A NON-NORMAL HEREDITARILY-SEPARABLE SPACE

BY

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Let us for the purposes of this paper use *S-space* to mean a hereditarily-separable regular Hausdorff space.

If an S-space is not normal, it is clearly not Lindelöf. Although both unfortunately depend on special set-theoretic assumptions, recently [1], [2] examples have been given of non-Lindelöf S-spaces; both happen to be normal.

So there is current vogue for the question, which Jones [3] says is an old one: *Is every S-space normal?* We prove here that the answer is at least conditionally *no*. Jones [3] shows a non-normal S-space can be used to construct a non-completely regular S-space. Thus it is consistent with the usual axioms of set theory that there be a non-completely regular S-space.

Let us call Σ an S^*-space provided Σ is an uncountable S-space with a basis for its topology consisting of sets which are open, closed, and countable. Clearly no S^*-space is Lindelöf.

The space described in [1] is an S^*-space and this space exists if there is a Souslin line.

In recent correspondence I. Juhász and J. Gerlits point out the following.

THEOREM 1. *If there is a Souslin line* (which is consistent with the axioms of set theory), *then there is a non-normal S-space.*

Proof. Let Σ be the S-space described in [1]. Let I be the closed unit interval. Using the precise technique given in [5] construct from Σ a normal space T such that $T \times I$ is not normal. Then $T \times I$ will be a non-normal S-space.

A perhaps more general construction gives the following.

THEOREM 2. *Assume that there is an S^*-space and $2^{\aleph_0} < 2^{\aleph_1}$.* (This combination is consistent with the usual axioms of set theory, being true, for instance, in $V = L$, Gödel's constructible model of the universe.) *Then there is a non-normal S-space.*

Proof. We use the following pretty lemma of F. B. Jones [4]. This whole paper is an excuse to restate this lemma.

LEMMA. *There exists a cardinality $\aleph_1$ subset A of the real numbers such that each countable subset B of A is a relative G_δ set. Observe that B countable implies $A - B$ is a G_δ but $2^{\aleph_0} < 2^{\aleph_1}$ implies there is a subset C of A which is not a G_δ.*

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In order to construct a non-normal S-space, which we call X as desired for Theorem 2, we assume the existence of an S^*-space Σ and subsets A and C of the reals as guaranteed by the lemma. Then our plan is to construct X by taking the disjoint union of two copies H and K of Σ and adding a discrete countable set Q to this union in such a way that H and K cannot be separated by disjoint open sets which do not meet in Q.

Since every uncountable subset of an S^*-space is an S^*-space, we assume without loss of generality that Σ has cardinality $\aleph_1$. Assume Y is the disjoint union of two copies H and K of Σ. Let N be the set of all positive integers. Let Q be the set of all open intervals of real numbers whose end points are rational; and, for $n \in N$, let Q_n be the set of all members of Q of length less than $1/n$. For $n \in N$ and $x \in A$, let $Q_n(x) = \{I \in Q_n \mid x \in I\}$.

Since H, K, C, and $A - C$ all have cardinality $\aleph_1$, there is a one to one function $f : Y \to A$ such that $f(H) = C$ and $f(K) = A - C$.

Topologize $X = Y \cup Q$ by defining U to be open in X provided both $U \cap Y$ is open in Y and $y \in Y \cap U$ implies there is an $n \in N$ such that $U \supset Q_n(f(y))$. Clearly $U \subset Q$ implies U is open.

(1) *Let us check that X is non-normal.*

Suppose there are disjoint open in X sets U and V containing H and K, respectively. For $n \in N$, define U_n to be the union of all the members of $Q_n \cap U$; clearly $\bigcap_{n \in N} U_n$ is a G_δ set. But $f(H) = C$ and $f(K) = A - C$, and U and V disjoint and open yields $\bigcap_{n \in N} U_n \cap A = C$. Thus C is a relative G_δ which is a contradiction.

(2) *Let us check that X is an S-space.*

Since Q is countable and H and K are each hereditarily-separable, Y is hereditarily-separable.

Clearly points are closed in X. To see that X is regular assume U is open in X and $x \in U$; we prove there is an open and closed subset of U containing x. This is clearly true if $x \in Q$ since $\{x\}$ is then open; so assume $x \in H$; the case $x \in K$ is symmetric.

Since H is an S^*-space there is a countable set V which is both open and closed in H such that $x \in V \subset U$. By the Lemma, $f(V)$ is a G_δ subset of A as is $A - f(V)$. So, for each $n \in N$, there exist open sets M_n and L_n in the reals such that $A \cap \bigcap_{n \in N} M_n = f(V)$ and $A \cap \bigcap_{n \in N} L_n = A - f(V)$. Assume $M_1 \supset M_2 \supset \cdots$ and $L_1 \supset L_2 \supset \cdots$. Define

$$M = \{I \in Q \mid \text{ for some } \ n, I \subset M_n \ \text{ and } \ I \cap (A - L_n) \neq \emptyset\}$$

and

$$L = \{I \in Q \mid \text{ for some } \ m, I \subset L_m \ \text{ and } \ I \cap (A - M_m) \neq \emptyset\}.$$

Clearly $L \cap M = \emptyset$ for $n \leq m$ contradicts $I \cap (A - L_n) \neq \emptyset$ and $I \subset L_m$, and $m \leq n$ contradicts $I \subset M_n$ and $I \cap (A - M_m) \neq \emptyset$.

We finally prove $V \cup (M \cap U)$ is both open and closed in X and thus has the properties we seek.

If $v \in V$ there is an $n \in N$ such that $f(v) \notin L_n$; so $v \in M_n$; thus there is a $k \in N$ such that $Q_k(v) \subset M$. Since $v \in U$ there is an $i \in N$ such that $Q_i(v) \subset U$. Thus $j = i + k$ implies $Q_j(v) \subset U \cap M$ and $V \cup (U \cap M)$ is open in X. If $y \in Y - V$ there is an $n \in N$ such that $f(y) \notin M_n$; so $y \in L_n$; thus there is a $k \in N$ such that $Q_k(y) \subset L$. Since $L \cap M = \emptyset$, $X - (V \cup (U \cap M))$ is thus open.

References

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