A CHARACTERIZATION OF PROJECTIVE SPACES IN TERMS OF h-ENCLOSABILITY[1]

BY

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Introduction

All manifolds considered shall be closed and connected in the category $\mathscr{C}$ = Diff or PL. Following the notation and terminology of [1] we say that a $\mathscr{C}$ – manifold L^r is *h-enclosable* with a point A, and write $M^n = [L^r, A]$ (mod $\mathscr{C}$) when L^r is a $\mathscr{C}$ – submanifold of M^n with $M^n - L^r$ contractible onto A and for which L^r is a deformation retract of $M^n - A$. For example, when $F = R, C, Q$ or H we have $FP^n = [FP^{n-1}, A]$ (mod $\mathscr{C}$) where FP^n is the projective space over F. The object of this paper is to prove that projective spaces are to a large extent characterized by this property of h-enclosability. More precisely, the following result will be proved.

THEOREM 1. *Suppose $M^n = [L^r, A]$ (mod $\mathscr{C}$) where $\mathscr{C} = PL$ or Diff and A is a single point.*

(A) *If either one of M^n or L^r is not orientable then $M^n \sim RP^n$ (homotopy equivalence) and $L^r \sim RP^r$ and $r = n - 1$.*

(B) *If both M^n and L^r are orientable then:*

(i) *M^n is a homotopy sphere and L^r is a single point, if n is odd.*

(ii) *If n is even, then the only possible values for r are 0, $n - 2$, $n - 4$, and $n - 8$. If $r = 0$, M^n is a homotopy sphere. If $r = n - 2$, $M^n \sim CP^{n/2}$ and $L^r \sim CP^{r/2}$. If $r = n - 4$ (resp. $r = n - 8$), M^n is a cohomology $QP^{n/4}$ (resp. $HP^{n/8}$) and L^r is a cohomology $QP^{r/4}$ (resp. $HP^{r/8}$).*

1. Cohomology of M^n

Throughout this paper A denotes a point and we write $X \sim Y$ to mean that X is homotopically equivalent to Y. We write $M^n = [L^r, A]$ (mod $\mathscr{C}^+$) to indicate that $M^n = [L^r, A]$ (mod $\mathscr{C}$) and that both M^n and L^r are orientable. Though we state results for Diff and PL we will give detailed proofs only in the case of Diff. For the PL case we have only to replace the normal bundle of L in M as it occurs in an argument by the regular neighborhood of L in M. Throughout this paper we will be making repeated use of the following result proved in [2].

PROPOSITION 2. *Let M^n, L^r be closed connected manifolds and $x \in M^n$. If $M^n - x \sim L^r$ then $r/n = l/(l + 1)$ for some integer $l \geq 0$.*

From now on $\mathscr{C}$ will stand for Diff or PL.

Received September 30, 1974.

[1] This research was partially supported by National Research Council grants.

PROPOSITION 3. *Suppose* $M^n = [L^r, A]$ (mod $\mathscr{C}^+$) *and* $r > 0$. *Then there exists an integer* $l \geq 1$ *such that* $r = ld$, $n = (l + 1)d$ *with* $d = n - r$. *Moreover*

$$H^*(M; Z) \simeq \frac{Z[u]}{u^{l+2}}, \quad H^*(L; Z) \simeq \frac{Z[v]}{v^{l+1}}$$

with $\deg u = d = \deg v$.

Proof. Proposition 2 immediately yields $r/n = l/(l + 1)$. Since $r > 0$ necessarily $l \geq 1$. For this l clearly $r = ld$ and $n = (l + 1)d$ with $d = n - r$. Also

$$H_i(L) \simeq H_i(M - A); \; H^i(M - A) \simeq H^i(L) \quad \text{for all } i$$

and because M is orientable for $i \leq n - 1$ we have

$$H_i(M - A) \simeq H_i(M) \quad \text{and} \quad H^i(M) \simeq H^i(M - A).$$

Combining these we have

$$H_i(L) \simeq H_i(M) \text{ and } H^i(M) \simeq H^i(L) \quad \text{for } i \leq n - 1. \tag{1}$$

From $H^i(M) \simeq H^i(L) = 0$ for $r + 1 \leq i < n$, $H^r(M) \simeq H^r(L) \simeq Z$ and Poincaré duality we immediately get $H_j(M) = 0$ for $1 \leq j < d$, $H_d(M) \simeq Z$. This in turn gives $H_j(L) = 0$ for $1 \leq j < d$ and $H_d(L) \simeq Z$. Now Poincaré duality for L gives $H^i(L) = 0$ for $r - d < i \leq r - 1$, $H^{r-d}(L) \simeq Z$. From this and (1) we get $H^i(M) = 0$ for $r - d < i \leq r - 1$, $H^{r-d}(M) \simeq Z$. Again Poincaré duality for M yields $H_j(M) = 0$ for $d < j < 2d$ and $H_{2d}(M) \simeq Z$. Proceeding thus we see that

$$\begin{aligned} H_{jd}(M) &\simeq Z \text{ for } 0 \leq j \leq l + 1, & H_q(M) &= 0 \text{ for all other } q, \\ H_{jd}(L) &\simeq Z \text{ for } 0 \leq j \leq l, & H_q(L) &= 0 \text{ for all other } q. \end{aligned} \tag{2}$$

From (2) we immediately see that

$$\begin{aligned} H^{jd}(M) &\simeq Z \text{ for } 0 \leq j \leq l + 1, & H^q(M) &= 0 \text{ for all other } q, \\ H^{jd}(L) &\simeq Z \text{ for } 0 \leq j \leq l, & H^q(M) &= 0 \text{ for all other } q. \end{aligned} \tag{3}$$

Let ν denote the normal bundle of L in M and $D(\nu)$ a normal disk bundle of L in M. Let $p_\nu: D(\nu) \to L$ be the projection, $s: L \to D(\nu)$ the zero cross-section, $i: D(\nu) \to M$, $j: L \to M$, $k: M \to (M, M - L)$, and $\mu: D(\nu) \to (D(\nu), D(\nu) - L)$ the respective inclusions. Let $\Phi: H^q(L) \to H^{q+d}(D(\nu), D(\nu) - L)$ denote the Thom isomorphism. Since $M - L$ is contractible the map

$$k^*: H^*(M, M - L) \to H^*(M)$$

is an isomorphism. In

$$H^0(L) \xrightarrow[\simeq]{\Phi} H^d(D(\nu), D(\nu) - L) \xleftarrow{i^*} H^d(M, M - L) \xrightarrow{k^*} H^d(M),$$

Φ, i^*, and k^* are all isomorphisms. Hence $U = k^* i^{*-1} \Phi(1)$ is a generator for

$H^d(M) \simeq Z$. Since $j^*: H^q(M) \to H^q(L)$ is an isomorphism for $q \leq n - 1$ (by (1)) we see that $V = j^*(U) \in H^d(L)$ is a generator for $H^d(L) \simeq Z$.

Now for any integer $q \geq 0$ we have

$$\Phi(V^q) = p_v^*(V^q) \cup \Phi(1) = p_v^* \circ j^*(U^q) \cup \Phi(1) = p_v^* \circ s^* \circ i^*(U^q) \cup \Phi(1)$$

since $j = i \circ s$. From the fact that $p_v: D(v) \to L$ and $s: L \to D(v)$ are homotopy inverses of one another we get $p_v^* \circ s^* = \mathrm{Id}_{H^*(D(v))}$. Hence

$$\Phi(V^q) = i^*(U^q) \cup \Phi(1).$$

From the commutativity of

$$\begin{tikzcd} H^d(D(v), D(v) - L) \arrow[r, "\mu^*"] & H^d(D(v)) \\ H^d(M, M - L) \arrow[u, "i^*", "\wr"'] \arrow[r, "k^*", "\simeq"'] & H^d(M) \arrow[u, "i^*"] \end{tikzcd}$$

we get $\mu^*(\Phi(1)) = i^*k^*i^{*-1}(\Phi(1)) = i^*(U)$. Hence $i^*(U^q) = \mu^*(\Phi(1)^q)$ and this yields $\Phi(V^q) = \mu^*(\Phi(1)^q) \cup \Phi(1)$. The commutativity of

$$\begin{tikzcd} H^{qd}(D(v), D(v) - L) \otimes H^d(D(v), D(v) - L) \arrow[d, "\mu^* \otimes \mathrm{Id}"] \arrow[dr, "\text{cup}"] & \\ H^{qd}(D(v)) \otimes H^d(D(v), D(v) - L) \arrow[r, "\text{cup}"'] & H^{qd+d}(D(v), D(v) - L) \end{tikzcd}$$

now yields $\Phi(V^q) = \mu^*(\Phi(1)^q) \cup \Phi(1) = \Phi(1)^{q+1}$ and this in turn yields

$$k^* \circ i^{*-1}\Phi(V^q) = k^* \circ i^{*-1}(\Phi(1)^{q+1}) = U^{q+1}. \tag{4}$$

From the fact V is a generator of $H^d(L) \simeq Z$ it now follows that U^2 is a generator of $H^{2d}(M)$ and hence $H^{2d}(L)$ has $j^*(U^2) = V^2$ as a generator. Iteration of this argument proves that U^q is a generator of $H^{qd}(M) \simeq Z$ for $0 \leq q \leq (l + 1)$ and that V^q is a generator of $H^{qd}(L) \simeq Z$ for $0 \leq q \leq l$.

Hence

$$H^*(M; Z) \simeq \frac{Z[u]}{u^{l+2}} \quad \text{and} \quad H^*(L; Z) \simeq \frac{Z[v]}{v^{l+1}}$$

with $\deg u = d = \deg v$.

2. The orientable case with n odd

THEOREM 4. *Suppose $M^n = [L^r, A]$ (mod $\mathscr{C}^+$) and n is odd. Then $r = 0$ and hence M^n is a homotopy sphere.*

Proof. If $r > 0$, by Proposition 3 we have $r = ld$, $n = (l + 1)d$ and

$$H^*(M; Z) \simeq \frac{Z[u]}{u^{l+2}}$$

for some $l \geq 1$, where $\deg u = d$. Since n is odd we have d odd and $u^2 \in H^{2d}(M) \simeq Z$ is an element of order 2. This implies $u^2 = 0$ contradicting

$$H^*(M) \simeq \frac{Z[u]}{u^{l+2}}$$

because $l + 2 \geq 3$.

This shows that $r = 0$ and hence M^n is a homotopy sphere.

3. The orientable case when $r > 0$

Proposition 5 below is valid in the nonorientable case also.

PROPOSITION 5. *If $\tilde{M^n} = [L^r, A] \pmod{\mathscr{C}}$, $n - r \geq 2$ and $n > 5$ then M^n is homeomorphic to the Thom space $T(\nu)$ of the normal bundle ν of L^r in M^n.*

Proof. Any map $f: S^1 \to M^n$ is homotopic to a map g transversal to L^r. If $n - r \geq 2$, $g(S^1) \subset M^n - L^r$. Since $M^n - L^r$ is contractible to A it follows that g is homotopically trivial. Hence $\pi_1(M) = 0$. Now, $\pi_1(L) \simeq \pi_1(M - A) \simeq \pi_1(M) = 0$. Let W^n be got by removing an open ball around A. Then L^r is a deformation retract of W^n, $\partial W = S^{n-1}$ is 1-connected and $\pi_1(L) = 0$. By Smale's Theorem W^n is a disk bundle over L. Clearly M is homeomorphic to the space got by collapsing the boundary S^{n-1} of W to a point. Uniqueness of tubular neighborhood implies that W is diffeomorphic ($\mathscr{C}$-equivalent) to the normal disk bundle $D(\nu)$ under an equivalence fixing L. It follows that M is homeomorphic to $T(\nu)$.

PROPOSITION 6. *Suppose $M^n = [L^r, A] \pmod{\mathscr{C}^+}$ and $r > 0$. Then $n - r = 2, 4,$ or 8.*

Proof. If $n - r = d$ is odd, then $r = ld$, $n = (l + 1)d$, $l \geq 1$, and

$$H^*(M) \simeq \frac{Z[u]}{u^{l+2}}$$

with $\deg u = d$ lead to a contradiction as in the proof of Theorem 4. It follows that d is even and since $r \leq n - 1$ we get $d \geq 2$. Also it follows that both n and r are even in this case.

Suppose $d \neq 2, 4, 8$. By Proposition 5, M^n is homeomorphic to the Thom space of the normal bundle of L in M and hence M is $n - r - 1 \geq 5$ connected. It follows that $M^n - A$ and hence L^r is at least 5-connected. Let $D(\nu)$ be a normal disk bundle of L in M and $\dot{D}(\nu)$ its boundary. From the fact that $\dot{D}(\nu)$ is an S^{d-1} bundle over L we see that $\dot{D}(\nu)$ is at least 4-connected. Now $M^n -$ Int $D(\nu)$ is contractible to the point at ∞ of the Thom space and $D(\nu)$ is the boundary of $M^n -$ Int $D(\nu)$. Since $\dot{D}(\nu)$ is simply connected it follows that $\dot{D}(\nu)$ is a homotopy sphere.

Now L is 1-connected and $H_j(L) = 0$ for $1 \leq j < d$, $H_d(L) \simeq Z$. By the Hurewicz Theorem $\pi_j(L) = 0$ for $j < d$, $\pi_d(L) \simeq H(L) \simeq Z$. Let $f\colon S^d \to L$ be a map representing the generator of $\pi_d(L) \simeq Z$ and let

$$\begin{tikzcd} E \arrow[r] \arrow[d, "h"'] & \dot{D}(v) \arrow[d, "p_v"] \\ S^d \arrow[r, "f"] & L \end{tikzcd}$$

be the pull-back. Then E is an S^{d-1} fibration over S^d. We will prove that E is a homotopy sphere.

From the Gysin exact sequence of the fibration $h\colon E \to S^d$ it is immediate that $H_j(E) = 0$ for $j \neq 0, d - 1, d$, and $2d - 1$. Moreover $H_{2d-1}(E) \simeq Z \simeq H_0(E)$ and E is 1-connected (as is clear from the homotopy exact sequence of h).

Consider the commutative diagram

$0 = H\ (S^d)$

$$\begin{tikzcd} & & & & & 0 \arrow[d, phantom, "\wr"] \\ {} \arrow[r] & H_d(E) \arrow[r] & H_d(S^d) \arrow[r] & H_0(S^d) \arrow[r] \arrow[d, "f_*"', "\wr"] & H_{d-1}(E) \arrow[r] \arrow[d, "f_*", "\wr"'] & H_{d-1}(S^d) \\ & H_d(\dot{D}(v)) \arrow[r] & H_d(L) \arrow[r] & H_0(L) \arrow[r] & H_{d-1}(\dot{D}(v)) \arrow[r] & {} \end{tikzcd}$$

arising from the Gysin sequence of h and p_v. Since $\dot{D}(v)$ is a homotopy $n - 1 = (l + 1)d - 1$ sphere it follows that $H_d(\dot{D}(v)) = 0 = H_{d-1}(\dot{D}(v))$. Hence $H_d(L) \simeq H_0(L)$. Since $f_*\colon H_d(S^d) \simeq H_d(L) \simeq Z$ by choice of f and

$$f_*\colon H_0(S^d) \simeq H_0(L) \simeq Z$$

it follows that $H_d(S^d) \to H_0(S^d)$ in the first sequence is an isomorphism. It follows immediately that $H_d(E) = 0 = H_{d-1}(E)$. Hence $h\colon E \to S^d$ is a fibration with E a homotopy S^{2d-1} and fiber an S^{d-1}. Since $d \neq 2, 4, 8$ this leads to a contradiction.

4. The case when at least one of M^n or L^r is nonorientable

THEOREM 7. *Suppose $M^n = [L^r, A]$ (mod $\mathscr{C}$) and at least one of M^n or L^r is nonorientable. Then $r = n - 1$, $M^n \sim RP^n$ and $L^{n-1} \sim RP^{n-1}$.*

Proof. Observe that $r > 0$ because if $r = 0$ we get $M^n \sim S^n$ in which case both M^n and L^r will be orientable. Let $d = n - r$ and suppose $d \neq 1$. Then

$$H_{n-1}(L) \simeq H_{n-1}(M - A) = 0.$$

The exactness of $H_{n-1}(M - A) \to H_{n-1}(M) \to H_{n-1}(M, M - A) = 0$ gives $H_{n-1}(M) = 0$. If X^s is a closed nonorientable manifold of dimension s then $H_{s-1}(X^s) \neq 0$ and $H_{s-1}(X^s; Z_2) \neq 0$. From $H_{n-1}(M) = 0$ we conclude that M

is orientable and that L is nonorientable. Since $n - 2 \geq r$ and $H_j(L^r) = 0$ for $j \geq r$ we get

$$0 = H_{n-2}(L) \simeq H_{n-2}(M - A) \simeq H_{n-2}(M).$$

Hence $H^{n-1}(M) = \text{Hom}\,(H_{n-1}(M);\, Z) \oplus \text{Ext}\,(H_{n-2}(M);\, Z) = 0$. By Poincaré duality for the orientable manifold M we get $H_1(M) = 0$. This yields

$$H^1(L; Z_2) \simeq H^1(M - A; Z_2) \simeq H^1(M; Z_2) = 0.$$

Poincaré duality for L now yields $H_{r-1}(L; Z_2) = 0$, a contradiction since L^r is nonorientable. Thus $r = n - 1$.

Case (i). Suppose L^{n-1} is nonorientable. Let $\beta\colon W^{n-1} \to L^{n-1}$ be the orientable double cover of L^{n-1}. Then $\pi_1(L^{n-1}) \simeq \pi_1(M^n - A) \simeq \pi_1(M^n)$ (since $r > 0$ implies $n \geq 2$). Denote the inclusion of L in M by j. Then the isomorphism $j_*\colon \pi_1(L) \simeq \pi_1(M)$ carries the subgroup $\pi_1(W)$ of $\pi_1(L)$ onto a subgroup G of index 2 in $\pi_1(M)$. Let $\alpha\colon \tilde{M} \to M$ be the covering corresponding to the subgroup G of $\pi_1(M)$. Then $\beta\colon W \to L$ is the pull-back of $\alpha\colon \tilde{M} \to M$ by means of the inclusion $j\colon L \to M$. Since $M^n - L^{n-1}$ is contractible it follows that $\alpha\colon \tilde{M} \to M$ restricted to $M^n - L^{n-1}$ consists of two disjoint copies each homeomorphic to $M - L$. In other words $\tilde{M} - W$ has 2 components each of which is contractible. An immediate application of the Van Kampen Theorem shows that $\tilde{M}$ is simply connected. Thus $G = \{1\}$ from which it follows immediately that $\pi_1(W) = 1$. Thus W is 1-connected and bounds a compact contractible manifold (the closure of one of the components of $M - \tilde{W}$). It follows immediately that $W \sim S^{n-1}$ and that $\tilde{M} \sim S^n$. These in turn yield $L \sim RP^{n-1}$ and $M \sim RP^n$.

Case (ii). Suppose M^n is not orientable. Let $g\colon \tilde{M} \to M$ be the orientable double covering of M and let $h\colon W \to L$ be the pull-back of $g\colon \tilde{M} \to M$ by means of j. $\tilde{M} - W$ is a double covering of $M - L$ and hence has two components each homeomorphic to $M - L$. The rest of the argument is the same as in Case (i).

COROLLARY 8. *Suppose n is odd, $M^n = [L^r, A]$ (mod $\mathscr{C}$) and M^n not a homotopy sphere. Then $r = n - 1$, $M^n \sim RP^n$ and $L^{n-1} \sim RP^{n-1}$.*

Proof. Immediate consequence of Theorems 4 and 7.

LEMMA 9. *Suppose X is a* 1-*connected* CW *complex of dimension* $2n$ *satisfying*

$$H^*(X) \simeq \frac{Z[u]}{u^{n+1}}$$

with $\deg u = 2$. *Then $X \sim CP^n$.*

Proof. Since CP^∞ is a $K(Z, 2)$ space there exists a map $\phi\colon X \to CP^\infty$ such that $\phi^*(i) = u$ where $i \in H^2(K(Z, 2); Z)$ is a characteristic element. By the cellular approximation theorem there exists a map $f\colon X \to CP^\infty$ with $f \sim \phi$ and

$f(X) \subset CP^n$. Then $f\colon X \to CP^n$ induces isomorphisms in cohomology and the spaces involved are 1-connected. By J. H. C. Whitehead's theorem f is a homotopy equivalence.

Proof of Theorem 1. It is an immediate consequence of Propositions 3 and 8, Theorems 4 and 9, and Lemma 11.

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