r = \tfrac{1}{2}(\lambda + \sqrt{\lambda^2 + 4(t - \lambda x)}) = r_2 .$$

If $r_1 < r_2$ we have the following situation: case A for $0 < r < r_0$, case B for $r_0 < r < r_1$, case C for $r_1 < r < r_2$, and case E for $r_2 < r$.

If $r_2 < r_1$, then we have case A for $0 < r < r_0$, case B for $r_0 < r < r_2$, case D for $r_2 < r < r_1$, and case E for $r_1 < r$.

The condition $r_1 < r_2$ is seen to be equivalent to $0 < t - \lambda x < 2\lambda^2$. With reasoning similar to that used in case B, we discover that

$$I = [2\lambda r + r^2 - 2(t - \lambda x)]/2\lambda r \quad \text{in case C}$$

$$I = 2[r^2 - (t - \lambda x)]/r^2 \quad \text{in case D}$$

and

$$I = [4\lambda r^3 - (t - \lambda x + \lambda r)^2]/2\lambda r^3 \quad \text{in case E.}$$

We thus obtain

$$\begin{aligned} S_\alpha \chi^2 &= \frac{1}{4}\lambda^{-2}\int_{r_0}^{r_1} [r^2 + \lambda r - (t - \lambda x)]^2 r^{-7-2\alpha}\, dr \\ &\quad + \frac{1}{4}\lambda^{-2}\int_{r_1}^{r_2} [2\lambda r + r^2 - 2(t - \lambda x)]^2 r^{-8-2\alpha}\, dr \qquad (1) \\ &\quad + \frac{1}{4}\lambda^{-2}\int_{r_2}^{\infty} [4\lambda r^3 - (t - \lambda x + \lambda r)^2]^2 r^{-7-2\alpha}\, dr \end{aligned}$$

when $0 < t - \lambda x < 2\lambda^2$ and

$$\begin{aligned} S_\alpha \chi^2 &= \frac{1}{4}\lambda^{-2}\int_{r_0}^{r_2} [r^2 + \lambda r - (t - \lambda x)]^4 r^{-7-2\alpha}\, dr \\ &\quad + 4\int_{r_2}^{r_1} [r^2 - (t - \lambda x)]^2 r^{-5-2\alpha}\, dr \qquad (2) \\ &\quad + \frac{1}{4}\lambda^{-2}\int_{r_1}^{\infty} [4\lambda r^3 - (t - \lambda x + \lambda r)^2]^2 r^{-7-2\alpha}\, dr \end{aligned}$$

when $2\lambda^2 < t - \lambda x$.

When $t - \lambda x < 0$, the situation is simpler. The rectangle is in case F for small r. It enters case E when $t = \lambda(x - r)$ or $r = \lambda^{-1}|t - \lambda x| = r_0$. It will enter case C at the first solution of $t - r^2 = \lambda(x - r)$, i.e.,

$$r = \tfrac{1}{2}(\lambda - \sqrt{\lambda^2 + 4(t - \lambda x)}) = r_3 .$$

It returns to case E at the second solution

$$r = \tfrac{1}{2}(\lambda + \sqrt{\lambda^2 + 4(t - \lambda x)}) = r_4 .$$

If $\lambda^2 + 4(t - \lambda x) < 0$, it remains in case E for all $r > r_0$.

We discover that

$$I = 0 \quad \text{in case F}$$

$$I = [\lambda r + t - \lambda x]^2/2\lambda r^3 \quad \text{in case E}$$

and

$$I = [2\lambda r - r^2 + 2(t - \lambda x)]/2\lambda r \quad \text{in case C.}$$

We thus obtain

$$S_\alpha \chi^2 = \frac{1}{4}\lambda^{-2}\int_{r_0}^{\infty}[\lambda r + t - \lambda x]^4 r^{-7-2\alpha}\,dr \tag{3}$$

for $t - \lambda x < -\lambda^2/4$ and

$$\begin{aligned} S_\alpha \chi^2 = {} & \frac{1}{4}\lambda^{-2}\int_{r_0}^{r_3}[\lambda r + t - \lambda x]^4 r^{-7-2\alpha}\,dr \\ & + \frac{1}{4}\lambda^{-2}\int_{r_3}^{r_4}[2\lambda r - r^2 + 2(t - \lambda x)]^2 r^{-3-2\alpha}\,dr \\ & + \frac{1}{4}\lambda^{-2}\int_{r_4}^{\infty}[\lambda r + t - \lambda x]^4 r^{-7-2\alpha}\,dr \end{aligned} \tag{4}$$

for $-\lambda^2/4 < t - \lambda x < 0$.

While each integral can be evaluated explicitly, such a computation does not display the dependence on λ and $|t - \lambda x|$ very well. A change of variables

$$r = \lambda^{-1}|t - \lambda x|\, r^*$$

is helpful. The quantity $\lambda^2|t - \lambda x|^{-1}$ occurs frequently; we denote this by σ^2.

We thus obtain

$$\begin{aligned} S_\alpha \chi^2 = {} & \frac{1}{4}\lambda^{2\alpha}|t - \lambda x|^{-2\alpha}\Big\{\sigma^4\int_{r_0^*}^{1}[\sigma^{-2}r^2 + r - 1]^4 r^{-7-2\alpha}\,dr \\ & + \int_1^{r_2^*}[2r + \sigma^{-2}r^2 - 2]^2 r^{-3-2\alpha}\,dr + \int_{r_2^*}^{\infty}[4r^3 - \sigma^2(1 + r)^2]^2 r^{-7-2\alpha}\,dr\Big\} \end{aligned} \tag{1'}$$

for $t - \lambda x > 0$, $\sigma^2 > \frac{1}{2}$;

$$\begin{aligned} S_\alpha \chi^2 = {} & \lambda^{2\alpha}|t - \lambda x|^{-2\alpha}\Big\{\frac{1}{4}\sigma^4\int_{r_0^*}^{r_2^*}[\sigma^{-2}r^2 + r - 1]^4 r^{-7-2\alpha}\,dr \\ & + 4\int_{r_2^*}^{1}[r^2 - \sigma^2]^2 r^{-5-2\alpha}\,dr + \frac{1}{4}\int_1 [4r^3 - \sigma^2(1 + r)^2]^2 r^{-7-2\alpha}\,dr\Big\} \end{aligned} \tag{2'}$$

for $t - \lambda x > 0$, $\sigma^2 < \frac{1}{2}$;

$$S_\alpha \chi^2 = \frac{1}{4}\sigma^2\lambda^{2\alpha}|t - \lambda x|^{-2\alpha}\int_1^{\infty}(r - 1)^4 r^{-7-2\alpha}\,dr \tag{3'}$$

for $t - \lambda x < 0, \quad \sigma^2 < 4$; and

$$(4') \qquad \begin{aligned} S_\alpha \chi^2 = \frac{1}{4}\lambda^{2\alpha}|t - \lambda x|^{-2\alpha}\Big\{\sigma^2 \int_1^{r_3^*} (r-1)^4 r^{-7-2\alpha}\,dr \\ + \int_{r_3^*}^{r_4^*} [2r - \sigma^{-2}r^2 - 2]^2 r^{-3-2\alpha}\,dr + \sigma^2 \int_{r_4^*}^{\infty} (r-1)^4 r^{-7-2\alpha}\,dr\Big\} \end{aligned}$$

for $t - \lambda x < 0, \quad \sigma^2 > 4$.

In the above,

$$r_0^* = \tfrac{1}{2}(-\sigma^2 + \sigma\sqrt{\sigma^2+4}), \qquad r_2^* = \tfrac{1}{2}(\sigma^2 + \sigma\sqrt{\sigma^2+4}),$$
$$r_3^* = \tfrac{1}{2}(\sigma^2 - \sigma\sqrt{\sigma^2-4}), \quad \text{and} \quad r_4^* = \tfrac{1}{2}(\sigma^2 + \sigma\sqrt{\sigma^2-4}).$$

For the first integral in (1′), note that

$$r_0^* = \tfrac{1}{2}\sigma^2(-1 + \sqrt{1+4\sigma^{-2}}) \geq \tfrac{1}{2}\sigma^2(-1 + 1 + 2\sigma^{-2} - 2\sigma^{-4}) = 1 - \sigma^{-2},$$

since $\sqrt{(1+a)} \geq 1 + a/2 - a^2/8$ for $0 \leq a \leq 8$. Thus

$$|\sigma^{-2}r^2 + r - 1| \leq \sigma^{-2} \quad \text{for} \quad r_0^* \leq r \leq 1.$$

As $r_0^* > 0$ for $\sigma > 0$ and $r_0^* \geq 1 - \sigma^{-2}$ for large σ, $r_0^* \geq c > 0$ for $\sigma^2 \geq \frac{1}{2}$. Thus

$$\int_{r_0^*}^{1} [\sigma^{-2}r^2 + r - 1]^4 r^{-7-2\alpha}\,dr \leq \sigma^{-8}\int_c^1 r^{-7-2\alpha}\,dr = c_\alpha \sigma^{-8}.$$

For the second integral in (1′), note that

$$r_2^* = \tfrac{1}{2}(\sigma^2 + \sigma\sqrt{\sigma^2+4}) \leq 2\sigma^2.$$

Thus

$$\int_1^{r_2^*} [2r + \sigma^{-2}r^2 - 2]^2 r^{-3-2\alpha}\,dr \leq \int_1^\infty [4r]^2 r^{-3-2\alpha}\,dr = c_\alpha.$$

For the third integral, note that $r_2^* \geq \sigma^2 \geq \frac{1}{2}$ and thus

$$\int_{r_2^*}^\infty [4r^3 - \sigma^2(1+r)^2]^2 r^{-7-2\alpha}\,dr \leq \int_{1/2}^\infty [4r^3 + r(1+r)^2]^2 r^{-7-2\alpha}\,dr = c_\alpha.$$

Thus from (1′) we obtain

$$S_\alpha X^2 \leq c_\alpha \lambda^{2\alpha}|t - \lambda x|^{-2\alpha} \quad \text{for} \quad 0 < t - \lambda x < 2\lambda^2.$$

Next we look at (2′). For the first integral, note $r_0^* \leq r \leq r_2^*$ implies

$$\tfrac{1}{2}(-\sigma + \sqrt{\sigma^2+4}) \leq \sigma^{-1}r \leq \tfrac{1}{2}(\sigma + \sqrt{\sigma^2+4}).$$

Squaring and subtracting 1, $-r_0^* \leq \sigma^{-2}r^2 - 1 \leq r_2^*$. Thus $|\sigma^{-2}r^2 + r - 1| \leq 2r_2^*$.

Now

$$\sigma^{-1}r_0^* = \tfrac{1}{2}(-\sigma + \sqrt{\sigma^2+4}) \geq \tfrac{1}{2} \quad \text{and} \quad \sigma^{-1}r_2^* = \tfrac{1}{2}(\sigma + \sqrt{\sigma^2+4}) \leq 2$$

for $\sigma^2 \leq \frac{1}{2}$. Thus $r_0^* \geq \frac{1}{2}\sigma$ and $r_2^* \leq 2\sigma$. Consequently

$$\int_{r_0^*}^{r_2^*} [\sigma^{-2}r^2 + r - 1]^4 r^{-7-2\alpha}\,dr \leq c\int_{\sigma/2}^{2\sigma} \sigma^4 r^{-7-2\alpha}\,dr = c_\alpha \sigma^{-2-2\alpha}.$$

For the second integral, since $r_2^* \geq \sigma$,

$$\begin{aligned}\int_{r_2^*}^1 [r^2 - \sigma^2]^2 r^{-5-2\alpha}\,dr &\leq \int_\sigma^1 [r^2 - \sigma^2]^2 r^{-5-2\alpha}\,dr \\ &= \sigma^{-2\alpha}\int_1^{\sigma^{-1}} [r^2 - 1]^2 r^{-5-2\alpha}\,dr \\ &\leq c_\alpha \sigma^{-2\alpha}.\end{aligned}$$

Since $\sigma^2 \leq \frac{1}{2}$, the third integral in (2′) is bounded by c_α. Thus from (2′) we obtain

$$\begin{aligned} S_\alpha \chi^2 &\leq c_\alpha \lambda^{2\alpha} \,|\, t - \lambda x \,|^{-2\alpha}(\sigma^{2-2\alpha} + \sigma^{-2\alpha} + 1) \\ &\leq c_\alpha \lambda^{2\alpha} \,|\, t - \lambda x \,|^{-2\alpha} + c_\alpha \,|\, t - \lambda x\,|^{-\alpha}\end{aligned}$$

for $2\lambda^2 < t - \lambda x$.

From (3′) we see immediately that

$$S_\alpha \chi^2 = c_\alpha \sigma^2 \lambda^{2\alpha} \,|\, t - \lambda x\,|^{-2\alpha} \leq c_\alpha \lambda^{2\alpha} \,|\, t - \lambda x\,|^{-2\alpha}$$

for $t - \lambda x < 0$, $\sigma^2 < 4$.

Finally we look at (4′). For $\sigma^2 > 4$ we have

$$r_3^* = \tfrac{1}{2}\sigma^2(1 - \sqrt{1 - 4\sigma^{-2}}) \leq \tfrac{1}{2}\sigma^2(1 - 1 + 2\sigma^{-2} + 8\sigma^{-4}) = 1 + 4\sigma^{-2}$$

since $\sqrt{(1-a)} \geq 1 - \frac{1}{2}a - \frac{1}{2}a^2$ for $0 \leq a \leq 1$. Hence

$$\int_1^{r_3^*} (r-1)^4 r^{-7-2\alpha}\,dr \leq \int_1^\infty (4\sigma^{-2})^4 r^{-7-2\alpha} = c_\alpha \sigma^{-8}.$$

To bound the second integral in (4′), we observe that $r_4^* \leq \sigma^2$ and hence $\sigma^{-2}r^2 \leq r$ for $r \leq r_4^*$. Thus

$$\int_{r_3^*}^{r_4^*} [2r - \sigma^{-2}r^2 - 2]^2 r^{-3-2\alpha}\,dr \leq \int_1^\infty [2r - 2]^2 r^{-3-2\alpha}\,dr = c_\alpha.$$

For the last integral, note $r_4^* \geq \frac{1}{2}\sigma^2$. Hence

$$\int_{r_4^*}^\infty (r-1)^4 r^{-7-2\alpha}\,dr \leq \int_{\sigma^2/2}^\infty r^{-3-2\alpha}\,dr = c_\alpha \sigma^{-4-4\alpha}.$$

Using these bound in (4′) yields

$$S_\alpha \chi^2 \leq \tfrac{1}{4}\lambda^{2\alpha} \mid t - \lambda x \mid^{-2\alpha} (c_\alpha \sigma^{-6} + c_\alpha + c_\alpha \sigma^{-2-4\alpha}) \leq c_\alpha \lambda^{2\alpha} \mid t - \lambda x \mid^{-2\alpha}$$

for $t - \lambda x < 0, \sigma^2 \geq 4$.

Consequently, we have in all cases

$$S_\alpha \chi \leq c_\alpha (\lambda^\alpha \mid t - \lambda x \mid^{-\alpha} + \mid t - \lambda x \mid^{\alpha/2})$$

as claimed previously.

Bibliography

1. R. J. Bagby, *Lebesgue spaces of parabolic potentials*, Illinois J. Math, vol. 15 (1971), pp. 610–634.
2. ———, *A difference quotient norm for spaces of quasi-homogeneous Bessel potentials*, Studia Math, vo.. 40 (1971), pp. 41–48.
3. G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1964.
4. B. F. Jones, Jr., *Lipschitz spaces and the heat equation*, J. Math. Mech., vol. 18 (1968), pp. 379–410.
5. C. H. Sampson, *A characterization of parabolic Lebesgue spaces*, Dissertation, Rice University, 1968.
6. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.
7. R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech., vol. 16 (1967), pp. 1031–1061.
8. A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operators*, J. Math. Pures Appl., vol. 35 (1956), pp. 223–248.

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