MAXIMAL SUBGROUPS OF $PSp_4(2^n)$ CONTAINING CENTRAL ELATIONS OR NONCENTERED SKEW ELATIONS[1]

BY

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1. Introduction

In [6] and [7], we laid the foundation for determining the maximal subgroups of $PSp_4(2^n)$. The purpose of this paper is to determine those maximal subgroups which contain either central elations or noncentered skew elations. Central elations are induced by transvections in $Sp_4(2^n)$, and noncentered skew elations are the duals of central elations. Other than the full symplectic groups over smaller fields, the maximal subgroups under consideration fall into six conjugacy classes, which are paired off by duality under the outer automorphism of $PSp_4(2^n)$.

The basic notation is that of [6] and [7]. By the Duality Theorem in [7], we need only look at subgroups of $PSp_4(q)$ which contain central elations. Repeated use will be made of the Center-Axis Theorem in [7]. See Huppert [12, pp. 191–214] for a discussion of the groups on a line. We will use I to denote the identity transformation or any identity matrix of appropriate rank.

THEOREM. *Let (V, f) be a nondegenerate, four-dimensional symplectic space over $GF(q)$, where $q = 2^n$; let δ be a duality on the incidence structure $PT(V, f)$ of points and totally isotropic lines.*

(i) *If G is a proper, superprimitive subgroup of $PSp_4(q)$ which contains a central elation, then G is the orthogonal group $GO(Q)$ for some nonmaximal index quadratic form Q on (V, f).*

(i*) *If G is a proper, superprimitive subgroup of $PSp_4(q)$ which contains a noncentered skew elation, then G is the dual $GO(Q)^\delta$ of the orthogonal group $GO(Q)$ for some nonmaximal index quadratic form Q on (V, f).*

COROLLARY. *The conjugacy classes of those maximal subgroups of $PSp_4(2^n)$ which contain central elations or noncentered skew elations are as follows:*

(a) *stabilizer of a point,*

(a*) *stabilizer of a totally isotropic line,*

(b) *maximal index orthogonal group,*

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(b*) *stabilizer of a pair of polar hyperbolic lines,*

(c) *nonmaximal index orthogonal group,*

(c*) *dual of nonmaximal index orthogonal group,*

(d_r) (*for each prime r dividing n*) *stabilizer of subgeometry over the maximal subfield* $GF(2^{n/r})$.

Proof of theorem. By the Duality Theorem in [7], it suffices to prove (i). Let G be a proper, superprimitive subgroup of $PSp_4(q)$ which contains a central elation. Recall that a *superprimitive* group is one which fixes no point, line, plane, pair of skew lines, tetrahedron, totally isotropic regulus, or subgeometry over a proper subfield of $F = GF(q)$.

For any central center P, the subgroup generated by the central elations in G is elementary abelian of order $2^{n'}$ for some n' no larger than n and independent of the choice of P. The proof of the theorem divides into two parts. The first part uses methods similar to those used by Mitchell [15] and Hartley [10].

2. Part A

Suppose $n' \geq 2$, or $n' = 1$ and no hyperbolic line contains more than three central centers. Let $q' = 2^{n'}$.

We will show there is a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) such that almost all rational points (over $GF(q')$) in $\langle x_1 \rangle^\perp$ and $\langle x_4 \rangle^\perp$ are central centers and that all central centers are rational points; hence the primitivity of G will yield $q = q'$. Then we will show that G is either $PSp_4(q)$ or $PGO_4(-1, 2)$.

LEMMA 1. *Assume the hypotheses of Part A. Let k be a hyperbolic line spanned by central centers and H the subgroup of G generated by the central elations in G with centers on k. Then there is a symplectic basis* $[x_1, \ldots, x_4]$ *for* (V, f) *with respect to which H is represented by*

$$\left\{ \begin{bmatrix} 1 & & \\ & A & \\ & & 1 \end{bmatrix} \;\middle|\; A \in SL_2(q') \right\},$$

and the central centers for G on k are precisely the rational points on k. Further, if X, Y, and Z are distinct central centers on k, then H contains a central elation g with center X such that $Z = g(Y)$.

Proof. Let $\bar{H}$ denote the action of H on the fixed line k for H. The group H fixes all points on $k^\perp$ with eigenvalue 1, since each of its generators does. Hence there is a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) with respect to which each element h in H has matrix of the form

$$\begin{bmatrix} 1 & & \\ & A & \\ & & 1 \end{bmatrix},$$

where A represents $\bar{h}$ and is in $SL_2(q)$. Thus, $\bar{H}$ is isomorphic to H. Direct computation shows that if $\bar{h}$ is an involution, then h is a central elation. Consequently, a Sylow 2-subgroup of $\bar{H}$ is the image of the subgroup consisting of central elations in G with a given center on k and has order $q' = 2^{n'}$.

Since $\bar{H}$ has no fixed points on k it is either dihedral of order $2 \cdot d$, where d is odd and at least 3, or isomorphic to $PSL_2(2^e)$ for some $e \geq 2$ [12, pp. 191–214]. The first case (in which $n' = 1$) leads to d central centers for G on k; hence the hypotheses of Part A imply $d = 3$, and $\bar{H}$ is isomorphic to $PSL_2(2)$. The latter case implies that $e = n'$. Thus, $\bar{H}$ is isomorphic to $PSL_2(q')$ in all cases, and there is an ordered basis $[u, v]$ for k with respect to which $\bar{H} = SL_2(q')$. Note that $\langle u \rangle$ and $\langle v \rangle$ might be any pair of preassigned central centers for G on k. There is then a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) such that $x_2 = (1/\sqrt{r})u$, $x_3 = (1/\sqrt{r})v$, and $r = f(u, v)$. With respect to $[x_1, \ldots, x_4]$, H is represented by

$$\left\{ \begin{bmatrix} 1 & & \\ & A & \\ & & 1 \end{bmatrix} \mid A \in SL_2(q') \right\},$$

and the central centers for G on k are precisely the rational points of $\langle x_2, x_3 \rangle$ over $GF(q')$. The last remark in the lemma is a consequence of the fact that a Sylow 2-subgroup of $PSL_2(q')$ acts regularly on the q' nonfixed points.

LEMMA 2. *Assume the hypotheses of Part A. Let k be a hyperbolic line which is spanned by central centers and does not lie in the polar of a given central center R. Then k meets $R^\perp$ in a central center.*

Proof. Let $T = k \cap R^\perp$ and $S = k^\perp \cap R^\perp$. Suppose T is not a central center. By Lemma 1, there is a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) such that $P = \langle x_2 \rangle$ and $Q = \langle x_3 \rangle$ are central centers. So $T = \langle x_2 + rx_3 \rangle$ for some r in $GF(q)^*$. Without loss of generality, x_1 is chosen to span S so that $R = \langle x_1 + x_2 + rx_3 \rangle$. The central elations in G with centers P, Q, and R respectively are

$$\begin{bmatrix} 1 & & & \\ & 1 & a & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & b & 1 & \\ & & & 1 \end{bmatrix} \text{ and } g_s = \begin{bmatrix} 1 & rs & s & s \\ & 1 + rs & s & s \\ & r^2 s & 1 + rs & rs \\ & & & 1 \end{bmatrix}$$

for all a and b in $GF(q')^*$, and for $q' - 1$ values s in F^*. Then $g_s(P)$ and $g_s(Q)$ are central centers on $\langle P, R \rangle$ and $\langle Q, R \rangle$, respectively. By Lemma 1, there are central elations g and h in G with centers P and Q, respectively, such that $g(R) = g_s(P)$ and $h(R) = g_s(Q)$. Hence there elements a and b in $GF(q')^*$ such that $rs(1 + ar) = 1 + rs$ and $s(b + r) = 1 + rs$. Consequently, $r^2 = (sa)^{-1}$. Thus, $s = b^{-1}$ and $r^2 = (sa)^{-1}$ are in $GF(q')$, r is in $GF(q')$, and T is a rational point and central center.

We continue the proof in Part A. Since G is primitive and acts transitively on its central centers, there exist nonorthogonal central centers P and Q, and a third central center R on neither $\langle P, Q\rangle$ nor $\langle P, Q\rangle^{\perp}$. By Lemma 2, we may assume, without loss of generality, that $P \perp R$. There is then a symplectic basis $[x_1, \ldots, x_4]$ such that $P = \langle x_2\rangle$, $Q = \langle x_3\rangle$, $S = R^{\perp} \cap \langle P, Q\rangle^{\perp} = \langle x_1\rangle$, $R = \langle x_1 + x_2\rangle$, and the central centers for G on $\langle P, Q\rangle$ and $\langle Q, R\rangle$ are precisely the rational points on these lines, with respect to

$$V^* = \{\sum a_i x_i \mid a_i \in GF(q')\}.$$

Since the central elations in G with center R have matrices

$$\begin{bmatrix} 1 & 0 & s & s \\ & 1 & s & s \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

for all s in $GF(q')^*$, and since the central centers different from P and on $\langle P, Q\rangle$ are the points $\langle ax_2 + x_3\rangle$ for all a in $GF(q')$, direct computation shows that all rational points in $S^{\perp} - \langle P, R\rangle$ are central centers for G. Application of the central elation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix}$$

in G with center Q to the rational points $\langle ax_1 + x_2\rangle$ for each a in $GF(q')$ shows that all rational points in $S^{\perp} - \{S\}$ are central centers for G.

Since G is primitive and transitive on its central centers, there is a central center R' for G on neither $S^{\perp}$ nor $\langle P, Q\rangle^{\perp}$. Since $R'^{\perp}$ meets $\langle P, Q\rangle$ in a single point, and since there are at least three central centers on $\langle P, Q\rangle$, we may assume that Q was chose to be a central center different from P and not orthogonal to R'. The lines $\langle P, Q\rangle$ and $\langle P, Q\rangle^{\perp}$ meet $R'^{\perp}$ in points P' and S', respectively; so $S' = \langle bx_1 + x_4\rangle$ for some b in F. Since the matrix

$$\begin{bmatrix} 1 & 0 & 0 & b \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

effects a symplectic coordinate change which fixes every vector in $S^{\perp}$ and centralizes the central elations with centers in $S^{\perp}$, we may assume, without loss of generality, that $b = 0$ and $S' = \langle x_4\rangle$.

By Lemmas 1 and 2, the point P' is spanned by $x_2 + ax_3$ for some a in $GF(q')$. So $R' = \langle x_2 + ax_3 + dx_4\rangle$ for some d in F. By Lemma 2, there is a rational point $\langle \alpha x_1 + \alpha x_2 + \beta x_3\rangle$ orthogonal to R'; hence $a + d = \beta/\alpha$. Thus, d is also in $GF(q')$ and R' is a rational point.

If g is a central elation with center R', then there is, by Lemma 1, a central elation with center Q mapping R' to $g(Q)$. The computation shows that all central elations in G with center R' have entries in $GF(q')$.

The calculation for applying the central elations in G with center R' to the central centers on $\langle P, Q\rangle$ shows that all rational points in $S'^{\perp} - \{S'\}$ are central centers for G.

We will now show that all central centers for G are rational points. Let R be an arbitrary central center for G and $k = \langle P, Q\rangle$. If R is on k, then we already know R is a rational point.

Suppose R lies on neither k nor $k^{\perp}$. Without loss of generality, $R \not\perp Q$. The unique totally isotropic transversal to $\{k, k^{\perp}\}$ containing R meets k in a central center $W = \langle x_2 + ax_3\rangle$ for some a in $GF(q')$ and meets $k^{\perp}$ in a point U equal to $\langle x_4\rangle$ or $\langle x_1 + bx_4\rangle$ for some b in F. In the latter case, $R = \langle [c, 1, a, bc]^t\rangle$ for some c in F^*; computation and application of Lemmas 1 and 2 to the point R and the lines $\langle x_1 + x_2, x_3\rangle$ and $\langle x_2 + x_4, x_3\rangle$ show that a, bc, and c lie in $GF(q')$. In the former case, $R = \langle [0, 1, a, c]^t\rangle$ for some c in F, and computations show that c is in $GF(q')$. In both cases, R is a rational point.

Suppose R lies on $k^{\perp}$ and is equal to $\langle ax_1 + x_4\rangle$ for some a in F^*. Lemma 1 applied to the line $\langle x_1 + x_2, x_3\rangle$ shows that

$$g = \begin{bmatrix} 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

is a central elation in G. So $g(R)$ is equal to $\langle [a + 1, 1, 0, 1]^t\rangle$ and is a central center not on k or $k^{\perp}$. Hence $g(R)$ is a rational point, $a + 1$ is in $GF(q')$, and R is a rational point.

It is easy to find explicitly five rational central centers for G, no four of which are coplanar. Since any projectivity is determined by the images of these five central centers [1, pp. 66–68], and since any element in G maps these centers to other central centers, which we have shown to be rational points, we conclude that G must stabilize the rational subgeometry V^*. Thus, $n' = n$, since G is superprimitive.

Knowing that $q' = q$, we have thus far shown that a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) can be chosen such that all points in $\langle x_1\rangle^{\perp}$ and $\langle x_4\rangle^{\perp}$ are central centers, except possibly $\langle x_1\rangle$ and $\langle x_4\rangle$, and that all the points on a hyperbolic line spanned by central centers are central centers.

Since $\langle ax_1 + x_2\rangle$ and $\langle x_2 + x_4\rangle$ are central centers for any a in F^*, all the points on $\langle x_1, x_4\rangle$ are central centers, except possibly $\langle x_1\rangle$ and $\langle x_4\rangle$. Two cases arise: (i) all points of $\langle x_1, x_4\rangle$ are central centers, or (ii) $q = 2$, and $\langle x_1 + x_4\rangle$ is the only central center on $\langle x_1, x_4\rangle$.

In the first case, the group generated by the central elations in G with centers on $\langle x_1, x_4\rangle$ maps $\langle x_1\rangle^{\perp}$ to the polar of any point on $\langle x_1, x_4\rangle$. Hence all points in V are central centers. Since any two central elations in G with nonorthogonal centers are conjugate in the dihedral group they generate, the subgroup of G generated by the central elations in G is transitive on its central centers and hence irreducible. A theorem of J. E. McLaughlin [14, p. 365] implies that G is the full symplectic group $PSp_4(q)$.

In the second case, we have shown all points to be central centers, except possibly the five points $\langle x_1\rangle$, $\langle x_4\rangle$, $\langle x_1 + x_2 + x_4\rangle$, $\langle x_1 + x_3 + x_4\rangle$, and $\langle x_1 + x_2 + x_3 + x_4\rangle$. If any of the five were a central center, then all would be, and the first case would apply. Hence none of these five points is a central center. It is now easy to verify that these five points form the quadric of the nonmaximal index quadratic form Q given by $Q([a, b, c, d]^t) = ad + b^2 + bc + c^2$. Since $GO_4(-1, 2)$ is generated by its ten transvections [3, p. 42], all of which are in G, the group G must actually be the orthogonal group $PGO_4(-1, 2)$.

3. Primitive subgroups of odd order

Before discussing Part B of the proof of the theorem, we show that the odd order primitive subgroups of $PSp_4(2^n)$ are precisely the subgroups of the Singer groups, which are contained in nonmaximal index orthogonal groups. Thus, $PSp_4(2^n)$ has no maximal subgroups of odd order.

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Let $F = GF(q)$ be the Galois field of order $q = 2^n$. The additive group of $GF(q^4)$ forms a four-dimensional vector space V over F. If d generates the multiplicative group $GF(q^4)^*$, then the function $T_d\colon V \to V$ given by $x \mapsto dx$, for all x in $GF(q^4)$, is an element of order $q^4 - 1$ in $GL(V)$ and induces an element $\overline{T}_d$ of order $(q^4 - 1)/(q - 1)$ in $PGL(V)$. The conjugates of the subgroups generated by T_d and $\overline{T}_d$ are called the *Singer groups* in $GL(V)$ and $PGL(V)$, respectively.

Let α generate the automorphism group of $GF(q^4)$ over F. Then the function $f\colon V \times V \to F$ given by $f(x, y) = \mathrm{Tr}\,(x\bar{y})$, where $\bar{z} = z^{\alpha^2}$ and $\mathrm{Tr}\,(z) = z + z^\alpha + \bar{z} + \bar{z}^\alpha$, is an alternate bilinear form on V. For a in F^*, the element $T_a\colon x \mapsto ax$ in $GL(V)$ is in $\mathrm{Sp}\,(f)$ if and only if $a\bar{a} = 1$. The Singer group induced by T_d in $PGL(V)$ intersects $PSp(f)$ in a group E of order $q^2 + 1$. The conjugates of E in $PSp(f)$ are Singer groups in $PSp(f)$.

LEMMA 3. (a) *The normalizer in $PSp_4(q)$ of any subgroup of a Singer group in $PSp_4(q)$ has order $4(q^2 + 1)$ and is the product of that Singer group and a normalizing flag-fixer.*

(b) *All nontrivial subgroups of the Singer groups in $PSp_4(q)$ are irreducible.*

(c) *Let G be a cyclic, irreducible subgroup of $PSp_4(q)$ of odd order. Then G is contained in a Singer group in $PSp_4(q)$.*

Proof. (a) Since $q^2 + 1$ and $q^2 - 1$ are relatively prime, this part follows from a theorem in Huppert [12, pp. 187–189], together with the direct computation showing that the generating automorphism α of $GF(q^4)$ over F of order 4 preserves the alternate bilinear form f given above.

(b) Let $T_e: x \to ex$ induce a nontrivial element of $PSp_4(q)$, where e is in $GF(q^4)$. Clearly, T_e has no fixed points. Suppose T_e fixes the line $\langle x, y\rangle$ for some x and y in $GF(q^4)$. Then $ex = ax + by$ and $ey = cx + dy$ for some a, b, c, and d in F. Computation shows that $X^2 + (a + d)X + (ad + bc)$ is in $F[X]$ and has root e. Hence e is in $GF(q^2)$, and $1 = e\bar{e}$ implies that $e = 1$, contrary to T_e being nontrivial.

(c) An examination of Table 1 in [7] shows that any nontrivial subgroup of an irreducible, cyclic, odd order subgroup of $PSp_4(q)$ must also be irreducible. So by part (a), we may assume that G has prime order r. Since G fixes no lines, the length of each orbit for G acting on hyperbolic lines is equal to r. Thus, r divides $(q^2 + 1)q^2$ and also $q^2 + 1$. A theorem of Wielandt [18] yields the conclusion.

PROPOSITION 1. *The normalizer in $PSp_4(q)$ of a Singer subgroup is contained in an orthogonal group $GO(Q)$ for some nonmaximal index quadratic form Q on (V, f).*

Proof. A Singer subgroup of $PSp_4(q)$ lies in such an orthogonal group by Theorem 5.6 in Hestenes [11, p. 513]. Further, direct computation shows that the above automorphism α of order 4 preserves the form Q in Hestenes. So Lemma 3 implies the result.

LEMMA 4. *Let G be an odd order, irreducible subgroup of $PSp_4(q)$ and H a normal subgroup of G with prime index r. Then H is also irreducible.*

Proof. Consider first the case in which G is absolutely irreducible. By Clifford's Theorem [2], H is completely reducible on V. If W is a proper irreducible H-submodule, which has dimension 1 or 2, then the inertial subgroup of G for W, which has odd index dividing dim $V = 4$, is equal to G. Thus, V is a direct sum of H-modules isomorphic to W, and hence H acts faithfully on W Drawing on an argument in Feit [4, p. 54], we extend the action on W to yield a faithful G-module. Let gH generate G/H. Since G is the inertial group for W, there is a matrix C such that $C^iT(h)C^{-i} = T^{g^i}(h)$, where T denotes the given representation of H on W and $T^g(h) = T(ghg^{-1})$. By Schur's Lemma, $C^{-r}T(g^r)$ is a scalar matrix S^*, and, after possible field extension, there is a scalar matrix S such that $S^r = S^*$. The definition $T(g^ih) = S^iC^iT(h)$ yields a representation of G itself on W, which is faithful since there is more than one choice for an rth root S of S^*. Dickson's classification of the groups on a line [12, p. 213] shows that G must be cyclic. Thus, H is irreducible by Lemma 3.

If G is not absolutely irreducible, then V is a direct sum of irreducible G-submodules of dimension 1 or 2 over some extension field $\tilde{F}$ of F. Dickson's theorem shows that G acts cyclically on each component. By possibly extending the field further, we may take all the components to be 1-dimensional. The

Galois group $G(\tilde{F}/F)$ acts naturally on V, using a basis for V with respect to which G is defined over F, and centralizes the action of G. The components of G on $V \otimes \tilde{F}$ are permuted transitively by $G(\tilde{F}/F)$ since G is irreducible on V (over F). So G acts faithfully on each component and hence is itself cyclic. Thus, H is irreducible by Lemma 3.

PROPOSITION 2. *Let G be a nontrivial, odd order subgroup of $PSp_4(q)$. Then G is primitive if and only if G is contained in a Singer subgroup of $PSp_4(q)$.*

Proof. First, note that a subgroup of $PSp_4(q)$ of odd order is primitive if and only if it is irreducible, since a group of odd order cannot act transitively on a pair of skew lines nor on the four vertices of a tetrahedron. Second, the reverse implication is Lemma 3(b).

Suppose now that G is an odd order, primitive subgroup of $PSp_4(q)$. By the Feit-Thompson Theorem [5], G is solvable, and there is a subnormal series $1 < G_1 < G_2 < \cdots < G_m = G$, whose quotients are of odd prime order. Successive application of Lemma 4 yields that G_1 is irreducible and hence contained in a Singer group in $PSp_4(q)$ by Lemma 3(c). If G_i is contained in a Singer group (for i between 1 and m), then by Lemma 3(a), G_{i+1} is in a Singer normalizer of order $4(q^2 + 1)$ and so has order dividing $q^2 + 1$. Wielandt's theorem implies that G_{i+1} lies in a Singer subgroup of $PSp_4(q)$, and iteration yields the same for G itself.

4. Part B

We return to the proof of the main theorem. Suppose $n' = 1$ and there is a hyperbolic line containing *more* than three central centers for G.

Immediately, we conclude that $q > 2$ and that no two central elations in G have the same center. For each central center X, let t_X denote the unique central elation in G with center X.

LEMMA 5. *The polar of a central center for G is spanned by central centers.*

Proof. There are three cases to consider.

Case 1. Suppose no two distinct central centers are orthogonal and G has centered skew elations. We show first that central centers are distinct from skew centers. If not, then the set of central centers coincides with the set of skew centers, since G is transitive on each of these sets. Since G is primitive, there are nonorthogonal centers P and Q. By the Center-Axis Theorem, there is a centered axis u through Q meeting $P^{\perp}$ in a center R, contrary to the hypothesis.

The Center-Axis Theorem implies that each central center is orthogonal to each skew center. Since there are nonorthogonal skew centers P and Q, all central centers lie on $\langle P, Q\rangle^{\perp}$, contrary to G being primitive. Thus, Case 1 cannot occur.

Case 2. Suppose no two distinct central centers are orthogonal, and G has no centered skew elations. Then G has pattern (0C) of [7]. A Sylow 2-subgroup S of G is cyclic of order 2 and is generated by a single central elation t. By Burnside's Theorem, G has a normal 2-complement C [8, p. 252]. If C were primitive, then by Proposition 2, G would lie in a Singer normalizer, which contains no central elations. Thus, C is not primitive and fixes a point P or a line k. Hence G acts on $\{P, t(P)\}$ or on $\{k, t(k)\}$, contrary to G being primitive. Thus, Case 2 cannot occur.

Case 3. Suppose G has orthogonal central centers, but the polar of a central center is not spanned by central centers. Let P and Q be distinct orthogonal central centers. Hence, $k = \langle P, Q \rangle$ is a centered axis, and every centered axis is spanned by central centers. The hypothesis implies there is a unique centered axis through each central center.

Let u be an arbitrary centered axis for G. If u contains P, then $u = k$. If u does not contain P, then the Center-Axis Theorem implies that u meets k. Hence the skew centers for the dual G^δ of G all lie in $K^\perp$, where $K = \delta(k)$, contrary to G^δ being superprimitive. Thus, Case 3 cannot occur, and the lemma is proved.

We will now examine carefully the polar of a central center. First, we remark that the number of central centers on any totally isotropic line spanned by central centers is a constant, say e. Indeed, if A and B are nonorthogonal central centers in the polar of a central center P, then A and B are conjugate in the dihedral subgroup of the point stabilizer G_P generated by the central elations in G with centers A and B. If A and B are orthogonal central centers in $P^\perp - \{P\}$, then there is by assumption a central center C not orthogonal to both A and B. Thus, G_P is transitive on the central centers in $P^\perp - \{P\}$, and the remark follows.

Let P be a central center, H the subgroup of G generated by the central elations in G with centers in $P^\perp$, $\bar{H}$ the action of H on $P^\perp/P$, and K the kernel of the action.

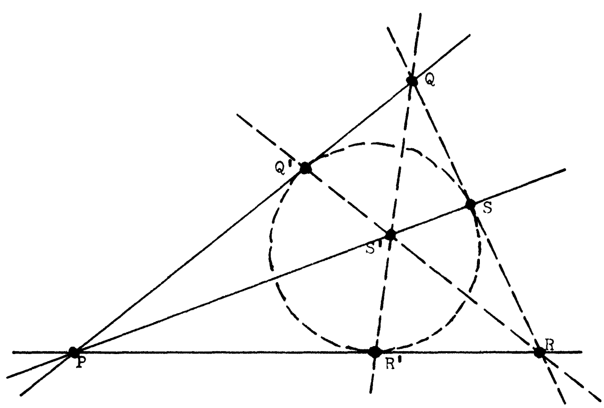
For X a central center in $P^\perp$, the central elation t_X is in K if and only if $X = P$. If Q and Q' are distinct, orthogonal central centers in $P^\perp - \{P\}$, then $t_Q t_{Q'}$ is a centered skew elation in G with axis $\langle Q, Q' \rangle$ and center different from Q and Q' [7, Table 2]. Thus, $\bar{t}_Q = \bar{t}_{Q'}$ if and only if $t_Q t_{Q'}$ has center P. If Q and R are nonorthogonal central centers in $P^\perp$, then $\langle t_Q, t_R \rangle$ acts faithfully on the hyperbolic line $\langle Q, R \rangle$, since t_Q and t_R each fix all vectors in $\langle Q, R \rangle^\perp$. Thus, $\langle t_Q, t_R \rangle$ acts faithfully on $P^\perp/P$ and is isomorphic to $\langle t_Q, t_R \rangle^-$. Further, $t_Q t_R$ has odd order d, and $\langle t_Q, t_R \rangle$ is dihedral of order $2 \cdot d$. If Q' and R' are central centers on $\langle P, Q \rangle$ and $\langle P, R \rangle$, respectively, such that $\bar{t}_Q = \bar{t}_{Q'}$ and $\bar{t}_R = \bar{t}_{R'}$, then $\langle t_Q, t_R \rangle$ is isomorphic to $\langle t_{Q'}, t_{R'} \rangle$.

LEMMA 6. *$\bar{H}$ is isomorphic to $PSL_2(2^{\bar{n}})$ for some $\bar{n} \geq 2$.*

Proof. Since $\bar{H}$ is isomorphic to a subgroup of $PSL_2(q)$ and has no fixed points, the lemma follows from [12, pp. 191–214], provided we show that $\bar{H}$ is not dihedral of order twice an odd integer. So suppose $\bar{H}$ is dihedral of order $2 \cdot d$, where d is an odd integer. Since a Sylow 2-subgroup of $\bar{H}$ has order 2, $\bar{t}_X = \bar{t}_{X'}$ for distinct, orthogonal central centers X and X' in $P^{\perp} - \{P\}$, and $t_X t_{X'}$ is a centered skew elation with center P. This implies that there cannot be four distinct, central centers on any totally isotropic line, since G is transitive on its central centers. Thus, $e = 2$ or $e = 3$.

Let Q and R be central centers in $P^{\perp}$ such that the involutions $\bar{t}_Q$ and $\bar{t}_R$ generate $\bar{H}$. If X is any central center in $P^{\perp} - \{P\}$, then the involution $\bar{t}_X$ in $\bar{H}$ can be lifted to a central elation t_Y in $\langle t_Q, t_R \rangle$. So $X \perp Y$, $\bar{t}_X = \bar{t}_Y$, and $\langle P, X \rangle$ meets $\langle Q, R \rangle$ in a central center. Call a totally isotropic line u through P *special* if it meets $\langle Q, R \rangle$ in a central center. Since $\bar{H}$ is dihedral of order $2 \cdot d$ and isomorphic to $\langle t_Q, t_R \rangle$, the line $\langle Q, R \rangle$ contains d central centers, and there are d special lines.

Suppose $e = 3$, that is, any totally isotropic line spanned by central centers has exactly three central centers. Let Q' and R' be the third central centers on $\langle P, Q \rangle$ and $\langle P, R \rangle$, respectively. Each of the hyperbolic lines $\langle Q, R' \rangle$, $\langle Q', R \rangle$, and $\langle Q', R' \rangle$ meets each special line in a central center, since, for example, $\langle t_{Q'}, t_{R'} \rangle$ is isomorphic to $\langle t_Q, t_R \rangle$. Let S be a third central center on $\langle Q, R \rangle$ and $S' = \langle P, S \rangle \cap \langle Q, R' \rangle$. It is easy to verify the incidence diagram in the figure.



If $d > 3$, then there is a central center X contained in $\langle Q, R \rangle$ and different from Q and S such that $\langle t_R, t_X \rangle$ is dihedral of order $2 \cdot d$. Since $\langle t_R, t_X \rangle$ is isomorphic to $\langle t_{R'}, t_X \rangle$, we conclude that $\langle R', X \rangle$ has d central centers and so meets each special line in a central center. Thus, $\langle R', X \rangle$ meets $\langle P, S \rangle$ in either S or S', both impossible since X is different from Q and S. Therefore, $d = 3$, and every hyperbolic line in the polar of a central center contains at most three central centers.

The contradiction to the assumption that $e = 3$ arises by showing that every hyperbolic line spanned by central centers lies in the polar of a central center. Since $t_Q t_{Q'}$ is a centered skew elation with center P, the central centers and the skew centers coincide. Let $\langle A, B\rangle$ be an arbitrary hyperbolic line spanned by central centers A and B, and k a centered axis through the skew center B. By the Center-Axis Theorem, k meets $A^\perp$ in a central center M, which is orthogonal to both A and B. Thus, $\langle A, B\rangle$ lies in $M^\perp$, and e cannot be 3.

Suppose $e = 2$, that is, any totally isotropic line spanned by central centers has exactly two central centers. Hence each centered axis contains exactly two central centers. Further, all central centers in $P^\perp - \{P\}$ must lie on the hyperbolic line $\langle Q, R\rangle$.

If G does not have pattern (3FC) or (3FCN), then by the Center-Axis Theorem any central center not in $P^\perp$ lies on $\langle Q, R\rangle^\perp$, and G fixes the set $\{\langle Q, R\rangle, \langle Q, R\rangle^\perp\}$, contrary to G being primitive. Thus, G must have pattern (3FC) or (3FCN).

For each of the d central centers X_i in $P^\perp - \{P\}$, the product $t_{X_i} t_P$ is a centered skew elation with center Y_i, different from P and X_i, and axis $\langle P, X_i\rangle$, for $i = 1, \ldots, d$ [7, Table 2]. The point Y_i is the only skew center on $\langle P, X_i\rangle$. We claim that there are exactly d skew centers in $P^\perp$, namely, $Y_1, \ldots, Y_d$. Suppose Y is a skew center in $P^\perp$ and k a centered axis containing Y. If k is different from $\langle P, Y\rangle$, then Table 2 in [7] yields a flag-fixer in G with axis $\langle P, Y\rangle$. In any case, $\langle P, Y\rangle$ is a centered axis and so must be one of the special lines $\langle P, X_i\rangle$. Since Y_i is the unique skew center on $\langle P, X_i\rangle$, we conclude that $Y = Y_i$.

Let Y_i and Y_j be distinct skew centers in $P^\perp$. There is a unique central center X on $\langle Q, R\rangle$ such that t_X interchanges Y_i and Y_j and fixes the hyperbolic line $\langle Y_i, Y_j\rangle$. Thus, $\langle Y_i, Y_j\rangle$ meets $\langle Q, R\rangle$ in the unique central center X. Since no two distinct central elations in the dihedral group $\langle t_Q, t_R\rangle$ interchange the same pair of central centers on $\langle Q, R\rangle$, we conclude that no three skew centers in $P^\perp$ are collinear.

Let Z be a central center not in $P^\perp$. The Center-Axis Theorem implies that $Z^\perp \cap P^\perp$ is a hyperbolic line containing d centers for G, at most two of which can be skew centers. Thus, $\langle Z, P\rangle^\perp = \langle Q, R\rangle$ or $d = 3$. If d were larger than 3, then each central center for G would lie on $\langle Q, R\rangle$ or $\langle Q, R\rangle^\perp$, contrary to G being primitive. So d must be 3.

By assumption, there is a hyperbolic line m which contains at least four distinct central centers $A_1, \ldots, A_4$. If A_i is different from P, then $A_i^\perp \cap P^\perp$ is a hyperbolic line containing three centers, that is, one of the four lines $\langle Q, R\rangle$, $\langle X_1, Y_3\rangle$, $\langle X_2, Y_1\rangle$, or $\langle X_3, Y_2\rangle$, where $Q = X_1$ and $R = X_3$. If m contains P, and P is different from A_j, then $A_j^\perp \cap P^\perp$ contains one of the central centers X_i, and m lies in $X_i^\perp$, an impossibility since no line in the polar of a central center has more than three central centers. So m does not contain P. Thus, the lines $\langle A_i, P\rangle^\perp$ ($i = 1, \ldots, 4$) are all distinct and must be exactly the four lines in $P^\perp$ listed above. Consideration of the cases yields that m must lie in the polar of a central center. Thus, the assumption that $e = 2$ also leads to a contradiction, and the lemma is proved.

Since $\bar{H}$ is isomorphic to $PSL_2(\bar{q})$, there is an ordered basis $[x_1, \ldots, x_4]$ for (V, f) such that $P = \langle x_1 \rangle$, the form f has matrix

$$\begin{bmatrix} & & & 1 \\ & & \varepsilon & \\ & \varepsilon & & \\ 1 & & & \end{bmatrix},$$

and the elements in $\bar{H}$ have matrices in $SL_2(\bar{q})$. Since the matrix

$$\begin{bmatrix} 1 & & & \\ & \sqrt{\varepsilon} & & \\ & & \sqrt{\varepsilon} & \\ & & & 1 \end{bmatrix}$$

effects a coordinate change which yields a symplectic basis for (V, f) and which centralizes the representation of H on $P^{\perp}/P$, we may assume, without loss of generality, that $[x_1, \ldots, x_4]$ is itself a symplectic basis, that is, $\varepsilon = 1$.

Let lower case Greek letters denote 2×1 matrices in $V_2(q)$, and for

$$\alpha = \begin{bmatrix} x \\ y \end{bmatrix},$$

let $\bar{\alpha} = [y, x]$. It is easy to verify that: (a) $\bar{\alpha}\alpha = 0$ for all α in $V_2(q)$, and (b) if A is in $SL_2(q)$ and α, β in $V_2(q)$, then $\bar{\alpha}AB = \bar{\beta}A^{-1}\alpha$. Further, the elements in H have matrices of the form

$$g = \begin{bmatrix} 1 & \bar{\alpha}A & z \\ & A & \alpha \\ & & 1 \end{bmatrix},$$

where α is in $V_2(q)$ and A runs over all matrices in $SL_2(\bar{q})$. Note that

$$g^{-1} = \begin{bmatrix} 1 & \bar{\alpha} & z \\ & A^{-1} & A^{-1}\alpha \\ & & 1 \end{bmatrix}.$$

The kernel K of the action of H on $P^{\perp}/P$ is represented by matrices g for which $A = I$. Since $\bar{\alpha}\alpha = 0$, computation shows that for g in K, the scalar z is determined additively modulo w by the vector α in $V_2(q)$, where

$$t_P = \begin{bmatrix} 1 & 0 & w \\ & I & 0 \\ & & 1 \end{bmatrix}$$

is the unique central elation in G with center P. Direct computation will also verify the following lemma.

LEMMA 7. *Let*

$$S = \begin{bmatrix} 1 & \bar{\varepsilon} & z \\ & I & \varepsilon \\ & & 1 \end{bmatrix}$$

be in K and A in $SL_2(\bar{q})$. Then there is an α in $V_2(q)$ and u in F such that

$$T = \begin{bmatrix} 1 & \bar{\alpha}A & u \\ & A & \alpha \\ & & 1 \end{bmatrix}$$

is in H, and further:

(i) $TST^{-1} = S^T = \begin{bmatrix} 1 & \bar{\varepsilon}A^{-1} & z \\ & I & A\varepsilon \\ & & 1 \end{bmatrix}$ *is in K, and*

(ii) $SS^T = \begin{bmatrix} 1 & \bar{\varepsilon}(A^{-1} + I) & \bar{\varepsilon}A\varepsilon \\ & I & (A + I)\varepsilon \\ & & 1 \end{bmatrix}$ *is in K.*

For

$$S = \begin{bmatrix} 1 & \bar{\alpha} & z \\ & I & \alpha \\ & & 1 \end{bmatrix}$$

in K, we have $z \equiv x^2 + xy + y^2 \pmod{w}$, where

$$\alpha = \begin{bmatrix} x \\ y \end{bmatrix},$$

since application of (ii) in Lemma 7 to S using

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

yields that

$$\begin{bmatrix} 1 & \bar{\alpha} & z' \\ & I & \alpha \\ & & 1 \end{bmatrix}$$

is in K, where $z' = x^2 + xy + y^2$.

We claim the $K = \langle t_P \rangle$. Suppose there is an

$$S = \begin{bmatrix} 1 & \bar{\alpha} & z \\ & I & \alpha \\ & & 1 \end{bmatrix}$$

in K with

$$\alpha = \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Without loss of generality, $y = 0$, otherwise application of (ii) in Lemma 7 using

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

yields a new S in K for which $y = 0$ and $x \neq 0$. Since t_P is also in K, we may assume that $z = x^2$. Since $\bar{q} \geq 4$, there are distinct elements r_1 and r_2 in $GF(\bar{q}) - \{0, 1\}$. Application of (i) in Lemma 7 to S using

$$A = \begin{bmatrix} r_i & \\ & 1/r_i \end{bmatrix}$$

yields $x^2 \equiv (r_i x)^2 \pmod{w}$ and hence $x^2(r_i^2 + 1) \equiv 0 \pmod{w}$ for $i = 1$ and 2. Since $x \neq 0$ and $r_i \neq 1$, we conclude that $x^2(r_i + 1) = w$ for both $i = 1$ and $i = 2$. This contradicts the choice of r_1 different from r_2, and shows that $K = \langle t_P \rangle$.

Since $K = \langle t_P \rangle$, direct computation shows that for

$$\begin{bmatrix} 1 & \bar{\alpha}A & z \\ & A & \alpha \\ & & 1 \end{bmatrix}$$

in H, $\alpha = \alpha(A)$ is determined by A, and z is determined modulo w by A. Further, $\alpha(AB) = A\alpha(B) + \alpha(A)$ for A and B in $SL_2(\bar{q})$. Thus, the function α is a derivation (crossed homomorphism or 1-cocycle) from $SL_2(\bar{q})$ to $V_2(q)$ as a natural $SL_2(\bar{q})$-module [13, pp. 105–108].

We claim that the homomorphism from H to $\bar{H}$ maps the central elations in $H - K$ one-to-one onto the set of involutions in $\bar{H}$. Indeed, if

$$T = \begin{bmatrix} 1 & \bar{\alpha}A & z \\ & A & \alpha \\ & & 1 \end{bmatrix}$$

is a central elation in $H - K$, then $A \neq I$, and $\bar{T} = A$ is an involution. Since all involutions in $SL_2(\bar{q})$ are conjugate, and since for each B in $SL_2(\bar{q})$ there is an S in H such that $\bar{S} = B$, we conclude that every involution in $\bar{H}$ is the image of a central elation in $H - K$. If t_Q and t_R are central elations in $H - K$ such that $\bar{t}_Q = \bar{t}_R$, then $t_Q t_R$ is in K, and $t_Q = t_R$ or $t_Q t_R = t_P$. The latter is impossible by Table 2 in [7]. Thus, the map is one-to-one.

Examination of the matrices for central elations in H shows that each central center for G in $P^\perp$ lies on one of the $\bar{q} + 1$ rational, totally isotropic lines in $P^\perp$. Conversely, for each rational, totally isotropic line m in $P^\perp$, which can be expressed as $\langle x_1, rx_2 + sx_3 \rangle$ for some r and s in $GF(\bar{q})$, there is an involution i in $SL_2(\bar{q})$ with center

$$\begin{bmatrix} r \\ s \end{bmatrix}.$$

By the last claim, there is a central elation t in H such that $\bar{t} = i$, and the center of t is different from P and lies on m. Since there are $\bar{q}^2 - 1$ involutions in

$SL_2(\bar{q})$, there are $\bar{q}^2 - 1$ central centers for G in $P^{\perp} - \{P\}$, and hence there are $e = \bar{q}$ central centers on each totally isotropic line spanned by central centers.

Let Diag denote the subgroup of diagonal matrices in $SL_2(\bar{q})$. If we restrict the function α to Diag, we obtain a derivation from Diag to $V_2(q)$ as a Diag-module. Since $V_2(q)$ has exponent 2, which is relatively prime to $\bar{q} - 1 = |\text{Diag}|$, all derivations from Diag to $V_2(q)$ are inner derivations. Hence there is a vector β in $V_2(q)$ such that $\alpha(D) = (D - I)\beta$ for all D in Diag. The matrix

$$\begin{bmatrix} 1 & \bar{\beta} & 0 \\ & I & \beta \\ & & 1 \end{bmatrix}$$

effects a symplectic change of coordinates which fixes each totally isotropic line through P, centralizes $\bar{H}$, and allows us to assume that

$$\alpha(D) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for all D in Diag.

If the unique central elation T_1 in H such that

$$\bar{T}_1 = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ has matrix } \begin{bmatrix} 1 & \bar{\alpha}A & z \\ & A & \alpha \\ & & 1 \end{bmatrix}, \quad \text{where } \alpha = \begin{bmatrix} x \\ y \end{bmatrix},$$

then $y = 0$, and the matrix

$$\begin{bmatrix} 1/x & & \\ & I & \\ & & x \end{bmatrix}$$

effects a symplectic coordinate change which allows us to assume that $x = 1$ and

$$\alpha\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For a derivation α, it is easy to verify that $\alpha(A^{-1}) = A^{-1}\alpha(A)$, and that $\alpha(BAB^{-1}) = (BAB^{-1} + I)\alpha(B) + B\alpha(A)$. So for d in $GF(\bar{q})^*$, we compute that

$$\alpha\left(\begin{bmatrix} 1 & d^2 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

and obtain a central elation T_d in H with center $\langle x_1 + dx_2 \rangle$. Since $\bar{q} > 2$, we can choose d in $GF(\bar{q}) - \{0, 1\}$. Then

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \alpha\left(\begin{bmatrix} d & 0 \\ 0 & 1/d \end{bmatrix}\right) &= \alpha\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/d & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} d + 1 & 0 \\ 0 & (1/d) + 1 \end{bmatrix} \alpha\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right), \end{aligned}$$

and hence

$$\alpha\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right) = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

Thus

$$\alpha\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right) = \begin{bmatrix}0\\ 1\end{bmatrix}$$

and H contains a central elation S with center $\langle x_1 + x_3\rangle$ such that

$$\bar{S} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}.$$

Since $SL_2(\bar{q})$ is generated by the involutions

$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1 & d^2\\ 0 & 1\end{bmatrix}$$

for all d in $GF(\bar{q})^*$, the subgroup H is generated by the central elations t_P, T_d, and S for all d in $GF(\bar{q})^*$. Since all entries in the matrices for T_d and S are from $GF(\bar{q})$, and since multiplication by t_P effects only the (1, 4) entry of any upper triangular matrix, we conclude that all matrices for elements in H have entries in $GF(\bar{q})$, except for the (1, 4) entry, and that all central centers in $P^\perp$ are rational points (with respect to $\{\sum a_i x_i \mid a_i \in GF(\bar{q})\}$).

The $\bar{q}$ central elations in G with centers on $\langle x_1, x_2\rangle$ are t_P with center $\langle x_1\rangle$ and T_d with center $\langle x_1 + dx_2\rangle$ for all d in $GF(\bar{q})^*$. For d in $GF(\bar{q}) - \{0, 1\}$, $T_1 T_d$ is a centered skew elation in G with center $\langle x_2\rangle$. Since G is transitive on its central centers and on its skew centers, we conclude that no point is both a skew center and a central center for G.

Since G is transitive on pairs of orthogonal central centers, there is in G an element g which fixes $\langle x_1 + x_2\rangle$ and maps $\langle x_1 + dx_2\rangle$ to $\langle x_1\rangle$, where $d \neq 0$ or 1. Since g stabilizes the set of central centers on $\langle x_1, x_2\rangle$ and hence fixes the rational subline $\langle x_1, x_2\rangle_{\bar{q}}$, we conclude that g must fix $\langle x_2\rangle$ and that $(T_1 T_d)^g$ is a centered skew elation in G with center $\langle x_2\rangle$. Computation of the matrix for $(T_1 T_d)^g = T_1 t_P$ shows that $w = 1$. It is consistent with the notation T_d to write T_0 for t_P.

Summarizing, we have found a symplectic basis $[x_1, \ldots, x_4]$ for (V, f) such that H is generated by the central elations S and T_d for all d in $GF(\bar{q})$. Further, all central centers in $P^\perp$ (where $P = \langle x_1\rangle$) are rational points. Every rational, totally isotropic line in $P^\perp$ is a centered axis, and its unique rational point which is not a central center is a skew center.

Since G is primitive, there is a central center Q not in $P^\perp$ such that t_P and t_Q generate the full dihedral group D generated by all central elations in G with center on $\langle P, Q\rangle$. For each of the $\bar{q} + 1$ rational, totally isotropic lines m through P, the Center-Axis Theorem implies that m meets $Q^\perp$ in a center X, which must be rational since all central centers in $P^\perp$ are rational, and since if G has pattern (3FC) or (3FCN), then X is the only skew center on m and hence rational. Thus, $\langle P, Q\rangle^\perp$ is a rational hyperbolic line, and any central center not in $P^\perp$ lies on a rational hyperbolic line through P.

Since $Q = \langle [a, b, c, 1]^t\rangle$ for some a in F and b, c in $GF(\bar{q})$, the matrix

$$\begin{bmatrix} 1 & c & b & a \\ & 1 & 0 & b \\ & & 1 & c \\ & & & 1 \end{bmatrix}$$

effects a symplectic basis change which sends Q to $\langle x_4\rangle$ and stabilizes the set of rational vectors in $P^\perp$. Without loss of generality, $Q = \langle x_4\rangle$; however, we can no longer say that $\langle x_2\rangle$ and $\langle x_3\rangle$ are skew centers.

Suppose t_Q maps x_1 to $x_1 + rx_4$, where r is in F. Since $\bar{q} > 2$, there is an a in $GF(\bar{q})^*$ such that $C = \langle x_1 + ax_2\rangle$ is a central center. Thus, $t_Q(C)$ is a central center and hence lies on a rational line through $\langle x_1\rangle$. Computation shows that r is in $GF(\bar{q})$.

Since P and Q are conjugate in the dihedral subgroup D of $PSp_4(\bar{q})$ generated by t_P and t_Q, we conclude that all central centers in $\langle x_4\rangle^\perp$ are rational points, and that every central center lies on a rational line through Q.

Let X be any central center for G. If X is not on $\langle P, Q\rangle$, then X is a rational point, since $\langle P, X\rangle$ and $\langle Q, X\rangle$ are distinct, rational lines. If X is on $\langle P, Q\rangle$, then X is also rational, since all central elations with center on $\langle P, Q\rangle$ are conjugate in D, which is contained in $PSp_4(\bar{q})$. Therefore, all central centers for G are rational points.

Since it is easy to find five central centers, no four of which are coplanar, and since a three-dimensional projectivity is determined by the images of five such points [1, pp. 66–68], we conclude that G stabilizes the set of rational points. By Proposition 4 in [7], G fixes the rational subgeometry $\{\sum a_i x_i \mid a_i \in GF(\bar{q})\}$, which is a contradiction to G being superprimitive, unless $\bar{q} = q$. Therefore, $\bar{q} = q$.

5. Construction of the quadric

Let $P = \langle x_1\rangle$ and let Q be the quadratic form on (V, f) given by

$$Q([a, b, c, d]^t) = a^2 + ad + d^2\varepsilon + bc,$$

where ε is in F^*. It is easy to verify that the generators S and T_d (all $d \in F$) of H are contained in the orthogonal group $GO(Q)$, as is H. Since both H and $GO(Q)$ have q^2 central elations with centers in $P^\perp$, and since the central elations in $GO(Q)$ have nonsingular centers, we conclude that the q^2 central centers for G in $P^\perp$ are precisely the nonsingular points for Q in $P^\perp$, and that the $q + 1$ skew centers for G in $P^\perp$ are precisely the singular points for Q in $P^\perp$. Thus, in the polar of any central center, the skew centers form an oval, and the central centers are the points off that oval.

Since $\langle x_1, x_4\rangle$ lies in the polar of a central center on $\langle x_2, x_3\rangle$, we conclude that $\langle x_1, x_4\rangle$ contains either $q - 1$ or $q + 1$ central centers. Let

$$W = \langle ax_1 + x_4\rangle$$

be a central center such that t_P and t_W generate the full dihedral group generated by the central elations in G with center on $\langle x_1, x_4\rangle$. Since the matrix

$$\begin{bmatrix} 1 & 0 & 0 & a \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

effects a symplectic coordinate change which maps W to $\langle x_4\rangle$, fixes the vectors in $\langle x_1\rangle^{\perp}$, and centralizes the matrices in H, we may assume that $W = \langle x_4\rangle$, and so

$$t_W = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ s & 0 & 0 & 1 \end{bmatrix} \quad \text{for some } s \text{ in } F^*.$$

Hence $w = t_P t_W$ has order $q - 1$ or $q + 1$ and acts regularly on the central centers on $\langle x_1, x_4\rangle$. Use $1/s$ for ε in Q. Then w is in $GO(Q)$, and $\langle x_2\rangle$ and $\langle x_3\rangle$ are singular points for Q and hence skew centers for G.

For each of the $q - 1$ central centers Z on $\langle x_2, x_3\rangle$, there is a unique skew center Z^* (different from Z and P) on the totally isotropic line $\langle P, Z\rangle$. The transformation w fixes Z, acts regularly on the $q - 1$ or $q + 1$ central centers on $\langle x_1, x_4\rangle$, and hence acts regularly on $q - 1$ or $q + 1$ totally isotropic lines through Z, each of which is spanned by central centers and has a unique skew center not on $\langle x_1, x_4\rangle$. This accounts for $(q - 1)^2$ or $q^2 - 1$ skew centers off $\langle x_1, x_4\rangle$, each of which is a singular point for Q, since Z^* is singular and w is in $GO(Q)$.

The transformation w acts faithfully on $\langle x_1, x_4\rangle$ as

$$\begin{bmatrix} 1 + s & 1 \\ s & 1 \end{bmatrix},$$

which has characteristic polynomial $X^2 + sX + 1$. Since the reducibility over F of $X^2 + sX + 1$ follows that of $X^2 + (\sqrt{s})X + 1$ and $X^2 + X + 1/s$, we conclude that $\langle x_1, x_4\rangle$ has $q - 1$ central centers if and only if Q has maximal index.

Suppose Q has maximal index. Then there are exactly two points X_1 and X_2 on $\langle x_1, x_4\rangle$ which are not central centers, namely $\langle a_i x_1 + x_4\rangle$, where a_1 and a_2 are the distinct roots in F to $X^2 + X + 1/s$. Computation shows that X_1 and X_2 are singular for Q.

Let $R_1 = \langle x_2\rangle$ and $R_2 = \langle x_3\rangle$. Then $\langle X_i, R_j\rangle$ $(i, j = 1, 2)$ are four totally isotropic lines each of which contains two points which are not central centers.

Hence $\langle X_i, R_j\rangle$ $(i, j = 1, 2)$ contains no central centers. Each of the other totally isotropic lines through R_1 or R_2 meets $\langle x_1, x_4\rangle$ in a central center, and hence has exactly R_1 or R_2 as its only point which is not a central center. Further, since X_i and R_j $(i, j = 1, 2)$ are orthogonal singular points for Q, all points on $\langle X_i, R_j\rangle$ are singular.

A count yields exactly $(q + 1)^2$ points which are not central centers. Therefore, the points which are not central centers are precisely the singular points for Q. Since G acts on its central centers, it stabilizes the quadric for Q, contrary to G fixing no totally isotropic regulus. Thus, Q must have nonmaximal index.

For each of the $q - 1$ central centers Z on $\langle x_2, x_3\rangle$, there are $q + 1$ points in $Z^\perp$ which are not central centers, and we have seen that all of these are singular for Q. Since each totally isotropic line through $\langle x_i\rangle$ $(i = 2, 3)$ meets $\langle x_1, x_4\rangle$ in a central center, $\langle x_i\rangle$ is the only point in $\langle x_i\rangle^\perp$ which is not a central center. Thus, there are $q^2 + 1$ points in (V, f) which are not central centers, all of which are singular for Q and hence form the quadric for Q. Since G stabilizes its central centers, it stabilizes the quadric for Q and lies in $GO(Q)$ [7, Proposition 5]. Since G contains a central elation at each nonsingular point, and since $GO_4(-1, q)$ is generated by its central elations [3, p. 42], we conclude that G is equal to $GO(Q)$, where Q is a nonmaximal index quadratic form on (V, f). This concludes the proof of the theorem.

6. Proof of corollary

By the theorem, the candidates for the maximal subgroups of $PSp_4(q)$ which contain central elations or noncentered skew elations are the nonmaximal index orthogonal groups, the stabilizers of the various geometric objects in the definition of superprimitive, and all the duals of the preceeding. The stabilizers of points (or equivalently of planes) and the stabilizers of totally isotropic lines are dual. The stabilizer of a hyperbolic line lies properly in the stabilizer of a polar pair, whose dual is a maximal index orthogonal group [7]. The stabilizer of a pair of skew totally isotropic lines fixes a totally isotropic regulus and lies in a maximal index orthogonal group. The stabilizer of a pair of distinct, nonpolar, hyperbolic lines fixes a unique totally isotropic line [6, Theorem 2]. The proof of the Duality Theorem [7] shows that if G fixes a tetrahedron, then its dual G^δ acts on a set of three points and hence has been considered above. Clearly, $PSp_4(q')$ cannot be maximal in $PSp_4(q)$ unless $GF(q')$ is maximal in $GF(q)$. We conclude that the only candidates for maximal subgroups of $PSp_4(q)$ which contain central elations or noncentered skew elations are those listed in the corollary. It remains to show that each of these is maximal and that all subgroups within a given category are conjugate. Only one category in each dual pair needs to be considered.

Let H be the stabilizer in $PSp_4(q)$ of a point P and G a subgroup of $PSp_4(q)$ which contains H properly. It is easy to verify that the orbits of H on the points of V are $\{P\}$, $P^{\perp} - \{P\}$, and $V - P^{\perp}$. Further, G must then be transitive on the points of V, hence irreducible. A theorem of McLaughlin [14, p. 365] implies that G is $PSp_4(q)$, and H is maximal.

Pollatsek [16] has shown that the orthogonal groups are all maximal in $PSp_4(q)$.

Let $F' = GF(q')$ be a maximal subfield of F and H the stabilizer in $PSp_4(q)$ of some subgeometry over F'. Suppose G is a maximal subgroup of $PSp_4(q)$ which properly contains H. By duality, we may assume, without loss of generality, that G is an orthogonal group or belongs to one of the categories (d_r) of the corollary, since H clearly fixes no point. If G is an orthogonal group (in which distinct central elations have distinct centers), then $q' = 2$, which is impossible, since no three skew centers (singular points) for an orthogonal group can be collinear. So G must be the stabilizer of some subgeometry over a maximal subfield $F'' = GF(q'')$ of F. If $q' = 2$, then F' is the only maximal subfield of F, and $F'' = F'$. If $q' > 2$, then the subgroups L' and L'' of H and G (resp.) generated by the central elations in H and G (resp.) with centers on a hyperbolic line spanned by central centers for H and G (resp.) are isomorphic to $PSL_2(q')$ and $PSL_2(q'')$ (resp.) with $L' \subseteq L''$; hence F' is a subfield of F'', and $F' = F''$. Therefore, $PSp_4(q')$ is a maximal subgroup of $PSp_4(q)$ whenever $GF(q')$ is maximal in $GF(q)$.

Since $Sp_4(q)$ is transitive on the symplectic bases, the groups in each of the categories (d_r), (a), and (dually) (a*) are conjugate. Since the symplectic transformation

$$\begin{bmatrix} 1 & 0 & 0 & \varepsilon \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

maps the quadric of Q_σ to the quadric of $Q_{(\varepsilon^2+\varepsilon+\sigma)}$, where

$$Q_\lambda(\textstyle\sum a_i x_i) = a_1^2 + a_1 a_4 + a_4^2 \lambda + a_2 a_3 \quad \text{for any } \lambda \text{ in } F,$$

and since Q_λ is of maximal index if and only if $X^2 + X + \lambda$ is reducible over F, we conclude that there are two classes of quadrics over $PSp_4(q)$ and only two classes of orthogonal groups in $PSp_4(q)$. Thus, all groups in each of the categories of the corollary are conjugate.

7. Other maximal subgroups

Since our result on the maximal subgroups of $PSp_4(q)$ which contain no central elations or noncentered skew elations is not as complete as in the corollary, we will state the theorem and at this time only give an outline of the proof.

THEOREM. *If M is a maximal subgroup of $PSp_4(2^n)$ which contains no central elations or noncentered skew elations, then either $q = 2$ and M is isomorphic to* *the alternating group on six letters ($PSp_4(2)$ is isomorphic to the symmetric group on six letters), or M contains normal subgroups M_1 and M_2 such that $M \geq M_1 > M_2 \geq \{1\}$, where M/M_1 and M_2 are of odd order, and M_1/M_2 is isomorphic to $PSL_2(q')$ or $Sz(q')$ (Suzuki group) for some power q' of 2.*

A few matrix computations yield contradictions to the existence of a primitive subgroup of $PSp_4(q)$ with pattern (2F). Dually, pattern (3F) is also ruled out for primitive subgroups. If G is a primitive subgroup of $PSp_4(q)$ and has pattern (1), (2), (3), or (1F), then it is easy to use the Sylow 2-Subgroup Theorem in [7] to verify that the Sylow 2-subgroups of G are TI sets; a theorem of Suzuki [17] then implies the last portion of the theorem above, with the additional possibility of $PSU_3(q')$. Ben Mwene observed that $PSU_3(q')$ cannot occur since $PSU_3(2^2)$ has a quaternion Sylow 2-subgroup, whereas $PSp_4(2^n)$ has no quaternion subgroups.

It remains to consider G a superprimitive subgroup of $PSp_4(q)$ with pattern (4F). By letting H be the subgroup of G generated by the flag-fixers in G with a given center P, and by considering the action of H on $P^\perp/P$, we are able to construct the actual matrices for elements in H and see that H is isomorphic to the symmetric group on four letters. By applying a theorem of Gorenstein and Walter [9] and considering the various cases, we can show that $G \cong PSL_2(9) \cong A_6$. Further computations show that $q = 2$, and that G is the obvious subgroup of $PSp_4(2) \cong S_6$.

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