ON THE HIGHER ORDER SECTIONAL CURVATURES

BY

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The riemannian (holomorphic) higher order sectional curvatures are invariants of the riemannian (kaehlerian) structure weaker than the riemannian (holomorphic) sectional curvature. The study of these invariants is very interesting as can be seen by the abundant bibliography on this subject; for example, the articles of Thorpe, Gray, Stehney, Hsiung, Levko, ...

If the riemannian sectional curvature of order two is bounded, Berger [1] gives an estimation of the curvature tensor components. Later, Karcher [2] gives an easy proof of this estimation. We shall prove in Section 1 a generalization of these results to the higher order riemannian curvature tensor components R_p when the sectional curvature of order p is also bounded.

Thorpe [6] gives the characterization of the constancy of the riemannian sectional curvature of order p and he concludes properties on the Pontrjagin classes of these manifolds. In an earlier article [4] we give a characterization of the constancy of the holomorphic sectional curvature of order p and we deduce properties on the Chern classes of the kaehlerian manifolds with constant holomorphic sectional curvature of order two. We shall generalize in Section 2 some results of Thorpe on riemannian sectional curvatures of order p to the holomorphic sectional curvatures of order p and we shall conclude some properties on the Chern classes of the kaehlerian manifolds with constant holomorphic sectional curvature of order p.

1. Higher order curvature tensor estimates

Let M be a riemannian manifold of even dimension n and let $\Lambda^p(M)$ denote the bundle of p-vectors of M. $\Lambda^p(M)$ is a riemannian vector bundle with inner product on the fiber $\Lambda^p(m)$ over m, $m \in M$, related to the inner product on the tangent space M_m of M at m by

$$g(u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p) = \det\{g(u_i, v_j)\}, \quad u_i, v_j \in M_m.$$

Let R denote the covariant curvature tensor of M. For each even $p > 0$ we define the pth curvature tensor R_p of M to be the covariant tensor field of order $2p$ given by

$$\begin{aligned} R_p(u_1, \ldots, u_p, v_1, \ldots, v_p) &= \frac{1}{2^{p/2}p!} \sum_{\alpha, \beta \in S_p} \varepsilon(\alpha)\varepsilon(\beta) R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \cdots \\ &\qquad R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)}) \end{aligned} \tag{1}$$

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where $u_i, v_j \in M_m$, S_p denotes the group of permutations of $(1, \ldots, p)$ and, for $\alpha \in S_p$, $\varepsilon(\alpha)$ is the sign of α.

It is evident that the tensor R_p has the following properties:

(i) It is alternating in the first p and in the last p variables.

(ii) It is invariant under the operation of interchanging the first p variables with the last p.

Hence, at each point $m \in M$, R_p may be regarded as a symmetric bilinear form on $\Lambda^p(M)$. By use of the inner product on $\Lambda^p(M)$, R_p at m may then be identified with a self-adjoint linear operator on $\Lambda^p(M)$. Explicitly, this identification is given by

$$g\{R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p\} \equiv R_p(u_1, \ldots, u_p, v_1, \ldots, v_p). \quad (2)$$

If $\{u_1, \ldots, u_n\}$ is an orthonormal basis of the tangent space, the sectional curvature of order p of the section generated by $u_{i_1}, \ldots, u_{i_p}$ is given by

$$K(u_{i_1}, \ldots, u_{i_p}) = R_p(u_{i_1}, \ldots, u_{i_p}, u_{i_1}, \ldots, u_{i_p}).$$

As is well known [7], R_p satisfies the generalized first Bianchi Identity

$$\sum_{k=1}^{p+1} (-1)^k R_p(v_1, \ldots, \hat{v}_k, \ldots, v_{p+1}, v_k, w_1, \ldots, w_{p-1}) = 0. \quad (3)$$

Let

$$\delta = \operatorname*{Min}_{u_{i_j} \in M_m} K(u_{i_1}, \ldots, u_{i_p}), \quad \Delta = \operatorname*{Max}_{u_{i_j} \in M_m} K(u_{i_1}, \ldots, u_{i_p}).$$

PROPOSITION 1. *If the sectional curvature of order p of a compact orientable riemannian manifold satisfies*

$$\delta \leq K(u_1, \ldots, u_p) \leq \Delta$$

then

$$|R_p(u, u_\alpha, u', u, u_\alpha, v)| \leq \frac{2^{p-\alpha-1}(\Delta - \delta)}{p - \alpha + 1} \quad (4)$$

where $u = (u_1, \ldots, u_{\alpha-1})$, $u' = (u_{\alpha+1}, \ldots, u_p)$, $v = (v_1, \ldots, v_{p-\alpha})$, *and* $u_1, \ldots, u_p, v_1, \ldots, v_{p-\alpha}$ *are orthonormal. The range of* α *is* $0 \leq \alpha \leq p - 1$.

Proof. We shall use induction; since

$$R_p(u_1, \ldots, u_{p-1}, u_p + x, u_1, \ldots, u_{p-1}, u_p + x) - R_p(u_1, \ldots, u_{p-1}, u_p - x, u_1, \ldots, u_{p-1}, u_p - x) = 4R_p(u_1, \ldots, u_{p-1}, u_p, u_1, \ldots, u_{p-1}, x)$$

for $u = (u_1, \ldots, u_{p-1})$, we have $|R_p(u, u_p, u, x)| \leq \frac{1}{2}(\Delta - \delta)$ for any unit vector x orthogonal to $u_1, \ldots, u_p$. Suppose

$$|R_p(u, u_\alpha, u', u, u_\alpha, v)| \leq \frac{2^{p-\alpha-1}(\Delta - \delta)}{p - \alpha + 1}.$$

Since R_p verifies the first Bianchi identity, we have

$$\begin{aligned} &R_p(u, u_\alpha + x, u_{\alpha+1}, \dots, u_p, u, v, u_\alpha + x) \\ &\qquad - R_p(u, u_\alpha - x, u_{\alpha+1}, \dots, u_p, u, v, u_\alpha - x) \\ &\qquad + R_p(u, u_\alpha, u_{\alpha+1} + x, u_{\alpha+2}, \dots, u_p, u, v, u_{\alpha+1} + x) \\ &\qquad - R_p(u, u_\alpha, u_{\alpha+1} - x, u_{\alpha+2}, \dots, u_p, u, v, u_{\alpha+1} - x) + \cdots \\ &\qquad + R_p(u, u_\alpha, \dots, u_{p-1}, u_p + x, u, v, u_p + x) \\ &\qquad - R_p(u, u_\alpha, \dots, u_{p-1}, u_p - x, u, v, u_p - x) \\ &\qquad\qquad = 2(p - \alpha + 2)R_p(u, u_\alpha, u', u, v, x). \end{aligned}$$

Thus for any unit vector x orthogonal to $u_1, \dots, u_p, v_1, \dots, v_{p-\alpha}$,

$$\begin{aligned} &R_p(u, u_\alpha, u', u, v, x) \\ &\quad = \frac{1}{p - \alpha + 2}\left\{R_p\left(u, \frac{u_\alpha + x}{2^{1/2}}, u_{\alpha+1}, \dots, u_p, u, v, \frac{u_\alpha + x}{2^{1/2}}\right)\right. \\ &\qquad - R_p\left(u, \frac{u_\alpha - x}{2^{1/2}}, u_{\alpha+1}, \dots, u_p, u, v, \frac{u_\alpha - x}{2^{1/2}}\right) + \cdots \\ &\qquad + R_p\left(u, \dots, u_{p-1}, \frac{u_p + x}{2^{1/2}}, u, v, \frac{u_p + x}{2^{1/2}}\right) \\ &\qquad \left. - R_p\left(u, \dots, u_{p-1}, \frac{u_p - x}{2^{1/2}}, u, v, \frac{u_p - x}{2^{1/2}}\right)\right\}. \end{aligned}$$

By the induction hypothesis, we conclude

$$|R_p(u, u_\alpha, u', u, v, x)| \leq \frac{2^{p-\alpha}(\Delta - \delta)}{p - \alpha + 2}.$$

2. The Chern classes of kaehlerian manifolds with constant holomorphic sectional curvature

Let M be a kaehlerian manifold; let $(z^1, \dots, z^n)$ be a complex coordinate system in M, $(Z_i = \partial/\partial z^i, Z_{\bar{j}} = \partial/\partial \bar{z}^j)$ a basis of the complex tangent spaces of M. Given a hermitian metric g on M, it is well known that there exists a unique extension to a complex symmetric bilinear form on the complex tangent space of M such that

$$g(Z_i, Z_j) = g(Z_{\bar{i}}, Z_{\bar{j}}) = 0 \quad \text{and} \quad g(Z_i, Z_{\bar{j}}) = g_{i\bar{j}}$$

are the components of a hermitian matrix. That extension permits definition of a symmetric bilinear form on the fiber $\Lambda^s(T_m^{\mathbf{C}}(M))$, $m \in M$, where $\Lambda^s(T^{\mathbf{C}}(M))$ is a complex vector bundle on M, by

$$g(Z_{A_1} \wedge \cdots \wedge Z_{A_s}, Z_{B_1} \wedge \cdots \wedge Z_{B_s}) = \det\{g(Z_{A_k}, Z_{B_l})\}$$

where A_k, $B_l \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. Moreover, for each coordinate neighborhood it is possible to take $g_{i\bar{j}} = \delta_{ij}$ at a fixed point.

LEMMA 1. *Let P be an oriented holomorphic p-plane with a complex basis $(Z_1, \ldots, Z_s, Z_{\bar{1}}, \ldots, Z_{\bar{s}})$, $p = 2s$. Then*

$$R_p(P) = \frac{2^s(-1)^{(1/2)s(s-1)}}{(2s)!} \sum_{\alpha, \beta \in S_s} \varepsilon(\alpha)\varepsilon(\beta) R(Z_{\alpha 1} \wedge Z_{\bar{\beta 1}}) \wedge \cdots \wedge R(Z_{\alpha s} \wedge Z_{\bar{\beta s}}) \tag{5}$$

Proof. Complete $(Z_1, \ldots, Z_s, Z_{\bar{1}}, \ldots, Z_{\bar{s}})$ to a complex basis $(Z_1, \ldots, Z_n, Z_{\bar{1}}, \ldots, Z_{\bar{n}})$. Since

$$R(Z_k \wedge Z_{\bar{l}}) = \sum_{i, j=1}^{n} g\{R(Z_k \wedge Z_{\bar{l}}), Z_i \wedge Z_{\bar{j}}\} Z_{\bar{i}} \wedge Z_j$$

(for a kaehlerian manifold

$$g\{R(Z_k \wedge Z_{\bar{l}}), Z_i \wedge Z_j\} = g\{R(Z_k \wedge Z_{\bar{l}}), Z_{\bar{i}} \wedge Z_{\bar{j}})\} = 0)$$

it is possible to write the right hand side of (5) as

$$D = \frac{2^s(-1)^{(1/2)s(s-1)}}{(2s)!} \sum_{(i), (j)} \sum_{\alpha, \beta \in S_s} \varepsilon(\alpha)\varepsilon(\beta) g\{R(Z_{\alpha 1} \wedge Z_{\bar{\beta 1}}), Z_{i_1} \wedge Z_{\bar{j}_1}\} \times \cdots$$
$$\times\, g\{R(Z_{\alpha s} \wedge Z_{\bar{\beta s}}), Z_{i_s} \wedge Z_{\bar{j}_s}\} Z_{\bar{i}_1} \wedge Z_{j_1} \wedge \cdots \wedge Z_{\bar{i}_s} \wedge Z_{j_s}$$

where $(i) = (i_1, \ldots, i_s)$, $(j) = (j_1, \ldots, j_s)$. Hence

$$g(D, Z_{l_1} \wedge Z_{\bar{k}_1} \wedge \cdots \wedge Z_{l_s} \wedge Z_{\bar{k}_s})$$
$$= \frac{2^s(-1)^{(1/2)s(s-1)}}{(2s)!} \sum_{(i), (j)} \sum_{\alpha, \beta, \gamma, \sigma \in S_s} \varepsilon(\alpha)\varepsilon(\beta) g\{R(Z_{\alpha 1} \wedge Z_{\bar{\beta 1}}), Z_{i_1} \wedge Z_{\bar{j}_1}\} \times \cdots$$
$$\times\, g\{R(Z_{\alpha s} \wedge Z_{\bar{\beta s}}), Z_{i_s} \wedge Z_{\bar{j}_s}\} \varepsilon(\gamma)\varepsilon(\sigma) \delta^{j_1}_{k_{\sigma 1}} \cdots \delta^{j_s}_{k_{\sigma s}} \delta^{i_1}_{l_{\gamma 1}} \cdots \delta^{i_s}_{l_{\gamma s}}$$
$$= \frac{2^s(-1)^{(1/2)s(s-1)}}{(2s)!} \sum_{\alpha, \beta, \gamma, \sigma \in S_s} \varepsilon(\alpha)\varepsilon(\beta)\varepsilon(\gamma)\varepsilon(\sigma)$$
$$\times\, g\{R(Z_{\alpha 1} \wedge Z_{\bar{\beta 1}}), Z_{l_{\gamma 1}} \wedge Z_{\bar{k}_{\sigma 1}}\} \times \cdots \times g\{R(Z_{\alpha s} \wedge Z_{\bar{\beta s}}), Z_{l_{\gamma s}} \wedge Z_{\bar{k}_{\sigma s}}\}$$
$$= (-1)^{(1/2)s(s-1)} g\{R_p(Z_1 \wedge \cdots \wedge Z_s \wedge Z_{\bar{1}} \wedge \cdots \wedge Z_{\bar{s}}), Z_{l_1} \wedge \cdots \wedge Z_{l_s} \wedge Z_{\bar{k}_1} \wedge \cdots \wedge Z_{\bar{k}_s}\}$$
$$= g\{R_p(Z_1 \wedge \cdots \wedge Z_s \wedge Z_{\bar{1}} \wedge \cdots \wedge Z_{\bar{s}}), Z_{l_1} \wedge Z_{\bar{k}_1} \wedge \cdots \wedge Z_{l_s} \wedge Z_{\bar{k}_s}\}.$$

This completes the proof, since with respect to any other p-vector spanned by elements of the complex basis, both sides of (5) have zero component.

Remark 1.

$$R_p(W_1 \wedge \cdots \wedge W_p) = \frac{1}{p!} \sum_{\alpha \in S_p} \varepsilon(\alpha) R(W_{\alpha 1} \wedge W_{\alpha 2}) \wedge \cdots \wedge R(W_{\alpha(p-1)} \wedge W_{\alpha p}) \tag{6}$$

holds in general, where $W_1, \ldots, W_p$ are arbitrary elements of the complex tangent space. We show the particular expression of (6) for (s, s)-planes.

COROLLARY 1. *Suppose $s \geq 0$ and $s' \geq 0$ are integers with $s + s' \leq n$. Let P be an oriented holomorphic $(2s + 2s')$-plane with an oriented complex basis*

$$(Z_1, \ldots, Z_{s+s'}, Z_{\bar{1}}, \ldots, Z_{(s+s')^-})$$

and let

$$\Gamma = \{Z_{i_1} \wedge \cdots \wedge Z_{i_s} \wedge Z_{\bar{j}_1} \wedge \cdots \wedge Z_{\bar{j}_s} : 1 \leq i_1 < \cdots < i_s \leq s + s', 1 \leq j_1 < \cdots < j_s \leq s + s'\}.$$

Then

$$R_{p+p'}(P) = \frac{(2s)!\,(2s')!}{(2s + 2s')!} \sum_{Q \in \Gamma} R_p(Q) \wedge R_{p'}(Q^*) \tag{7}$$

where $p = 2s$, $p' = 2s'$ and Q^* is the oriented complement of Q in P spanned by elements of the preferred basis.

Proof. By Lemma 1,

$$R_{p+p'}(P) = \frac{2^{s+s'}(-1)^{(1/2)(s+s')(s+s'-1)}}{(2s + 2s')!} \sum_{\gamma, \delta \in S_{s+s'}} \varepsilon(\gamma)\varepsilon(\delta) R(Z_{\gamma 1} \wedge Z_{\bar{\delta 1}}) \wedge \cdots \wedge R(Z_{\gamma(s+s')} \wedge Z_{(\delta(s+s'))^-}).$$

For each pair $(i) = (i_1 < \cdots < i_s)$, $(j) = (j_1 < \cdots < j_s)$, we choose a pair $(i_{s+1}, \ldots, i_{s+s'})$, $(j_{s+1}, \ldots, j_{s+s'})$ such that $(i_1, \ldots, i_{s+s'})$ and $(j_1, \ldots, j_{s+s'})$ are even permutations of $(1, \ldots, s + s')$. Then

$$\begin{aligned} R_{p+p'}(P) &= \frac{2^{s+s'}(-1)^{(1/2)(s+s')(s+s'-1)}}{(2s + 2s')!} \\ &\quad \times \sum_{(i),(j)} \left\{ \sum_{\alpha, \beta \in S_s} \varepsilon(\alpha)\varepsilon(\beta) R(Z_{i_{\alpha 1}} \wedge Z_{\bar{j}_{\beta 1}}) \wedge \cdots \wedge R(Z_{i_{\alpha s}} \wedge Z_{\bar{j}_{\beta s}}) \right\} \\ &\quad \wedge \left\{ \sum_{\rho, \tau \in S_{s'}} \varepsilon(\rho)\varepsilon(\tau) R(Z_{i_{\rho(s+1)}} \wedge Z_{\bar{j}_{\tau(s+1)}}) \wedge \cdots \wedge R(Z_{i_{\rho(s+s')}} \wedge Z_{\bar{j}_{\tau(s+s')}}) \right\} \\ &= \frac{(2s)!\,(2s')!}{(2s + 2s')!} \sum_{Q \in \Gamma} R_p(Q) \wedge R_{p'}(Q^*). \end{aligned}$$

The statements of Lemma 1 and Corollary 1 have an equivalent formulation through higher order curvature forms $\Psi^{i_1\cdots i_s}_{j_1\cdots j_s}$ regarding these forms as the components of a tensorial form R_p on M with values in the bundle of complex p-vectors as follows: If $W_1, \ldots, W_{2s}$ are vectors in the complex tangent space and $z = (m, Z_1, \ldots, Z_n, Z_{\bar{1}}, \ldots, Z_{\bar{n}})$ is a complex frame, if $W'_1, \ldots, W'_{2s}$ are complex tangent vectors on the bundle of complex frames such that $d\Pi W'_j = W_j$, $1 \leq j \leq 2s$, then

$$R_p(W_1, \ldots, W_{2s}) = \sum_{(i),(j)} \Psi^{i_1\cdots i_s}_{j_1\cdots j_s}(W'_1, \ldots, W'_{2s}) Z_{i_1} \wedge \cdots \wedge Z_{i_s} \wedge Z_{\bar{j}_1} \wedge \cdots \wedge Z_{\bar{j}_s}$$

where $1 \leq i_1 < \cdots < i_s \leq n$, $1 \leq j_1 < \cdots < j_s \leq n$. (5), (7) take the form

$$\Psi^{i_1\cdots i_s}_{j_1\cdots j_s} = \frac{2^s(-1)^{(1/2)s(s-1)}}{(2s)!} \sum_{\alpha,\beta\in S_s} \varepsilon(\alpha)\varepsilon(\beta)\Psi^{i_{\alpha 1}}_{j_{\beta 1}} \wedge \cdots \wedge \Psi^{i_{\alpha s}}_{j_{\beta s}}, \tag{8}$$

$$\Psi^{i_1\cdots i_{s+s'}}_{j_1\cdots j_{s+s'}} = \frac{(2s)!\,(2s')!\,(-1)^{ss'}}{(s!)^2(s'!)^2(2s+2s')!} \sum_{\alpha,\beta\in S_{s+s'}} \varepsilon(\alpha)\varepsilon(\beta)\Psi^{i_{\alpha 1}\cdots i_{\alpha s}}_{j_{\beta 1}\cdots j_{\beta s}} \wedge \Psi^{i_{\alpha(s+1)}\cdots i_{\alpha(s+s')}}_{j_{\beta(s+1)}\cdots j_{\beta(s+s')}} \tag{9}$$

As we know, the holomorphic sectional curvature of order $2s$ in a kaehlerian manifold M of the holomorphic $2s$-plane generated by

$$(X_1, \ldots, X_s, JX_1, \ldots, JX_s)$$

is given by

$$K_p(P) = R_p(X_1, \ldots, X_s, JX_1, \ldots, JX_s, X_1, \ldots, X_s, JX_1, \ldots, JX_s).$$

If $\theta = (\theta^1, \ldots, \theta^{2n})$ is the canonical form on the bundle of unitary frames, set $\phi^i = \theta^i + i\theta^{n+i}$; we have the following Proposition [4] that characterizes the constant holomorphic sectional curvatures.

PROPOSITION 2. *Let M be a kaehlerian manifold with constant holomorphic sectional curvature of order p, K_p. Then the curvature form of order p is given by*

$$\begin{aligned}\Psi^{i_1\cdots i_s}_{j_1\cdots j_s} = \frac{1}{(s+1)!} K_p \Big\{ & s!\, \phi^{i_1} \wedge \cdots \wedge \phi^{i_s} \wedge \bar{\phi}^{j_1} \wedge \cdots \wedge \bar{\phi}^{j_s} \\ & + (s-1)! \sum_k \delta^{i_k}_{j_k} \sum_{\lambda_k} \phi^{i_1} \wedge \cdots \wedge \phi^{i_{k-1}} \wedge \phi^{\lambda_k} \\ & \wedge \phi^{i_{k+1}} \wedge \cdots \wedge \phi^{i_s} \wedge \bar{\phi}^{j_1} \wedge \cdots \wedge \bar{\phi}^{j_{k-1}} \wedge \bar{\phi}^{\lambda_k} \\ & \wedge \bar{\phi}^{j_{k+1}} \wedge \cdots \wedge \bar{\phi}^{j_s} + \cdots + \delta^{i_1}_{j_1} \cdots \delta^{i_s}_{j_s} \sum_{\lambda_1\cdots\lambda_s} \phi^{\lambda_1} \wedge \cdots \\ & \wedge \phi^{\lambda_s} \wedge \bar{\phi}^{\lambda_1} \wedge \cdots \wedge \bar{\phi}^{\lambda_s} \Big\}. \end{aligned} \tag{10}$$

Remark 2. In Proposition 2, we suppose, without loss of generality, that if $\#(i_1, \ldots, i_s) \cap (j_1, \ldots, j_s) = r$, then $i_1 = j_1, \ldots, i_r = j_r$, $0 \leq r \leq s$.

PROPOSITION 3. *Let M be a kaehlerian manifold of dimension n. Assume that M has constant pth holomorphic sectional curvature K_p and constant qth holomorphic sectional curvature K_q for some even p and q with $p + q = 2s + 2s' \leq n$. Then M has constant $(p + q)$th holomorphic sectional curvature cK_pK_q, where c is given by*

$$c = \frac{\{(s + s')!\}^3 (s + s' + 1)(2s)!(2s')!}{(s)!(2s + 2s')!(s + 1)!(s' + 1)!(s')!}$$

Proof. By Proposition 2, it suffices to show that

$$\begin{aligned}\Psi^{i_1 \cdots i_{s+s'}}_{j_1 \cdots j_{s+s'}} = \frac{c}{(s + s' + 1)!} K_p K_q & \\ \Big\{ (s + s')!\, \phi^{i_1} \wedge \cdots \wedge \phi^{i_{s+s'}} \wedge \bar{\phi}^{j_1} \wedge \cdots \wedge \bar{\phi}^{j_{s+s'}} & \\ + (s + s' - 1)! \sum_k \delta^{i_k}_{j_k} \sum_{\lambda_k} \phi^{i_1} \wedge \cdots \wedge \phi^{i_{k-1}} \wedge \phi^{\lambda_k} & \qquad (11) \\ \wedge \phi^{i_{k+1}} \wedge \cdots \wedge \phi^{i_{s+s'}} \wedge \bar{\phi}^{j_1} \wedge \cdots \wedge \bar{\phi}^{j_{k-1}} \wedge \bar{\phi}^{\lambda_k} & \\ \wedge \bar{\phi}^{j_{K+1}} \wedge \cdots \wedge \bar{\phi}^{j_{s+s'}} + \cdots + \delta^{i_1}_{j_1} \cdots \delta^{i_{s+s'}}_{j_{s+s'}} \sum_{\lambda_1 \cdots \lambda_{s+s'}} & \\ \phi^{\lambda_1} \wedge \cdots \wedge \phi^{\lambda_{s+s'}} \wedge \bar{\phi}^{\lambda_1} \wedge \cdots \wedge \bar{\phi}^{\lambda_{s+s'}} \Big\} &\end{aligned}$$

but that is a consequence of (9) and (10).

PROPOSITION 4. *Let M be a kaehlerian manifold with pth holomorphic sectional curvature K_p identically zero for some even p. Then M has qth holomorphic sectional curvature identically zero for all $q \geq p$.*

The proof follows from Proposition 2 and (9).

PROPOSITION 5. *Let M be a kaehlerian manifold with pth holomorphic sectional curvature constant K_p. Then the Chern classes $c_{2s}(M)$, $c_{3s}(M), \ldots$ are generated by $c_s(M)$.*

Proof. Since $c_s(M)$ is represented, up to a constant factor, by (see [3]) $\sum \Psi^{i_1 \cdots i_s}_{i_1 \cdots i_s}$ where summation is over all s-tuples $(i_1, \ldots, i_s)$, $1 \leq i_j \leq n$, it suffices to show that $\sum \Psi^{i_1 \cdots i_{ms}}_{i_1 \cdots i_{ms}}$ is a multiple of

$$\sum \Psi^{i_1 \cdots i_s}_{i_1 \cdots i_s} \wedge \cdots \wedge \sum \Psi^{i_1 \cdots i_s}_{i_1 \cdots i_s}.$$

Indeed, by Proposition 3, M has mpth holomorphic sectional curvature constant for all integers $m \geq 1$. It is possible to verify the following by inspection of the formula in Proposition 2: If the kth holomorphic sectional curvature is constant, then the coefficient of $\phi^{j_1} \wedge \cdots \wedge \phi^{j_k} \wedge \bar{\phi}^{j_1} \wedge \cdots \wedge \bar{\phi}^{j_k}$ in $\sum \Psi^{i_1 \cdots i_k}_{i_1 \cdots i_k}$ is independent of the choice $j_1 \leq \cdots \leq j_k$. It follows that $\sum \Psi^{i_1 \cdots i_k}_{i_1 \cdots i_k}$ is a multiple of

$$\sum \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \bar{\phi}^{i_1} \wedge \cdots \wedge \bar{\phi}^{i_k}.$$

Setting $k = mp$ and $k = p$ here, we quickly obtain the claim of the preceding paragraph.

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