A THEOREM ON INTEGRAL-VALUED ADDITIVE FUNCTIONS

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1. Introduction

Let f be an integral-valued additive function.

It is known that, if $f(p) = 0$ for almost all primes, in the sense that

$$\sum_{f(p)\neq 0} 1/p < +\infty,$$

then for every integer q the set of those positive integers n for which $f(n) = q$ possesses a density.[1]

If $f(p) = 1$ for all primes, and if $f(n) > 0$ for all n, then for every positive integer q the number of the n's not greater than x for which $f(n) = q$ is asymptotic to

$$\frac{x(\log \log x)^{q-1}}{(q-1)! \log x}$$

as x tends to infinity.[2]

Here we consider a case when $f(p) = 0$ for many primes and also $f(p) = 1$ for many primes. Moreover we assume that $f(n) \geq 0$ for all n.

As usual the letter p always denotes a prime, while the letters m, n, k, q, r, ν denote integers. m, n, k are always positive integers.

We denote by N the set of all positive integers. γ is Euler's constant.

An empty sum is assumed to be zero and a product which has no factor is assumed to be 1.

The following theorem will be proved:

THEOREM. *Let f be an integral-valued additive function satisfying $f(n) \geq 0$ for every $n \in N$.*

Given a non-negative integer q and an infinite subset S of N, denote by $\nu_q(x)$ the number of those $n \in S$ which do not exceed x and satisfy $f(n) = q$.

Suppose that:

(i) *The characteristic function of S is multiplicative;*

(ii) *As x tends to infinity $\sum_{p \leq x, p\in S, f(p)=0} (\log p)/p \sim \alpha \log x$, where α is a positive constant;*

(iii) $\sum_{p\in S, f(p)=1} 1/p = +\infty$ *and, for every $r > 1$,*

$$\sum_{p \leq x, p\in S, f(p)=r} 1/p = o(\{\sum_{p \leq x, p\in S, f(p)=1} 1/p\}^r) \quad (x \to +\infty).$$

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[1] J. Kubilius, *Probabilistic methods in the theory of numbers* (Translations of Mathematical Monographs), p. 93.

[2] A. Wintner, *The distribution of primes*, Duke Math. J., vol. 9 (1942), pp. 423–430.

Then as x tends to infinity

$$\nu_0(x) \sim \frac{e^{-\gamma\alpha}}{\Gamma(\alpha)} \cdot \frac{x}{\log x} \prod_{p \le x} \left(1 + \sum_{p^r \in S, r \ge 1, f(p^r) = 0} \frac{1}{p^r}\right)$$

and, for $q \ge 1$,

$$\nu_q(x) \sim \nu_0(x)(1/q!)(\sum_{p \le x, p \in S, f(p) = 1} 1/p)^q.$$

The set S may be, for instance, the set of squarefree integers.

Hypothesis (iii) is obviously satisfied if

$$\sum_{p \le x, p \in S, f(p) = 1} 1/p \sim \beta \log \log x,$$

where β is a positive constant.

It is to be noticed that the result for $\nu_0(x)$ follows at once from "Satz 1.1." of Wirsing's paper: "Das asymptotische Verhalten von Summen über multiplikative Funktionen II" [3], for the characteristic function of the set of those $n \in S$ for which $f(n) = 0$ is obviously multiplicative.

We shall also use the work of Wirsing for the proof of the general result.

2. Six Lemmas

For the proof of our theorem we need Lemmas 1, 3, 4 and 6 below.

Lemma 2 is used in the proof of Lemma 3, and Lemma 5, which is a deep tauberian theorem (due to Wirsing), is used in the proof of Lemma 6.

The statements of Lemmas 5 and 6 involve a slowly oscillating function.

Let us recall that a real- or complex-valued function L of one real variable is said to be slowly oscillating if:

(1) There exists a real x_0 such that $L(x)$ exists and is not zero for all $x > x_0$.

(2) We have $\lim_{x \to +\infty} L(\lambda x)/L(x) = 1$ for every positive λ.

It is well known[4] that, if L is measurable, then the limit must be uniform in λ on every interval $[\lambda_1, \lambda_2]$, where $0 < \lambda_1 < \lambda_2 < +\infty$.

It then follows very easily that in this case we have[5]

$$L(x) = o(x^\varepsilon) \quad \text{for every positive } \varepsilon$$

(which is obviously equivalent to $L(x) = O(x^\varepsilon)$ for every positive ε).

In fact, given a positive ε, there exists a positive X such that

$$L(x) \neq o \quad \text{and} \quad \left|\frac{L(\lambda x)}{L(x)}\right| \le e^\varepsilon \quad \text{for} \quad 1 < \lambda \le e \quad \text{if} \quad x \ge X.$$

[3] Acta Math. Acad. Sci. Hungar., vol. 18 (1967), pp. 411–467.

[4] J. Korevaar, T. Van Aardenne-Ehrenfest and N. G. de Brujn, *A note on slowly oscillating functions*, Nieuw Arch. Wisk, vol. 23 (1949), pp. 77–86, and H. Delange, *Sur un théorème de Karamata*, Bull. Sci. Math. (2), vol. 79 (1955), pp. 9–12.

[5] Since $1/L$ is also slowly oscillating, we also have $1/L(x) = o(x^\varepsilon)$.

Then we immediately see that

$$|L(y)/L(x)| \leq e^{\varepsilon}(y/x)^{\varepsilon} \quad \text{for} \quad X \leq x < y.$$

In particular, taking $x = X$, we have

$$|L(y)| \leq e^{\varepsilon}(L(X)/X^{\varepsilon})y^{\varepsilon} \quad \text{for} \quad y > X.$$

Let us mention that the following well known result, that we shall use later on, can be derived very simply from these remarks:

Let L be a real-valued function defined on the interval $[0, +\infty]$.

If L is non-negative, non-decreasing and slowly oscillating, then the Laplace-integral $\int_0^{+\infty} e^{-st}L(t)\,dt$ converges for Re $s > 0$ and, as s tends to zero through positive values,

$$s\int_0^{+\infty} e^{-st}L(t)\,dt \sim L\left(\frac{1}{s}\right).$$ [6]

The integral converges for Re $s > 0$ because $L(t) = o(t^{\varepsilon})$ for every positive ε.

When s is real and small enough for $L(1/s)$ to be >0, we may write

$$(1) \qquad L(1/s)^{-1}\, s\int_0^{+\infty} e^{-st}L(t)\,dt = \int_0^{+\infty} e^{-u}\{L(u/s)/L(1/s)\}\,du.$$

For every positive u, $L(u/s)/L(1/s)$ tends to 1 as s tends to zero. Moreover, if X is chosen as above, then we have for $0 < s \leq 1/X$

$$o \leq L(u/s)/L(1/s) \leq 1 \quad \text{if} \quad u \leq 1,$$
$$\leq e^{\varepsilon}u^{\varepsilon} \quad \text{if} \quad u > 1.$$

It follows that the right-hand side of (1) tends to 1 as s tends to zero.

2.1. LEMMA 1. *Let $u_1, u_2, \cdots, u_n, \cdots$ and $v_1, v_2, \cdots, v_n, \cdots$ be complex-valued functions whose domain is a fixed set D.*

Let E be any non-empty subset of D.

Suppose that for every $n \in N$ and every $x \in E$

$$|u_n(x)| \leq U_n \quad \text{and} \quad |u_n(x) - v_n(x)| \leq V_n,$$

where the U_n's and the V_n's are positive constants satisfying

$$\sum_{n=1}^{+\infty} U_n^2 < +\infty \quad \text{and} \quad \sum_{n=1}^{+\infty} V_n < +\infty.$$

Then the infinite product $\prod_{n=1}^{+\infty}\{1 + u_n(x)\}\, e^{-v_n(x)}$ is uniformly convergent for $x \in E$.

Proof. There exists a positive U such that $U_n \leq U$ and $V_n \leq U$ for every $n \in N$.

[6] The result actually holds if L is not supposed to be real-valued, non-negative and non-decreasing, but only to be measurable and bounded on every interval $[0, T]$, where $0 < T < +\infty$.

Since $((1+u)e^{-u}-1)/u^2$ and $(e^u-1)/u$ are entire functions of u (if taken equal to $-1/2$ and 1 respectively for $u=0$), there exists a positive M such that

$$|(1+u)e^{-u}-1| \leq M|u|^2 \quad \text{and} \quad |e^u-1| \leq M|u| \quad \text{for} \quad |u| \leq U.$$

Now set $(1+u_n(x))e^{-v_n(x)} = 1+w_n(x)$.

We have for every $n \in N$ and every $x \in E$

$$w_n(x) = \{(1+u_n(x))e^{-u_n(x)}-1\}\, e^{u_n(x)-v_n(x)} + e^{u_n(x)-v_n(x)} - 1$$

and, since $|u_n(x)| \leq U_n \leq U$ and $|u_n(x)-v_n(x)| \leq V_n \leq U$,

$$|w_n(x)| \leq M|u_n(x)|^2 e^{|u_n(x)-v_n(x)|} + M|u_n(x)-v_n(x)|$$
$$\leq W_n \quad \text{where} \quad W_n = Me^U U_n^2 + MV_n.$$

We see that $\sum_{n=1}^{+\infty} W_n < +\infty$, and it follows that the infinite product

$$\prod_{n=1}^{+\infty}\{1+w_n(x)\}, \quad \text{i.e.} \quad \prod_{n=1}^{+\infty}\{1+u_n(x)\}\, e^{-v_n(x)},$$

is uniformly convergent for $x \in E$.

2.1.1. *Remark.* We may consider a product of the form

$$\prod\{1+u_p(x)\}\, e^{-v_p(x)},$$

where p runs through the sequence of prime numbers.

This product could be written as

$$\prod_{n=1}^{+\infty}\{1+u_{p_n}(x)\}\exp\{-v_{p_n}(x)\},$$

where $p_1, p_2, \cdots, p_n, \cdots$ is the sequence of prime numbers.

The lemma shows that, if we have for every prime p and every $x \in E$,

$$|u_p(x)| \leq U_p \quad \text{and} \quad |u_p(x)-v_p(x)| \leq V_p,$$

where $\sum_p U_p^2 < +\infty$ and $\sum_p V_p < +\infty$, then the product is uniformly convergent for $x \in E$.

2.2. Lemma 2. *Let* $g(n) = \prod_{p|n,\, p^2 \nmid n} p$ (*so that* $1 \leq g(n) \leq n$). *Then as* x *tends to infinity* $\sum_{n\leq x}\log(n/g(n)) = O(x)$.

Proof. For each n, $\log(n/g(n)) = \sum_{p^2|n}\log p + \sum_{p,r,p^r|n,r>1}\log p$. It follows that

$$\sum_{n\leq x}\log\frac{n}{g(n)} \leq \sum_{p\leq\sqrt{x}}\frac{x}{p^2}\log p + \sum_{p\leq\sqrt{x}}\sum_{p^r\leq x,r>1}\frac{x}{p^r}\log p$$
$$\leq x\left(\sum_p\frac{\log p}{p^2} + \sum_p\frac{\log p}{p(p-1)}\right).$$

2.3. Lemma 3. *Let* f *be an integral-valued additive function and let* χ *be a bounded multiplicative function.*

Then we have for each integer q,

$$\sum_{n\le x, f(n)=q} \chi(n)\log n = \sum_{m,p,mp\le x, f(m)+f(p)=q} \chi(m)\chi(p)\log p + O(x).$$

Proof. We suppose that $|\chi(n)| \le M$ for every $n \in N$.

We have

$$\begin{aligned}&\sum_{m,p,mp\le x, f(m)+f(p)=q} \chi(m)\chi(p)\log p\\ &= \sum_{mp\le x, p\nmid m, f(mp)=q} \chi(mp)\log p + \sum_{mp\le x, p\mid m, f(m)+f(p)=q} \chi(m)\chi(p)\log p.\end{aligned}$$

Grouping together the pairs $[m, p]$ for which the product mp has the same value, we obtain

$$\begin{aligned}\sum_{mp\le x, p\nmid m, f(mp)=q} \chi(mp)\log p &= \sum_{n\le x, f(n)=q} \chi(n)\sum_{p\mid n, p^2\nmid n}\log p\\ &= \sum_{n\le x, f(n)=q} \chi(n)\log g(n)\\ &= \sum_{n\le x, f(n)=q} \chi(n)\log n\\ &\qquad - \sum_{n\le x, f(n)=q} \chi(n)\log(n/g(n))\\ &= \sum_{n\le x, f(n)=q} \chi(n)\log n\\ &\qquad + O(x) \quad \text{by Lemma 2}\end{aligned}$$

for

$$\left|\sum_{n\le x, f(n)=q} \chi(n)\log(n/g(n))\right| \le M\sum_{n\le x}\log(n/g(n)).$$

Also, since $p \mid m$ is equivalent to $m = kp$, we have

$$\sum_{mp\le x, p\mid m, f(m)+f(p)=q} \chi(m)\chi(p)\log p = \sum_{kp^2\le x, f(kp)+f(p)=q} \chi(kp)\chi(p)\log p$$

and therefore

$$\begin{aligned}\left|\sum_{mp\le x, p\mid m, f(m)+f(p)=q} \chi(m)\chi(p)\log p\right| &\le M^2\sum_{kp^2\le x}\log p\\ &= M^2\sum_{p\le\sqrt{x}}[x/p^2]\log p \le M^2 x\sum_p (\log p)/p^2.\end{aligned}$$

Thus we see that

$$\sum_{m,p,mp\le x, f(m)+f(p)=q} \chi(m)\chi(p)\log p = \sum_{n\le x, f(n)=q} \chi(n)\log n + O(x),$$

which is the desired result.

2.4. LEMMA 4. *Let ρ be a (real- or complex-valued) function whose domain is the set of prime numbers.*

Suppose that as x tends to infinity

$$\sum_{p\le x} \rho(p)(\log p)/p = \alpha\log x + o(\log x),$$

where α is a constant.

Set $R(t) = \sum_{p\le e^t}(\rho(p)/p) - \alpha\log t$ $(t > 0)$. Then:

1. *There exist positive constants K_1 and K_2 such that we have for every positive λ and every positive t*

$$|R(\lambda t) - R(t)| \le K_1|\log\lambda| + K_2; \tag{2}$$

2. *We have for every positive* λ

$$\lim_{t\to+\infty}(R(\lambda t)-R(t))=0. \tag{3}$$

Proof. It is obviously sufficient to prove (2) and (3) for $\lambda > 1$, for, if they hold for $\lambda = \lambda_0$, then they also hold for $\lambda = 1/\lambda_0$.

Set $\sum_{p\le x}\rho(p)(\log p)/p = \Phi(x) = \alpha\log x + \eta(x)$.

We have

$$\eta(x) = o(\log x) \quad \text{as } x \text{ tends to infinity.} \tag{4}$$

Moreover, given any $X > 1$, $\eta(x)/\log x$ is obviously bounded for $1 < x \le X$. It follows that there exists a positive K_1 such that

$$|\eta(x)| \le K_1\log x \quad \text{for every} \quad x > 1. \tag{5}$$

Now we have for every $\lambda > 1$ and every positive t,

$$\begin{aligned}\sum_{e^t<p\le e^{\lambda t}}\frac{\rho(p)}{p} &= \int_{e^t}^{e^{\lambda t}}\frac{d\Phi(x)}{\log x}\\ &= \alpha\int_{e^t}^{e^{\lambda t}}\frac{dx}{x\log x}+\int_{e^t}^{e^{\lambda t}}\frac{d\eta(x)}{\log x}\\ &= \alpha\log\lambda+\frac{\eta(e^{\lambda t})}{\lambda t}-\frac{\eta(e^t)}{t}+\int_{e^t}^{e^{\lambda t}}\frac{\eta(x)\,dx}{x(\log x)^2},\end{aligned}$$

and therefore

$$R(\lambda t)-R(t)=\frac{\eta(e^{\lambda t})}{\lambda t}-\frac{\eta(e^t)}{t}+\int_{e^t}^{e^{\lambda t}}\frac{\eta(x)\,dx}{x(\log x)^2}. \tag{6}$$

(6) with (5) yields

$$|R(\lambda t)-R(t)|\le 2K_1+K_1\log\lambda,$$

so that we have (2) with $K_2 = 2K_1$.

(6) with (4) shows that for $\lambda > 1$,

$$\lim_{t\to+\infty}(R(\lambda t)-R(t))=0.$$

2.5. LEMMA 5. *Let f and g be two real- or complex-valued functions of the non-negative variable x.*

Suppose that $f\in L^2(0, X)$ for every $X > 0$ and that g is bounded on $[0, +\infty[$ *and measurable.*

Suppose moreover that as x tends to infinity

$$\int_0^x g(t)\,dt\sim\int_0^x|g(t)|\,dt\sim x,\qquad \int_0^x f(t)\,dt\sim x^\alpha L(x)$$

and

$$xf(x)=\alpha\int_0^x f(x-u)g(u)\,du+o(x^\alpha L(x)),$$

where α is a positive constant and L a measurable slowly oscillating function. Then as x tends to infinity $f(x) \sim \alpha x^{\alpha-1}L(x)$.

This is "satz 3.3." of the above quoted paper of Wirsing.

2.6. LEMMA 6. *Let a be a real-valued arithmetical function satisfying* $a(n) \geq 0$ *for every* $n \in N$, *and let b be a real-valued function of the prime p satisfying* $0 \leq b(p) \leq M$ *for every p.*

Suppose that we have as x tends to infinity,

$$\sum_{p \leq x} b(p)(\log p)/p \sim \alpha \log x, \tag{7}$$

$$\sum_{n \leq x} a(n)/n \sim (\log x)^{\alpha}L(\log x), \tag{8}$$

and

$$\sum_{n \leq x} a(n) \log n = \sum_{m,p,mp \leq x} a(m)b(p) \log p + o(x(\log x)^{\alpha}L(\log x)), \tag{9}$$

where α is a positive constant and L a measurable slowly oscillating function. Then as x tends to infinity

$$\sum_{n \leq x} a(n) \sim \alpha x(\log x)^{\alpha-1}L(\log x).$$

Proof. We use the same method as Wirsing in §§4.3. to 4.5. of the above quoted paper for the proof of his "satz 1.1.".

We set $A(x) = \sum_{n \leq x} a(n)$.

2.6.1. We first prove that

$$A(x) = o(x(\log x)^{\alpha}L(\log x)) \quad (x \to +\infty). \tag{10}$$

Let ε be any positive number <1.

We obviously have

$$A(\varepsilon x) \leq \varepsilon x \sum_{n \leq \varepsilon x} a(n)/n \quad \text{and} \quad A(x) - A(\varepsilon x) \leq x \sum_{\varepsilon x < n \leq x} a(n)/n.$$

Therefore

$$x^{-1}(\log x)^{-\alpha}L(\log x)^{-1}A(x) \leq \varepsilon(\log x)^{-\alpha}L(\log x)^{-1}\sum_{n \leq \varepsilon x} a(n)/n + (\log x)^{-\alpha}L(\log x)^{-1}\left(\sum_{n \leq x} a(n)/n - \sum_{n \leq \varepsilon x} a(n)/n\right)$$

and, by (8), it follows that

$$\limsup_{x \to +\infty} x^{-1}(\log x)^{-\alpha}L(\log x)^{-1}A(x) \leq \varepsilon.$$

2.6.2. Now we prove that (10) implies

$$\int_1^x (A(t)/t)\, dt = o(x(\log x)^{\alpha}L(\log x)). \tag{11}$$

For this purpose, choose a real number ω satisfying $o < \omega < 1$.

First, since $L(\lambda u)/L(u)$ tends uniformly to 1 for $\omega \leq \lambda \leq 1$ as u tends to infinity, (10) implies that, given any $\varepsilon > 0$, we have

$$A(t) \leq \varepsilon t(\log t)^{\alpha} L(\log x) \quad \text{for} \quad x^{\omega} \leq t \leq x$$

when x is large enough. Then

$$\int_{x^{\omega}}^{x} (A(t)/t)\, dt \leq \varepsilon L(\log x) \int_{x^{\omega}}^{x} (\log t)^{\alpha}\, dt \leq \varepsilon L(\log x) \int_{1}^{x} (\log t)^{\alpha}\, dt.$$

Since $\int_1^x (\log t)^{\alpha}\, dt \sim x(\log x)^{\alpha}$ as x tends to infinity, it follows that

$$\limsup_{x \to +\infty} x^{-1}(\log x)^{-\alpha} L(\log x)^{-1} \int_{x^{\omega}}^{x} (A(t)/t)\, dt \leq \varepsilon.$$

This proves that $\int_{x^{\omega}}^{x} (A(t)/t)\, dt = o(x(\log x)^{\alpha} L(\log x))$.

Now, since $L(u) = O(u)$ as u tends to infinity, (10) implies

$$A(x) = o(x(\log x)^{\alpha+1}) \quad (x \to +\infty),$$

which in turn implies

$$\int_{1}^{X} (A(t)/t)\, dt = o(X(\log X)^{\alpha+1}) \quad (X \to +\infty)$$

Taking $X = x^{\omega}$ we have

$$\int_{1}^{x^{\omega}} (A(t)/t)\, dt = o(x^{\omega}(\log x)^{\alpha+1}) = o(x(\log x)^{\alpha} L(\log x))$$

for $(x^{\omega-1} \log x)/L(\log x) = o(1)$.

2.6.3. Now, since $\sum_{n \leq x} a(n) \log n = A(x) \log x - \int_1^x (A(t)/t)\, dt$, it follows from (11) that

$$\sum_{n \leq x} a(n) \log n = A(x) \log x + o(x\, (\log x)^{\alpha} L(\log x)).$$

2.6.4. Thus (9) yields

$$A(x) \log x = \sum_{m,p,mp \leq x} a(m)b(p) \log p + o(x(\log x)^{\alpha} L(\log x)).$$

Replacing x by e^{ξ}, we see that as ξ tends to infinity

$$\xi A(e^{\xi}) = \sum_{m,p,\log m + \log p \leq \xi} a(m)b(p) \log p + o(e^{\xi}\xi^{\alpha} L(\xi)). \tag{12}$$

2.6.5. Setting $K(\xi) = \sum_{\log p \leq \xi} b(p)(\log p)/p$, we see that

$$\sum_{m,p,\log m + \log p \leq \xi} a(m)b(p) \log p = \sum_{\log m \leq \xi} a(m) \int_{0}^{\xi - \log m} e^{u}\, dK(u). \tag{13}$$

Now construct an increasing sequence of real numbers $\xi_0, \xi_1, \cdots, \xi_{\nu}, \cdots$ such that

$$\xi_0 = 0, \quad \lim_{\nu \to +\infty} \xi_{\nu} = +\infty, \quad \lim_{\nu \to +\infty} (\xi_{\nu+1} - \xi_{\nu}) = 0$$

and

$$\lim_{\nu \to +\infty} \xi_{\nu+1}(\xi_{\nu+1} - \xi_{\nu}) = +\infty.$$

This can be achieved for instance by taking $\xi_0 = 0$ and, for each $\nu \geq 0$,

$$\xi_{\nu+1} = \xi_\nu + 1/\sqrt{(1 + \xi_\nu)}.$$

Define a function h on the interval $[0, +\infty[$ by

$$h(\xi) = (K(\xi_{\nu+1}) - K(\xi_\nu))/(\xi_{\nu+1} - \xi_\nu) \quad \text{for} \quad \xi_\nu \leq \xi < \xi_{\nu+1}, \quad \nu = 0, 1, 2, \cdots,$$

so that h is a step function on every bounded interval.

Let $H(\xi) = \int_0^\xi h(u)\, du$ $(\xi \geq 0)$.
Obviously $H(\xi_\nu) = K(\xi_\nu)$ for $\nu \geq 0$.
For $\xi_\nu \leq \xi < \xi_{\nu+1}$ we have

$$\begin{aligned} | H(\xi) - K(\xi) | &\leq | H(\xi) - K(\xi_\nu) | + | K(\xi) - K(\xi_\nu) |, \\ &\leq | K(\xi_{\nu+1}) - K(\xi_\nu) | + | K(\xi) - K(\xi_\nu) |, \\ &\leq 2M \textstyle\sum_{\xi_\nu < \log p \leq \xi_{\nu+1}} (\log p)/p \leq 2Me^{-\xi_\nu} \sum_{e^{\xi_\nu} < p \leq e^{\xi_{\nu+1}}} \log p. \end{aligned}$$

But it is well known that

$$\textstyle\sum_{x<p\leq y} \log p \leq 2(y - x) + O(y/\log y)$$

as x and y tend to infinity with $x < y$.
Therefore as ν tends to infinity we have for $\xi_\nu \leq \xi < \xi_{\nu+1}$,

$$| H(\xi) - K(\xi) | \leq 4M(e^{\xi_{\nu+1}-\xi_\nu} - 1) + O(e^{\xi_{\nu+1}-\xi_\nu}/\xi_{\nu+1}),$$

and it follows that

$$H(\xi) - K(\xi) = o(1) \quad \text{as } \xi \text{ tends to infinity.}$$

Set $\delta(\xi) = \int_0^\xi e^u\, d(K(u) - H(u)) = \int_0^\xi e^u\, dK(u) - \int_0^\xi e^u h(u)\, du$.
We have

$$\delta(\xi) = e^\xi(K(\xi) - H(\xi)) - \int_0^\xi (K(u) - H(u))e^u\, du = o(e^\xi) \quad (\xi \to +\infty).$$

Moreover $e^{-\xi}\delta(\xi)$ is obviously bounded on every bounded interval.

2.6.6. Now (13) yields

$$\begin{aligned} &\textstyle\sum_{m,p,\log m+\log p\leq\xi} a(m)b(p) \log p \\ &\quad = \textstyle\sum_{\log m\leq\xi} a(m) \displaystyle\int_0^{\xi-\log m} e^u h(u)\, du + \textstyle\sum_{\log m\leq\xi} a(m)\delta(\xi - \log m). \end{aligned}$$

The last sum is $o(e^\xi \xi^\alpha L(\xi))$.
In fact, given $\varepsilon > 0$, there exists $X > 0$ such that

$$| \delta(\xi) | \leq \varepsilon e^\xi \quad \text{for} \quad \xi \geq X.$$

For $0 \leq \xi \leq X$, $| \delta(\xi) | \leq M_X$.

Then, for $\xi > X$,

$$
\begin{aligned}
\left| \sum_{\log m \le \xi} a(m)\delta(\xi - \log m) \right| &\le \varepsilon \sum_{\log m \le \xi - X} a(m) e^{\xi - \log m} \\
&\quad + M_X \sum_{\xi - X < \log m \le \xi} a(m) e^{\xi - \log m} \\
&\le \varepsilon e^{\xi} \sum_{\log m \le \xi} a(m)/m \\
&\quad + M_X e^{\xi} \sum_{\xi - X < \log m \le \xi} a(m)/m.
\end{aligned}
$$

By (8) this implies

$$\limsup_{\xi \to +\infty} (1/e^{\xi}\xi^{\alpha}L(\xi)) \left| \sum_{\log m \le \xi} a(m)\delta(\xi - \log m) \right| \le \varepsilon.$$

Now

$$
\begin{aligned}
&\sum_{\log m \le \xi} a(m) \int_0^{\xi - \log m} e^u h(u)\, du \\
&\qquad = \sum_{\log m \le \xi} a(m) \int_0^{\xi} e^u h(u) Y(\xi - \log m - u)\, du,
\end{aligned}
$$

where

$$
\begin{aligned}
Y(t) &= 1 \quad \text{if} \quad t \ge 0 \\
&= 0 \quad \text{if} \quad t < 0,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\sum_{\log m \le \xi} a(m) \int_0^{\xi - \log m} e^u h(u)\, du \\
&\qquad = \int_0^{\xi} e^u h(u) \Big(\sum_{\log m \le \xi} a(m) Y(\xi - \log m - u) \Big)\, du \\
&\qquad = \int_0^{\xi} A(e^{\xi - u}) e^u h(u)\, du.
\end{aligned}
$$

Thus (12) yields

$$\xi A(e^{\xi}) = \int_0^{\xi} A(e^{\xi - u}) e^u h(u)\, du + o(e^{\xi}\xi^{\alpha}L(\xi)),$$

or, setting $\Phi(\xi) = e^{-\xi}A(e^{\xi})$ and $h_1(\xi) = (1/\alpha)h(\xi)$,

$$\xi\Phi(\xi) = \alpha \int_0^{\xi} \Phi(\xi - u) h_1(u)\, du + o(\xi^{\alpha}L(\xi)). \tag{14}$$

2.6.7. Now we shall apply Lemma 5.

Φ is obviously bounded and measurable on every bounded interval, and therefore $\Phi \in L^2(0, X)$ for every $X > 0$.

h_1 is obviously measurable and we shall see presently that it is bounded.

In fact, if $\xi_\nu \le \xi < \xi_{\nu+1}$, then

$$
\begin{aligned}
|h(\xi)| &\le \frac{|K(\xi_{\nu+1}) - K(\xi_\nu)|}{\xi_{\nu+1} - \xi_\nu} \le \frac{M}{\xi_{\nu+1} - \xi_\nu} \sum_{\xi_\nu < \log p \le \xi_{\nu+1}} \frac{\log p}{p} \\
&\le \frac{M e^{-\xi_\nu}}{\xi_{\nu+1} - \xi_\nu} \sum_{\xi_\nu < \log p \le \xi_{\nu+1}} \log p.
\end{aligned}
$$

But, as ν tends to infinity, we have

$$\sum_{\xi_\nu < \log p \leq \xi_{\nu+1}} \log p \leq 2(e^{\xi_{\nu+1}} - e^{\xi_\nu}) + O(e^{\xi_{\nu+1}}/\xi_{\nu+1})$$

and therefore

$$\frac{Me^{-\xi_\nu}}{\xi_{\nu+1} - \xi_\nu} \sum_{\xi_\nu < \log p \leq \xi_{\nu+1}} \log p \leq 2M \frac{e^{\xi_{\nu+1}-\xi_\nu} - 1}{\xi_{\nu+1} - \xi_\nu} + O\left(\frac{e^{\xi_{\nu+1}-\xi_\nu}}{\xi_{\nu+1}(\xi_{\nu+1} - \xi_\nu)}\right),$$

and the last expression tends to $2M$.

We have

$$\int_0^x h_1(y)\, dy = (1/\alpha)H(x) = (1/\alpha)K(x) + o(1) \sim x$$

as x tends to infinity.

Since $h_1(y) \geq 0$, $\int_0^x | h_1(y) | \, dy = \int_0^x h_1(y)\, dy$.

Now $\int_0^x \Phi(y)\, dy = \int_0^x e^{-\xi} A(e^\xi)\, d\xi = \int_1^{e^x} A(t)/t^2\, dt$.

But, for $y \geq 1$, $\sum_{n \leq y} a(n)/n = A(y)/y + \int_1^y A(t)/t^2\, dt$.

Thus

$$\int_0^x \Phi(y)\, dy = \sum_{n \leq e^x} a(n)/n - A(e^x)/e^x \sim x^\alpha L(x) \quad (x \to +\infty).$$

Finally Lemma 5 gives

$$\Phi(x) \sim \alpha x^{\alpha-1} L(x),$$

i.e.

$$e^{-x} A(e^x) \sim \alpha x^{\alpha-1} L(x) \quad (x \to +\infty).$$

Replacing x by $\log x$, we obtain

$$A(x) \sim \alpha x (\log x)^{\alpha-1} L(\log x),$$

which is the desired result.

3. Proof of the theorem

Let χ be the characteristic function of the set S. By hypothesis (i), χ is multiplicative.

3.1. If z is a complex number satisfying $| z | < 1$ and if $\operatorname{Re} s > 1$, then the series $\sum_{n=1}^{+\infty} \chi(n) z^{f(n)}/n^s$ is obviously absolutely convergent, and we have

$$\sum_{n=1}^{+\infty} \chi(n) z^{f(n)}/n^s = \prod_p (1 + \sum_{r=1}^{+\infty} \chi(p^r) z^{f(p^r)}/p^{rs}),$$

where the infinite product is absolutely convergent.

3.1.1. Now we observe that for each prime p the series $\sum_{r=1}^{+\infty} \chi(p^r) z^{f(p^r)}/p^{rs}$ is absolutely convergent for $| z | < 1$ and $\operatorname{Re} s > 0$.

Moreover, given $\sigma_0 > 1/2$, the infinite product

$$\prod_p (1 + \sum_{r=1}^{+\infty} \chi(p^r) z^{f(p^r)}/p^{rs}) \exp(-\chi(p) z^{f(p)}/p^s)$$

is uniformly convergent for $| z | < 1$ and $\operatorname{Re} s \geq \sigma_0$.

This follows from Lemma 1, where $x = (s, z)$, by writing this product as

$$\prod_p (1 + u_p(s, z))e^{-v_p(s,z)},$$

where $u_p(s, z) = \sum_{r=1}^{+\infty} \chi(p^r)z^{f(p^r)}/p^{rs}$ and $v_p(s, z) = \chi(p)z^{f(p)}/p^s$.

In fact we have for every p and every pair (s, z) satisfying $|z| < 1$ and $\text{Re } s \geq \sigma_0$,

$$|u_p(s, z)| \leq \sum_{r=1}^{+\infty} 1/p^{r\sigma_0} = 1/(p^{\sigma_0} - 1)$$

and

$$|u_p(s, z) - v_p(s, z)| = |\sum_{r=2}^{+\infty} \chi(p^r)z^{f(p^r)}/p^{rs}| \leq \sum_{r=2}^{+\infty} 1/p^{r\sigma_0} = 1/p^{\sigma_0}(p^{\sigma_0} - 1).$$

Therefore we may define $H(s, z)$ for $\text{Re } s > 1/2$ and $|z| < 1$ by

$$H(s, z) = \prod_p (1 + \sum_{r=1}^{+\infty} \chi(p^r)z^{f(p^r)}/p^{rs}) \exp(-\chi(p)z^{f(p)}/p^s),$$

and the function H is analytic in s and z for $\text{Re } s > 1/2$ and $|z| < 1$.

It is to be noticed that $H(s, z) > 0$ when s is real and z is real ≥ 0, for then all factors of the product are > 0.

When $\text{Re } s > 1$ and $|z| < 1$, both the infinite product

$$\prod_p (1 + \sum_{r=1}^{+\infty} \chi(p^r)z^{f(p^r)}/p^{rs})$$

and the series $\sum_p \chi(p)z^{f(p)}/p^s$ are absolutely convergent, and we have

$$H(s, z) = \{\prod_p (1 + \sum_{r=1}^{+\infty} \chi(p^r)z^{f(p^r)}/p^{rs})\} \exp(-\sum_p \chi(p)z^{f(p)}/p^s).$$

Since the series $\sum_p \chi(p)z^{f(p)}/p^s$ is absolutely convergent, we may write

$$\sum_p \chi(p)z^{f(p)}/p^s = \sum_{r=0}^{+\infty} (\sum_{f(p)=r} \chi(p)z^{f(p)}/p^s).$$

Thus, if we define $F_r(s)$ for $\text{Re } s > 1$ by

$$F_r(s) = \sum_{f(p)=r} \chi(p)/p^s,[7]$$

then we have for $\text{Re } s > 1$ and $|z| < 1$,

$$H(s, z) = \{\prod_p (1 + \sum_{r=1}^{+\infty} \chi(p^r)z^{f(p^r)}/p^{rs})\} \exp(-\sum_{r=0}^{+\infty} F_r(s)z^r).$$

3.1.2. Thus we can restate the result of §3.1. as follows:

The series $\sum_{n=1}^{+\infty} \chi(n)z^{f(n)}/n^s$ is absolutely convergent for $\text{Re } s > 1$ and $|z| < 1$ and we have for these values of s and z

$$\sum_{n=1}^{+\infty} \chi(n)z^{f(n)}/n^s = H(s, z) \exp\{\sum_{r=0}^{+\infty} F_r(s)z^r\},$$

or equivalently

$$\sum_{n=1}^{+\infty} \chi(n)z^{f(n)}/n^s = H(s, z)e^{F_0(s)} \exp\{\sum_{r=1}^{+\infty} F_r(s)z^r\}. \tag{15}$$

Now the left-hand side of (15) may be written as

$$\sum_{q=0}^{+\infty} (\sum_{f(n)=q} \chi(n)/n^s)z^q.$$

[7] It is to be noticed that hypothesis (iii) implies $F_1(s) > 0$ for s real > 1.

We shall obtain the value of $\sum_{f(n)=q} \chi(n)/n^s$ for Re $s > 1$ by expanding the right-hand side in powers of z and taking the coefficient of z^q.

We have for Re $s > 1/2$ and $|z| < 1$,

$$H(s, z) = \sum_{r=0}^{+\infty} C_r(s) z^r,$$

where the C_r's are analytic for Re $s > 1/2$, and $C_0(s) = H(s, 0)$.

We see that for Re $s > 1$,

$$\begin{aligned}\sum_{f(n)=q} \chi(n)/n^s &= e^{F_0(s)} \times \text{coefficient of } z^q \text{ in } \left(\sum_{r=0}^{+\infty} C_r(s) z^r\right) \exp\left(\sum_{r=1}^{+\infty} F_r(s) z^r\right),\\ &= e^{F_0(s)} \times \text{coefficient of } z^q \text{ in } \left(\sum_{r=0}^{q} C_r(s) z^r\right) \exp\left(\sum_{r=1}^{q} F_r(s) z^r\right).\end{aligned}$$

Changing s to $1 + s$ we see that, for Re $s > 0$,

$$\begin{aligned}\sum_{f(n)=q} \frac{\chi(n)}{n^{1+s}} &= e^{F_0(1+s)} \times \text{coefficient of } z^q \text{ in } \left(\sum_{r=0}^{q} C_r(1+s) z^r\right) \exp\left(\sum_{r=1}^{q} F_r(1+s) z^r\right)\\ &= F_1(1+s)^q e^{F_0(1+s)} \times \text{coefficient of } Z^q \text{ in } \left(\sum_{r=0}^{q} \frac{C_r(1+s)}{F_1(1+s)^r} Z^r\right) \exp\left(\sum_{r=1}^{q} \frac{F_r(1+s)}{F_1(1+s)^r} Z^r\right).\end{aligned}$$ [8]

3.1.3. Now we observe that, for Re $s > 0$,

$$F_r(1 + s) = \sum_{f(p)=r} (\chi(p)/p)(1/p^s) = s \int_0^{+\infty} e^{-st} l_r(t)\, dt,$$

where
$$l_r(t) = \sum_{\log p \le t, f(p)=r} \chi(p)/p.$$

l_1 is a slowly oscillating function, for $l_1(t)$ tends to infinity as t tends to infinity (by hypothesis (iii)) and, given any $\lambda > 1$, we have, for every positive t,

$$|l_1(\lambda t) - l_1(t)| \le \sum_{e^t < p \le e^{\lambda t}} 1/p,$$

which tends to log λ as t tends to infinity.

It follows that, as s tends to zero through positive values, $F_1(1 + s) \sim l_1(1/s)$ (and therefore $F_1(1 + s)$ tends to infinity).

Besides, for $r > 1$, since $l_r(t) = o(l_1(t)^r)$ as t tends to infinity, we have

$$F_r(1 + s) = o\left(s \int_0^{+\infty} e^{-st} l_1(t)^r\, dt\right) = o(l_1(1/s)^r).$$

It follows that, for $r > 1$, $F_r(1 + s)/F_1(1 + s)^r$ tends to zero as s tends to zero through positive values.

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[8] See Note (7), page 368.

Thus we see that, as s tends to zero through positive values,

$$\sum_{f(n)=q} \frac{\chi(n)}{n^{1+s}} \sim \frac{C_0(1)}{q!} e^{F_0(1+s)} l_1\left(\frac{1}{s}\right)^q$$

(we have to remember that $C_0(1) = H(1, 0) > 0$).

3.2. Now set

$$R(t) = l_0(t) - \alpha \log t = \textstyle\sum_{p \leq e^t, f(p)=0} \chi(p)/p - \alpha \log t.$$

We have for s real > 0,

$$\begin{aligned} F_0(1 + s) &= s \int_0^{+\infty} e^{-st} l_0(t)\, dt = \int_0^{+\infty} e^{-u} l_0(u/s)\, du \\ &= \int_0^{+\infty} e^{-u} R(u/s)\, du + \alpha \int_0^{+\infty} e^{-u} \log(u/s)\, du \\ &= R(1/s) + \int_0^{+\infty} e^{-u}(R(u/s) - R(1/s))\, du + \alpha \log(1/s) - \gamma\alpha. \end{aligned}$$

We see that $\int_0^{+\infty} e^{-u}(R(u/s) - R(1/s))\, du$ tends to zero as s tends to zero through positive values for, by Lemma 4 (where $\rho(p) = \chi(p)$ if $f(p) = 0$ and $\rho(p) = 0$ otherwise), we have $| R(u/s) - R(1/s) | \leq K_1 | \log u | + K_2$ for every positive s and every positive u, and, for every positive u, $R(u/s) - R(1/s)$ tends to zero as s tends to zero.

Thus, as s tends to zero through positive values,

$$e^{F_0(1+s)} \sim e^{-\gamma\alpha}(1/s)^\alpha \exp R(1/s).$$

It follows that

$$\textstyle\sum_{f(n)=q} \chi(n)/n^{1+s} \sim \Gamma(\alpha + 1)(1/s)^\alpha L_q(1/s),$$

where

$$L_q(t) = \frac{C_0(1)}{q!} \cdot \frac{e^{-\gamma\alpha}}{\Gamma(\alpha + 1)} (\exp R(t))\, l_1(t)^q.$$

Since l_1 is a slowly oscillating function and $\lim_{t\to+\infty} (R(\lambda t) - R(t)) = 0$ for every positive λ, L_q is a slowly oscillating function.

Since, for s real > 0,

$$\sum_{f(n)=q} \frac{\chi(n)}{n^{1+s}} = \sum_{f(n)=q} \frac{\chi(n)}{n} \cdot \frac{1}{n^s},$$

a well known tauberian theorem for Dirichlet series with non-negative coefficients shows that, as t tends to infinity,

$$\textstyle\sum_{\log n \leq t, f(n)=q} \frac{\chi(n)}{n} \sim t^\alpha L_q(t).$$

It follows that, as x tends to infinity,

$$\sum_{n \le x, f(n)=q} \chi(n)/n \sim (\log x)^{\alpha} L_q(\log x).$$

3.3. Now we shall see that, as x tends to infinity,

$$\sum_{n \le x, f(n)=q} \chi(n) \log n = \sum_{m,p,mp \le x, f(m)=q, f(p)=0} \chi(m)\chi(p) \log p + o(x(\log x)^{\alpha} L_q(\log x)).$$

3.3.1. For $q = 0$ this follows immediately from Lemma 3. In fact, since $f(n) \ge 0$ for all $n \epsilon N$, Lemma 3 gives for $q = 0$,

$$\sum_{n \le x, f(n)=0} \chi(n) \log n = \sum_{m,p,mp \le x, f(m)=0, f(p)=0} \chi(m)\chi(p) \log p + O(x).$$

But $x = o(x(\log x)^{\alpha} L_0(\log x))$ for, as t tends to infinity, $1/L_0(t) = o(t^{\alpha})$.

3.3.2. For $q \ge 1$ Lemma 3 gives

$$\sum_{n \le x, f(n)=q} \chi(n) \log n = \sum_{m,p,mp \le x, f(m)=q, f(p)=0} \chi(m)\chi(p) \log p + \sum_{r=0}^{q-1} \Big(\sum_{m,p,mp \le x, f(m)=r, f(p)=q-r} \chi(m)\chi(p) \log p\Big) + O(x).$$

Again $x = o(x(\log x)^{\alpha} L_q(\log x))$.

Moreover, for each $r \ge 0$ and $\le q - 1$,

$$\Big| \sum_{m,p,mp \le x, f(m)=r, f(p)=q-r} \chi(m)\chi(p) \log p \Big| \le \sum_{m,p,mp \le x, f(m)=r} \chi(m) \log p = \sum_{m \le x, f(m)=r} \chi(m)\theta(x/m)$$

and therefore

$$\begin{aligned} \sum_{m,p,mp \le x, f(m)=r, f(p)=q-r} \chi(m)\chi(p) \log p &= O(x \textstyle\sum_{m \le x, f(m)=r} \chi(m)/m) \\ &= O(x(\log x)^{\alpha} L_r(\log x)) \\ &= o(x(\log x)^{\alpha} L_q(\log x)). \end{aligned}$$

3.4. If we set

$$\begin{aligned} a(n) &= \chi(n) && \text{if } f(n) = q, \\ &= 0 && \text{otherwise,} \\ b(p) &= \chi(p) && \text{if } f(p) = 0, \\ &= 0 && \text{otherwise,} \end{aligned}$$

and $L(t) = L_q(t)$, then all hypotheses of Lemma 6 are satisfied.

Lemma 6 yields

$$\sum_{n \le x} a(n) \sim \alpha x(\log x)^{\alpha-1} L_q(\log x),$$

that is

$$(16)\qquad \nu_q(x) \sim \frac{C_0(1)}{q!} \cdot \frac{e^{-\gamma\alpha}}{\Gamma(\alpha)} x(\log x)^{\alpha-1} l_1(\log x)^q \exp R(\log x).$$

But $C_0(1) = H(1, 0)$, which is the limit as x tends to infinity of

$$\{\textstyle\prod_{p \leq x} (1 + \sum_{r \geq 1, f(p^r)=0} \chi(p^r)/p^r)\} \exp(-\sum_{p \leq x, f(p)=0} \chi(p)/p).$$

Therefore we may replace $C_0(1)$ by this expression in (16).

Since, by the definition of $R(t)$,[9]

$$\exp(-\textstyle\sum_{p \leq x, f(p)=0} \chi(p)/p) = (\log x)^{-\alpha} \exp\{-R(\log x)\},$$

we obtain

$$\nu_q(x) \sim \frac{e^{-\gamma\alpha}}{\Gamma(\alpha)} \cdot \frac{x}{\log x} \left\{\prod_{p \leq x} \left(1 + \sum_{\substack{r \geq 1 \\ f(p^r)=0}} \chi(p^r)/p^r\right)\right\} \frac{1}{q!} l_1(\log x)^q,$$

which is the desired result.

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[9] See §3.2.