ON TORSION IN LOOP SPACES OF *H*-SPACES

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A finite H-complex is an H-space that has the homotopy type of a finite CW complex. If X is a connected finite H-complex, then its reduced cohomology with rational coefficients is an exterior algebra on odd dimensional generators. If it is generated by $x_1, \cdots, x_n$, in dimensions $d_1, \cdots, d_n$, respectively, with $d_i \leq d_{i+1}$, then X is said to have rank n and have type $(d_1, \cdots, d_n)$. For any space Y if $H^i(Y, Z)$ contains an element of order p for some i and some prime p, then X is said to have p-torsion.

For a compact connected Lie group G it is well known that the loop space of G is torsion free, i.e., no p-torsion for any p [3]. We shall show:

THEOREM 1. *Let X be an arcwise connected finite H-complex. If X has no p-torsion, then ΩX has no p-torsion.*

At the end of [9] a question was raised whether or not the condition on torsion in the loop space of X can be eliminated. The theorem in [8] shows that this condition is superfluous and here we shall prove that the loop space is in fact torsion free:

THEOREM 2. *Let X be an arcwise connected finite H-complex of rank 2 and its mod 2 cohomology be primitively generated. Then ΩX is torsion free.*

Proof of Theorem 1. We divide into two cases: (i) X is simply connected, (ii) X is not simply connected.

(i) Since X is a finite H-complex, ΩX is of finite type. Thus p-torsion in cohomology is equivalent to p-torsion in homology (defined similarly). Suppose that ΩX has p-torsion. Then by the Universal Coefficient Theorem,

$$H_n(\Omega X; Z_p) \cong (H_n(\Omega X; Z) \otimes Z_p) \oplus (H_{n-1}(\Omega X; Z) * Z_p),$$

we have that if $H_n(\Omega X; Z)$ has an element of order p, then

$$H_n(\Omega X; Z_p) \neq 0 \quad \text{and} \quad H_{n+1}(\Omega X; Z_p) \neq 0.$$

This means that $H_i(\Omega X; Z_p) \neq 0$ for some positive odd integer i. This implies that

$$H_*(\Omega X; Z_p) \neq Z_p[y_i, \cdots, y_m, \cdots],$$

where $\dim y_i = 2n_i$; hence

$$H^*(X; Z_p) \neq \wedge(x_1, \cdots, x_m, \cdots),$$

an exterior algebra on generators x_i, where $\dim x_i = 2n_i + 1$ [5, Thm. 5.15].

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Thus X has p-torsion by a theorem of Borel [1]. This contradicts the hypothesis that X has no p-torsion.

(ii) Suppose that X is not simply connected. Then consider the universal covering space $\tilde{X}$ of X; $\tilde{X}$ is a simply connected finite H-complex [12]. If X has no p-torsion, then $\tilde{X}$ has no p-torsion [4] and by (i) above $\Omega\tilde{X}$ has no p-torsion. We shall show that ΩX has no p-torsion. Let $(\Omega X)_*$ be the path connected component containing the base point $*$ and let $p : \tilde{X} \to X$ be the covering projection. Since $\Omega\tilde{X}$ is path connected and $(\Omega X)_*$ is also path connected, the map $\Omega p : \Omega\tilde{X} \to (\Omega X)_*$ induces a one-one correspondence between the path components of $\Omega\tilde{X}$ and $(\Omega X)_*$. Since $p_\#: \Pi_i(\tilde{X}) \cong \Pi_i(X)$ for $i \geq 2$ and $\Pi_i(Y) \cong \Pi_{i-1}(\Omega Y)$ for any Y, we have that

$$\Pi_{i-1}(\Omega\tilde{X}) \cong \Pi_{i-1}(\Omega X) = \Pi_{i-1}((\Omega X)_*)$$

and that the isomorphism is $(\Omega p)_\#$. Thus $\Omega\tilde{X}$ has the homotopy type of $(\Omega X)_*$. Since $\Omega\tilde{X}$ has no p-torsion, $(\Omega X)_*$ has no p-torsion. Now, the cohomology of ΩX is the direct sum of the cohomology of the path components of ΩX and since all path components of ΩX have the homotopy type of $(\Omega X)_*$, it follows that ΩX has no p-torsion. This completes the proof of the theorem.

Proof of Theorem 2. By [8] we have that X has no p-torsion for $p \geq 5$; hence ΩX has no p-torsion for $p \geq 5$ by Theorem 1 above. We divided the rest of the proof into two cases: (i) X has no 2-torsion, and (ii) X has 2-torsion.

(i) If X has no 2-torsion, then by the exact reasoning of part (i) of [8], X has no 3-torsion, i.e., X is torsion free. From Theorem 1 above it follows that ΩX is torsion free.

(ii) If X has 2-torsion, again we subdivide into two cases: (a) X is simply connected, and (b) X is not simply connected.

(a) If X has 2-torsion and if X is simply connected, then by [11, Thm. 2.1 (ii)] we have that X has no p-torsion for all odd primes p. Thus ΩX has no p-torsion for $p \geq 3$. If we can show that ΩX has no 2-torsion, then we are done. Since X has 2-torsion and is simply connected, we have by [11, Thm. 2.1 (i)],

$$H^*(X; Z_2) \cong H^*(G_2; Z_2),$$

where the cohomology ring $H^*(G_2; Z_2)$ has one generator in dimension 3 and one generator in dimension 5. Suppose that ΩX has 2-torsion. By the exact reasoning of part (i) in Theorem 1, $H_i(\Omega X; Z_2)$ contains an element of order 2 for some positive odd integer i. Let m be the smallest such i. Then $H_m(\Omega X; Z_2)$ contains an indecomposable element. Since by [5, Thm. 5.13],

$$s_m : Q(H_m(\Omega X; Z_2)) \to P(H_{m+1}(X; Z_2))$$

is a monomorphism, we see that $H_{m+1}(X; Z_2)$ contains a primitive element. Since by [10],

$$P(H_{m+1}(X; Z_2)) \cong (Q(H^{m+1}(X; Z_2)))^*,$$

we see that $H^{m+1}(X : Z_2)$ has an indecomposable element; hence a generator. But $m + 1$ is even, a contradiction to the fact that $H^*(X; Z_2)$ has generators only in dimensions 3 and 5.

(b) If X is not simply connected, then consider the universal covering space $\tilde{X}$ of X. $\tilde{X}$ is of either rank one or rank 2 [4]. If $\tilde{X}$ is of rank one, then $\tilde{X}$ has the homotopy type of S^3 or S^7 [6]. It is well known that ΩS^3 or ΩS^7 is torsion free. If $\tilde{X}$ is of rank two, then by part (a) above $\Omega\tilde{X}$ is torsion free. Thus in any case $\Omega\tilde{X}$ is torsion free, and by the exact reasoning of part (ii) in Theorem 1, we have that ΩX is torsion free. This completes the proof of the theorem.

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