APPLICATIONS OF NULL-HOMOTOPIC SURGERY

BY

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In the first part of this paper we study a class of manifolds obtained by attaching $S^{n-1} \times D^n$ to $D^n \times S^{n-1}$ with a diffeomorphism of the boundary. The class we study is those manifolds with the attaching diffeomorphism of the following form. Let $f : S^{n-1} \to SO(n)$ and $g : S^{n-1} \to SO(n)$; then

$$h : S^{n-1} \times S^{n-1} \to S^{n-1} \times S^{n-1}$$

is defined by $h(x, y) = ([f(g(x)y)]^{-1}x, g(x)y)$. We denote the resulting manifold by $M(f, g)$. This class is motivated by looking at the boundary of the manifold obtained when two copies of the tangent disk bundle of S^n are plumbed together. We then study the manifolds which result when surgery is done on the S^{n-1} factor of the right hand side, the natural framing being twisted by a function $t : S^{n-1} \to SO(n)$. Using these results and a generalization of a theorem we proved in [3], we obtain an elementary proof of a result which generalizes the following theorem due to E. H. Brown, Jr. and B. Steer [1].

THEOREM. *Suppose n is odd, $n \neq 1, 3, 7$, V_n is the Stiefel manifold of unit tangent vectors to S^n, and Σ^{2n-1} is the sphere obtained by plumbing two copies of the tangent disk bundle of S^n. Then V_n and $V_n \# \Sigma^{2n-1}$ are diffeomorphic.*

In the last section of the paper we show how the ideas and techniques of the preceding sections can be used to give an elementary proof of some of the results of Tamura [5].

The proofs in this paper are more elementary than the ones given in the above cited papers in that we are able to explicitly define diffeomorphisms between manifolds whose existence in the papers was obtained by the h-cobordism theorem.

1. The homology of $M(f, g)$

In this section we will analyze the homology of the manifold obtained from $D^n \times S^{n-1} \cup S^{n-1} \times D^n$ where the identification on the boundary is given by

$$(x, y) \to ([f(g(x)y)]^{-1}x, g(x)y).$$

Here as in the introduction $f, g : S^{n-1} \to SO(n)$. We denote the resulting manifold by $M(f, g)$. Since we have the fibration

$$SO(n-1) \to SO(n) \xrightarrow{\pi} S^{n-1}.$$

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the composition of f and π for any $f : S^{n-1} \to SO(n)$ is a map from S^{n-1} into S^{n-1}. We will denote the degree of this map by $\pi(f)$. It follows that $\pi(f)$ is zero if and only if the $SO(n)$-bundle over S^n determined by f has a section.

THEOREM 1. *$H_{n-1}(M(f, g); Z)$ is isomorphic to the integers modulo* $| 1 - \pi(f)\pi(g) |$.

Proof. Using the Mayer-Vietoris sequence with integer coefficients to calculate the homology of $M(f, g)$, it reduces to

$$0 \to H_n(M(f, g)) \to H_{n-1}(S^{n-1} \times S^{n-1})$$
$$\to H_{n-1}(D^n \times S^{n-1}) \oplus H_{n-1}(S^{n-1} \times D^n) \to H_{n-1}(M(f, g)) \to 0$$

Now each of

$$H_{n-1}(S^{n-1} \times S^{n-1}) \quad \text{and} \quad H_{n-1}(D^n \times S^{n-1}) \oplus H_{n-1}(S^{n-1} \times D^n)$$

is isomorphic to $Z \oplus Z$ and we will take the naturally embedded S^{n-1}'s to represent the generators of the respective groups.

The homomorphism from

$$H_{n-1}(S^{n-1} \times S^{n-1}) \to H_{n-1}(D^n \times S^{n-1}) \oplus H_{n-1}(S^{n-1} \times D^n)$$

can be represented by the matrix $(\deg (h_{i,j}))i, j = 1, 2$, where $h_{i,j} : S^{n-1} \to S^{n-1}$ are defined as follows ($*$ denotes north pole of S^{n-1}):

$$h_{1,1}(x) = *, \qquad h_{1,2}(x) = x,$$
$$h_{2,1}(x) = [f(g(x)*)]^{-1}x, \qquad h_{2,2}(x) = [f(g(*)x)]^{-1}*.$$

Now $\deg (h_{1,1}) = 0$ and $\deg (h_{1,2}) = 1$. If we let $g(*) = \text{Identity}$ then $\deg (h_{2,2}) = -\pi(f)$.

To compute $\deg (h_{2,1})$ we observe the following. If $\alpha : S^{n-1} \to S^{n-1}$ and $\beta : S^{n-1} \to SO(n)$ then

$$\deg (x \to \beta(x)[\alpha(x)]) = \deg (\alpha) + \pi(\beta).$$

This is proven by letting β map northern hemisphere to identity and letting α map southern hemisphere to north pole. Thus $\beta(x)[\alpha(x)]$ is constant on the equator so that its degree is the sum of the degrees of the northern and southern hemisphere which is $\deg (\alpha) + \pi(\beta)$. Hence

$$\deg (h_{2,1}) = 1 + \pi[f(g(x)*)]^{-1} = 1 - \pi(f(g(x)*)).$$

But

$$\pi(f(g(x)*)) = \deg (\pi \circ f \circ \pi \circ g) = \deg (\pi \circ f) \deg (\pi \circ g) = \pi(f)\pi(g).$$

Thus this matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 - \pi(f)\pi(g) & -\pi(f) \end{pmatrix}.$$

Recall that $\pi(f)$ denotes the degree of the composition

$$S^{n-1} \xrightarrow{f} SO(n) \xrightarrow{\pi} S^{n-1}.$$

By making suitable changes of the bases, the map can be represented by the matrix

$$\begin{pmatrix} 1 - \pi(f)\pi(g) & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that the group $H_{n-1}(M(f, g))$ is isomorphic to $Z_{|1-\pi(f)\pi(g)|}$. It is easy to see that the image of the generator of the second factor of

$$H_{n-1}(D^n \times S^{n-1}) \oplus H_{n-1}(S^{n-1} \times D^n)$$

in $H_{n-1}(M(f, g))$ is a generator of $H_{n-1}(M(f, g))$. Hence we see that a necessary condition for $M(f, g)$ to be diffeomorphic to $M(f', g')$ is that $|1 - \pi(f)\pi(g)|$ equal $|1 - \pi(f')\pi(g')|$. We observe that if $\pi(f)$ or $\pi(g)$ is zero then $M(f, g)$ is a homotopy sphere. Also, if f and g are the classifying map for the tangent bundle of S^n and n is odd, then $M(f, g)$ is the Kervaire sphere.

We now want to study the manifold obtained by doing surgery on the $S^{n-1} \times \{0\}$ naturally embedded in the second factor of $M(f, g)$, with its induced framing. If $h : S^{n-1} \to SO(n)$ and the surgery is done after twisting the framing using h, we denote the resulting manifold by $M(f, g, h)$. $M(f, g, h)$ is realized by identifying $D^n \times S^{n-1}$ and $S^{n-1} \times D^n$ using the homeomorphism

$$(x, y) \to (f^{-1}(g(x)y)x, h(f^{-1}(g(x)y)x)g(x)y).$$

A calculation similar to the previous one shows that the $(n - 1)$-dimensional homology of $M(f, g, h)$ is given by the integers modulo

$$|\pi(g) + \pi(h) - \pi(f)\pi(g)\pi(h)|.$$

2. Construction of the diffeomorphism

THEOREM 2. *If f, g and h map S^{n-1} into $SO(n)$ and I is the map of S^{n-1} into $SO(n)$ which takes each element to the identity matrix, then $M(f, I, g)$ is diffeomorphic to $M(h, g, I)$.*

Proof. Map the left hand factor of $M(h, g, I)$ to the left hand factor of $M(f, I, g)$ by

$$(x, y) \to (f(y)x, y)$$

and the right hand factor of $M(f, I, g)$ to the right hand factor of $M(h, g, I)$ by

$$(x, y) \to (h^{-1}(y)x, y).$$

It is easily checked that these maps will define a diffeomorphism between $M(f, I, g)$ and $M(h, g, I)$.

THEOREM 3. *Suppose M^n is a manifold, g is an embedding of $D^{k+1} \times D^{n-k-1}$ into M and F is a framing of a normal bundle of $g(\partial D^{k+1} \times \{0\})$. The mapping g induces a framing g' of the normal bundle of $g(\partial D^{k+1} \times \{0\})$ by taking a framing of $\{0\} \times D^{n-k}$ together with the inward pointing vector along the boundary of D^{k+1}. For some map $\lambda : S^k \to SO(n-k)$, $F = \lambda \cdot g'$. Then the manifold obtained by doing surgery on $(\partial D^{k+1} \times \{0\}, F)$ is diffeomorphic to $M^n \# K$, where K is the $(n-k-1)$-sphere bundle over S^{k+1} determined by λ.*

COROLLARY 4. *Suppose there exists an S^k embedded in M^n with non-trivial normal bundle, and $\lambda : S^{k-1} \to SO(n-k)$ classifies the normal bundle of S^k. Let F denote a framing of the normal bundle of the equator S^{k-1} in S^k. Then the manifold obtained by doing surgery on (S^{k-1}, F) is diffeomorphic to $M^n \# K$ and diffeomorphic to $M^n \# K'$ where the classifying maps for the $(n-k)$-sphere bundles K and K' differ by λ.*

Proof of Corollary 4. Let g embed $D^k \times D^{n-k}$ into M such that $D^k \times \{0\}$ is mapped to the northern hemisphere of S^k and let g' embed $D^k \times D^{n-k}$ into M such that $D^k \times \{0\}$ is mapped to the southern hemisphere of S^k. Then the framings induced by g and g' differ by a map homotopic to λ. The result now foilows easily from Theorem 3.

Proof of Theorem 3. Since the subset of M on which the surgery takes place is contained in an n-cell, we may assume that M^n is written as $M^n \# S^n$ and that the surgery takes place in S^n. Hence it may be assumed that $M^n = S^n$ in the proof of the theorem.

Since $g(\partial D^{k+1} \times \{0\})$ is unknotted in S^n, the result of doing surgery on $(\partial D^{k+1} \times \{0\}, F)$ is

$$D^{k+1} \times S^{n-k-1} \cup D^{k+1} \times S^{n-k-1}$$

where the identification is given by

$$(x, y) \to (x, \lambda(x) \cdot y).$$

The resulting space is just K, the $(n-k-1)$-sphere bundle over S^{k+1} determined by λ and K is diffeomorphic to $S^n \# K$.

We now obtain as a corollary the following theorem of E. H. Brown, Jr. and B. Steer [1]. We let V_n denote the unit sphere bundle of the tangent bundle of S^n, i.e., the Stiefel manifold of unit tangent vectors to S^n, and Σ^{2n-1} is the Kervaire sphere.

COROLLARY 5. *V_n and $\Sigma^{2n-1} \# V_n$ are diffeomorphic, for n odd*

Proof. As in Theorem 2 we will let I denote the constant map from S^{n-1} to the identity matrix in $SO(n)$. We will let T denote a map from S^{n-1} into $SO(n)$ which classifies the tangent bundle of S^n. Since our constructions only depend on the homotopy class of the map, we can assume that T maps a neighborhood of the north pole of S^{n-1} to the identity matrix, and the image of T always fixes the north pole of S^{n-1}. Now it is easy to verify that $M(I, I, T)$ is diffeomorphic to V_n. By Theorem 2, $M(I, I, T)$ is diffeomorphic to $M(T, T, I)$. $M(T, T, I)$ is obtained by doing surgery on the sphere $S^{n-1} \times \{0\}$ in the second factor of the identification space with the framing induced by the natural framing of $S^{n-1} \times D^n$. This is equivalent to doing the surgery on $S^{n-1} \times \{\text{north pole}\}$ with the natural framing on S^{n-1} and the inward pointing normal vector. Since T is the identity on a neighborhood of the north pole, $S^{n-1} \times \{\text{north pole}\}$ in the first factor bounds the n-cell $D^n \times \{\text{north pole}\}$ with the natural framing induced by the inward pointing vector of D^n and the framing of S^{n-1}. This framing clearly extends over the n-cell $D^n \times \{\text{north pole}\}$.

We now need to compare the framing of $S^{n-1} \times \{\text{north pole}\}$ in the second factor with the framing of the S^{n-1} in the first factor which extends over D^n. The inverse of the identification map is given by

$$(x, y) \to (T(y)x, T^{-1}(T(y)x)y),$$

and since we are interested only when y is in a neighborhood of the north pole, we can assume $T(y)$ is the identity matrix. Hence this map is

$$(x, y) \to (x, T^{-1}(x)y).$$

This carries the $(n - 1)$-sphere over the north pole of D^n in the second factor to the north pole over each point of the boundary of D^n in the first factor. The framing is easily seen to correspond to the map T^{-1}.

Hence $M(T, T, I)$ is diffeomorphic to $M(T, T) \# K(T^{-1})$ by Theorem 3. $M(T, T) \# K(T^{-1})$ is diffeomorphic to $M(T, T) \# K(T)$ and $M(T, T)$ is the Kervaire sphere while $K(T)$ is V_n. This concludes the proof of Corollary 5.

3. Another application of the techniques

In this section we will prove some of the results of Tamura using the elementary techniques of the first sections. We do not prove as general a theorem as Tamura, but hope to illustrate that many apparently different results can be proved with the above techniques.

Manifolds which are 3-sphere bundles over the 4-sphere have been studied by many authors. In [4] Milnor initiated the study and showed that some non-trivial 3-sphere bundles over S^4 where homeomorphic to S^7 but not diffeomorphic to S^7 with the standard differentiable structure. For further results of this type the reader should consult [2] and [5]. Since 3-sphere bundles over S^4 can be thought of as $D^4 \times S^3 \cup D^4 \times S^3$ with an identification, the resulting bundles are classified by homotopy classes of maps from S^3 into $SO(4)$ and $\pi_3(SO(4))$ is isomorphic to $Z \oplus Z$. A specific isomorphism is obtained between these groups as follows. For each $(h, j) \in Z \oplus Z$ let $f_{hj} : S^3 \to SO(4)$ be defined by $f_{hj}(u) \cdot v = u^h v u^j$, for all $v \in R^4$. Quaternionic multiplication is understood on the right. $B_{m,n}$ is used to denote the total space of the bundle determined by f_{mn}.

The following result is proved by Tamura in [5]. The manifold

$$B_{mn+(n^2+n)/2,1-n^2} \# B_{m,1}$$

is diffeomorphic to $B_{m+(n^2+n)/2,1-n^2}$. We will prove the special case of this theorem when $n = 1$ to illustrate the techniques.

THEOREM 6. *The manifold* $B_{m+1,0} \# B_{m,1}$ *is diffeomorphic to* $B_{m+1,0}$.

Proof. By definition, $B_{m,1}$ is the manifold obtained from $D^4 \times S^3 \cup D^4 \times S^3$ where the identification is given by $(u, v) \to (u, u^m(uv)u^{-m})$. We are interested in the manifold $B_{m+2,0} \# B_{m,1}$, so in terms of Theorem 3, we will obtain a three sphere in $B_{m,1}$ with a canonical framing of its normal bundle, then do surgery on this sphere after twisting the framing by the element corresponding to $(m + 1)\sigma + 0\rho$ in $\pi_3\, SO(4)$. Using the ideas of the above proofs, it is not difficult to show that $\{0\} \times S^3$ in the second factor bounds a 4-cell in $B_{m,1}$ and that the "natural" framing of this 3-sphere in $B_{m,1}$ corresponds to the element $m\sigma + (+1)\rho$. Hence twisting this framing by the element $1\sigma + (-1)\rho$ and doing surgery corresponds to the following space. (Here we are using σ and ρ to denote generators of the factors of $Z + Z$.) $D^4 \times S^3 \cup S^3 \times D^4$ with the identification

$$(u, v) \to ((u^m(uv)u^{-m})^0 u(u^m(uv)u^{-m})^{-1}, u^m(uv)u^{-m})$$
$$= (u^{m+1}v^{-1}u^{-1}u^{-m}, u^m(uv)u^{-m}).$$

Or equivalently, $D^4 \times S^3 \cup D^4 \times S^3$ with identification

$$(u, v) \to (u^m(uv)u^{-m}, u^{m+1}v^{-1}u^{-m-1}).$$

The theorem will follow if we show that this is diffeomorphic to $B_{m+1,0}$. Recalling that $B_{m+1,0}$ is the space $D^4 \times S^3 \cup D^4 \times S^3$ with identification given by

$$(u, v) \to (u, u^{m+1}vu^{-m-1}),$$

it is easily checked that $B_{m+1,0}$ is diffeomorphic to $B_{m+1,0} \# B_{m,1}$ by the following map: the identity map on the first factor of $D^4 \times S^3$ and on the second factor use the map $(u, v) \to (vu, v^{-1})$.

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