ALGEBRAIC EQUIVALENTS OF FLOW DISJOINTNESS

BY

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We discuss the implications of the techniques introduced in [2] for the theory of disjointness [3] of minimal transformation groups (called flows here).

We were motivated by the question: (i) given flows X and Y with no common factor, under what conditions are they disjoint? (I.e., when is $X \times Y$ minimal?) Since our techniques are algebraic in character, the question must be stated in terms of algebras rather than flows. This introduces the possibility of confusion since algebras correspond to pointed flows not to flows. Thus suppose that Z is a common factor of the flows X and Y. This means that there are epimorphisms $\phi: X \to Z$ and $\psi: Y \to Z$. To translate this into the language of algebras we pick base points $x_0 \in X$ and $y_0 \in Y$ with $x_0 u = x_0$ and $y_0 u = y_0$ (see Section 1 for details). This allows us to correspond algebras $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ to the pointed flows (X, x_0), (Y, y_0), $(Z, \phi(x_0))$ and $(Z, \psi(y_0))$ respectively. The existence of the homomorphisms ϕ and ψ implies that $\mathscr{A} \supset \mathscr{C}$ and $\mathscr{B} \supset \mathscr{D}$. In general $\mathscr{C} \neq \mathscr{D}$. However, the theory guarantees that $\mathscr{D} = \mathscr{C}\alpha$ for some $\alpha \in G$.

The above discussion shows that the proper translation of (i) is (ii): given algebras $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{A} \cap \mathscr{B}\alpha = \mathbf{C}$ (the constants) ($\alpha \in G$), under what conditions are they disjoint? (I.e., when is $|\mathscr{A}| \times |\mathscr{B}|$ minimal?)

When stated in this way the original question naturally gives rise to (iii): given algebras $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{A} \cap \mathscr{B} = \mathbf{C}$, under what conditions are they disjoint?

The hypothesis of (iii) is a priori weaker than that of (ii) (whence the theorems obtained in answer to (iii) are stronger). However, it is easy to see that problems (ii) and (iii) coincide when one of the algebras involved is regular.

The results obtained in this paper are answers to (iii). Since these results are stronger than the ones occurring in the literature apropos (i), these latter are also true under the weaker hypotheses of (iii).

The conjecture is that no common factor implies disjointness when T is abelian, i.e., no conditions need be added when T is abelian. (If T is not abelian then it is known that no common factor does not imply disjointness [7].) Along these lines we show that if T is abelian, $\mathscr{A} \cap \mathscr{B} = \mathbf{C}$ and $AB \supset G_\infty$, then $\mathscr{A}$ and $\mathscr{B}$ are disjoint. This is the strongest result known so far.

The above result gives rise to some subsidiary questions: (1) If $\mathscr{A} \cap \mathscr{B} = \mathbf{C}$ will $\mathfrak{A}(A) \cap \mathfrak{A}(B)$ equal $\mathbf{C}$? (2) When is AB a group? (3) Does AB a group imply that the answer to (1) is affirmative?

In Section 1 we review the definitions and notation used in the paper. Section

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2 contains results about the groups $\mathfrak{G}(\mathscr{A})$ which will be used to obtain results on disjointness. In Section 3 we present some conditions equivalent to disjointness. Sections 2 and 3 are independent of each other. In Section 4 we use results of the previous two sections to describe some classes of flows for which $\mathscr{A} \cap \mathscr{B} = \mathbf{C}$ implies disjointness.

1. Definitions

We will assume the notations and conventions of [2, Section 1], briefly reviewing the most important definitions as they arise.

Two flows $\mathscr{A}$, $\mathscr{B}$ are *disjoint* (written $\mathscr{A} \perp \mathscr{B}$) if $|\mathscr{A}| \times |\mathscr{B}|$ is minimal. In considering the concept of disjointness it will be useful to relativize it and to consider: $\mathscr{A}$ and $\mathscr{B}$ are disjoint over $\mathscr{F}$ (written $\mathscr{A} \perp^{\mathscr{F}} \mathscr{B}$) if $\mathscr{A} \cap \mathscr{B} = \mathscr{F}$ and given $x \in |\mathscr{A}|$, $y \in |\mathscr{B}|$ with $x \mid \mathscr{F} = y \mid \mathscr{F}$, there exists $z \in |\mathscr{A} \vee \mathscr{B}|$ with $z \mid \mathscr{A} = x$ and $z \mid \mathscr{B} = y$.

If $\mathbf{C}$ denotes the constant functions, then clearly $\mathscr{A} \perp^{\mathbf{C}} \mathscr{B}$ iff $\mathscr{A} \perp \mathscr{B}$. It is also clear that $\mathscr{A} \perp^{\mathscr{F}} \mathscr{B}$ iff given $x \in |\mathscr{A}|$ and $y \in |\mathscr{B}|$ with $x \mid \mathscr{F} = y \mid \mathscr{F}$, there is a net $r_n \in \beta T$ with $r_n \mid \mathscr{A} \to x \mid \mathscr{A}$ and $r_n \mid \mathscr{B} \to y \mid \mathscr{B}$. It is this latter condition which we will use.

The symbols $\mathscr{A}$, $\mathscr{B}$, $\mathscr{F}$ etc. refer to T-subalgebras of some fixed $\mathfrak{A}(u)$ and A, B, F etc. refer to their groups $\mathfrak{G}(\mathscr{A})$, $\mathfrak{G}(\mathscr{B})$, etc.

2. When does $AB = F$?

We shall see that what is involved in proving disjointness of algebras $\mathscr{A}$ and $\mathscr{B}$ (or more generally disjointness over the intersection $\mathscr{F}$) is showing that $AB = G$ ($AB = F$). This in turn involves the groups $H(F, \mathscr{A})$.

2.1 PROPOSITION. *Let $AB \supset F$. Then*

$$AH(B, \tau) \supset H(F, \tau) \quad \text{and} \quad H(A, \tau)B \supset H(F, \tau).$$

Proof. The inclusion $AH(B, \tau) \supset H(F, \tau)$ is proved in [2, 3.12]. (Note that in that proposition, the assumption $F \supset AB$ is not used in the proof that $AH(B, \tau) \supset H(F, \tau)$.) The second inclusion follows from the first by taking inverses.

As in [2, 7.2 and 7.7.5] we define $F_1 = H(F, \tau)$, $F_{\alpha+1} = H(F_\alpha, \tau)$ for ordinals α, and $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$ for limit ordinals α. We set $F_\infty = F_\nu$ where ν is the first ordinal such that $F_\nu = F_{\nu+1}$. Then $\mathfrak{A}(F_\infty)$ is the largest PI extension of $\mathscr{F}$. We will often use the fact [2, 3.9] that $\bigcap_\alpha BF_\alpha = BF_\infty$ for any τ-closed subset B.

2.2 COROLLARY. *Let $AB \supset F_\infty$. Then $A_\infty B_\infty \supset F_\infty$, Thus if $F \supset AB \supset F_\infty$, we have $A_\infty B_\infty = F_\infty$.*

2.3 PROPOSITION. *Let $F = ABH(F, \tau)$. Then $ABF_\infty = F$. Thus if in addition $AB \supset F_\infty$, then $AB = F$.*

Proof. Set $F_1 = H(F, \tau)$ as above. It is proved in [1, 14.6] that F_1 is normal in F. Since $B \subseteq F$, BF_1 is a subgroup of G. Thus we may apply 2.1 with BF_1 in place of both AB and F, then left multiply by A, to get

$$ABH(F_1, \tau) \supset AH(BF_1, \tau).$$

Thus we have

$$ABF_2 = ABH(F_1, \tau) \supset AH(BF_1, \tau) \supset H(F, \tau) = F_1,$$

where the last inclusion is obtained by applying 2.1 with BF_1 in place of B.

It follows from a slight generalization of [2, 3.13] that F_2 is normal in F, so BF_2 is a group, and equals BF_2B; hence,

$$ABF_2 = ABF_2B \supset AF_1B = ABF_1 = F.$$

By transfinite induction we obtain $ABF_\alpha = F$ for all α, so $ABF_\infty = F$.

2.4 PROPOSITION. *Let $BF_1 = F = AF_1$ and $BA \supset F_\infty$. Then $(B \cap A)F_1 = F$. The same is true for F_α (α any ordinal) instead of F_1.*

Proof. Notice that since $BF_1 = F$, $BF_2 \supset F_1$ by 2.2 and so $BF_2 \supset BF_1 = F$. Hence by transfinite induction $BF_\infty = BF_\alpha = F$ for all α. Now $BA \supset F_\infty$ implies that $BA \supset BF_\infty = F$. Thus the assumption $BA \supset F_\infty$ is only apparently weaker than $BA \supset F$.

By 2.1, $BA_1 \supset F_1$ whence $BA_1 \supset BF_1 = F$. Hence by induction $BA_\infty = F$. Since $A \subset F$, we conclude $A \subset BA_\infty$.

Let $a \in A$. Then $a = ba_\infty$ with $b \in B$ and $a_\infty \in A_\infty \subset A$. Hence $b = aa_\infty^{-1} \in A$. Thus $b \in B \cap A$ and so $A \subset (B \cap A)A_\infty$. Now $A \subseteq F$ implies $A_\infty \subset F_\infty \subset F_1$, hence $A_\infty F_1 = F_1$ and $(B \cap A)A_\infty F_1 = (B \cap A)F_1$. Finally we get

$$F = AF_1 \subset (B \cap A)A_\infty F_1 = (B \cap A)F_1.$$

The reverse inclusion is clear.

The algebra $\mathscr{K}$ is defined by

$$\mathscr{K} = \{f \in \mathfrak{A}(u) : ft \in \mathfrak{A}(u)\ (t \in T)\}.$$

Notice that any point distal flow is in $\mathscr{K}$, and that $\mathscr{A} \subseteq \mathscr{K}$ iff $G \mid \mathscr{A}$ is a T-invariant set. Thus if $\mathscr{A} \subseteq \mathscr{K}$, $G \mid \mathscr{A}$ is dense in $|\mathscr{A}|$.

We denote by $\mathscr{E}$ the algebra of equicontinuous functions.

2.5 LEMMA. *If $\mathscr{A} \subseteq \mathscr{K}$ then $AE = AH(G, \tau)$.*

Proof. By [1, 15.13], $E = H(G, \mathscr{K})$. By [2, 3.10], $E = KH(G, \tau)$. Hence if $\mathscr{A} \subseteq \mathscr{K}$ then $A \supseteq K$ so $AE = AKH(G, \tau) = AH(G, \tau)$.

3. Disjointness and the group of a flow

The first theorem relating groups to disjointness was [5] that if T is abelian, then $\mathscr{A} \perp \mathscr{B}$ iff $AB = G$. The following sequence of results generalizes both that theorem and [4, Theorem 2.3].

For any τ-closed A, the algebra $\mathfrak{A}(A, \mathscr{B})$ is [2, Section 5] the norm-closed subalgebra of $\mathfrak{A}(u)$ generated by the functions $\{f^A : f \in \mathscr{B}\}$. The flow $|\mathfrak{A}(A, \mathscr{B})|$ is isomorphic to

$$\{(A \mid \mathscr{B}) \circ p : p \in M\},$$

which is a subflow of $2^{|\mathscr{B}|}$. For a full discussion of this flow and the circle operation see [2, Section 5]. The following properties of this operation will be used extensively in this section: $(A \mid \mathscr{B}) \circ p = (A \circ p) \mid \mathscr{B}$ and $(A \mid \mathscr{B}) \circ pr = ((A \mid \mathscr{B}) \circ p) \circ r$ for all subsets A of G, subalgebras $\mathscr{B}$ of $\mathfrak{A}(u)$ elements p, r of M.

3.1 THEOREM. *For arbitrary τ-closed subgroups A and T-subalgebras $\mathscr{B}$, $\mathfrak{G}(\mathfrak{A}(A, \mathscr{B}))$ is the largest τ-closed subgroup F satisfying $A \subseteq F \subseteq BA$.*

Proof. First we show $A \subseteq \mathfrak{G}(\mathfrak{A}(A, \mathscr{B})) \subseteq BA$. Since $A \circ \alpha = A \circ u$ $(\alpha \in A)$, $A \subset \mathfrak{G}(\mathfrak{A}(A, B))$. If $\gamma \in \mathfrak{G}(\mathfrak{A}(A, \mathscr{B}))$ then $(A \circ \gamma) \mid \mathscr{B} = (A \circ u) \mid \mathscr{B}$ and since $\gamma \in A \circ \gamma$, we know

$$\gamma \mid \mathscr{B} \in (A \circ \gamma) \mid \mathscr{B} = (A \circ u) \mid \mathscr{B} = (A \mid \mathscr{B}) \circ u.$$

Then by [2, 2.4], $\gamma \in \operatorname{cls}_{\tau(\mathscr{B})} A$. Now by [2, 3.2], $\operatorname{cls}_{\tau(\mathscr{B})} A = B \operatorname{cls}_\tau A = BA$, hence $\gamma \in BA$.

Now suppose $A \subseteq F \subseteq BA$ for some τ-closed subgroup F. We will show that $F \subseteq \mathfrak{G}(\mathfrak{A}(A, \mathscr{B}))$. First note that since $A \subseteq F$ and $BF = BA$, we have

$$\begin{aligned}(A \circ u) \mid \mathscr{B} \subset (F \circ u) \mid \mathscr{B} &= (F \mid \mathscr{B}) \circ u = (BF \mid \mathscr{B}) \circ u = (BA \circ u) \mid \mathscr{B} \\ &= (A \circ u) \mid \mathscr{B},\end{aligned}$$

so $(A \circ u) \mid \mathscr{B} = (F \circ u) \mid \mathscr{B}$. Now if $\gamma \in F$ then $F \circ \gamma = F \circ u$, thus

$$\begin{aligned}(A \circ \gamma) \mid \mathscr{B} &= (A \circ u \circ \gamma) \mid \mathscr{B} = (F \circ u \circ \gamma) \mid \mathscr{B} = (F \circ \gamma) \mid \mathscr{B} \\ &= (F \circ u) \mid \mathscr{B} = (A \circ u) \mid \mathscr{B}.\end{aligned}$$

This shows $F \subseteq \mathfrak{G}(\mathfrak{A}(A, \mathscr{B}))$.

3.2 COROLLARY. *If $\mathscr{A}$, $\mathscr{B}$ are given algebras, then $BA \supseteq G_\infty$ iff $\mathfrak{A}(A, B)$ is a $\mathscr{P}\mathscr{I}$ flow.*

Proof. If $\mathfrak{A}(A, \mathscr{B})$ is a $\mathscr{P}\mathscr{I}$ flow, then $\mathfrak{G}(\mathfrak{A}(A, \mathscr{B})) \supseteq G_\infty$, so $BA \supseteq G_\infty$.

Suppose $BA \supseteq G_\infty$. Then $BA \supseteq G_\infty A$. Since G_∞ is normal in G, $G_\infty A$ is a group. Thus 3.1 implies $\mathfrak{G}(\mathfrak{A}(A, \mathscr{B})) \supseteq G_\infty A$. Hence $\mathfrak{G}(\mathfrak{A}(A, \mathscr{B})) \supseteq G_\infty$ and $\mathfrak{A}(A, \mathscr{B})$ is $\mathscr{P}\mathscr{I}$.

The next lemma is proved in [1, 18.4].

3.3 LEMMA. *If $\mathscr{A}$ and $\mathscr{B}$ are disjoint over $\mathscr{F}$, then $AB = F$.*

3.4 THEOREM. *Suppose $\mathscr{A} \cap \mathscr{B} = \mathscr{F}$ and $\mathscr{F} = \mathfrak{A}(F, \mathscr{A})$. Then $\mathscr{A}$ is disjoint from $\mathscr{B}$ over $\mathscr{F}$ iff $AB = F$.*

Proof. Suppose $AB = F$ and let $p \mid \mathscr{F} = q \mid \mathscr{F}$. Since $\mathscr{F} = \mathscr{A}(F, \mathscr{A})$ we have $p \in (F \mid \mathscr{A}) \circ q$, hence $p \in (AB \mid \mathscr{A}) \circ q = (B \mid \mathscr{A}) \circ q$. This implies the existence of nets $t_n \in T$ and $\beta_n \in B$ such that $t_n \to q$ and $\beta_n t_n \mid \mathscr{A} \to p \mid \mathscr{A}$. Since $\beta_n \mid \mathscr{B} = u \mid \mathscr{B}$, we also have

$$\beta_n t_n \mid \mathscr{B} = ut_n \mid \mathscr{B} \to uq \mid \mathscr{B} = q \mid \mathscr{B}.$$

This proves disjointness.

Remark. The condition that $\mathscr{F} = \mathfrak{A}(F, \mathscr{A})$ is equivalent to the condition that $\mathscr{A} \supseteq \mathscr{F}$ be a RIC extension [2, 5.10].

3.5 LEMMA. *Suppose A and B are τ-closed subgroups of G. Then*

$$\mathfrak{A}(A) \cap \mathfrak{A}(B) = \mathfrak{A}(F),$$

where F is the τ-closed subgroup generated by A and B.

Proof. Define $\mathscr{F} = \mathfrak{A}(A) \cap \mathfrak{A}(B)$ and $F = \mathfrak{G}(\mathscr{F})$. Then since $\mathscr{F} \subseteq \mathfrak{A}(A)$, we have $A \subseteq F$, and so $\mathfrak{A}(F) \subseteq \mathfrak{A}(A)$. Similarly $\mathfrak{A}(F) \subseteq \mathfrak{A}(B)$, so

$$\mathscr{F} \subseteq \mathfrak{A}(F) \subseteq \mathfrak{A}(A) \cap \mathfrak{A}(B),$$

and equality holds.

Now let H denote the τ-closed subgroup generated by A and B. Then

$$\mathfrak{A}(H) \subseteq \mathfrak{A}(A) \cap \mathfrak{A}(B) = \mathfrak{A}(F)$$

so $F \subseteq H$. Also

$$F = \mathfrak{G}(\mathfrak{A}(A) \cap \mathfrak{A}(B)) \supseteq AB,$$

so $F \supseteq H$.

3.6 THEOREM. *Fix $\mathscr{A}$ and $\mathscr{B}$. Then the following are equivalent.*

(a) *BA is a group*
(b) $\mathfrak{G}(\mathfrak{A}(A, \mathscr{B})) = BA$
(c) *$\mathfrak{A}(A)$ and $\mathfrak{A}(B)$ are disjoint over their intersection.*

In particular when A or B is normal, $\mathfrak{A}(A)$ and $\mathfrak{A}(B)$ are disjoint over their intersection.

Proof. (a) is equivalent to (b) by 3.1.

(b) $\Rightarrow$ (c). If (b) holds then BA is a group, whence by 3.5 we have

$$\mathfrak{A}(A) \cap \mathfrak{A}(B) = \mathfrak{A}(BA).$$

By [2, 5.5.3] and 3.1 $\mathfrak{A}(BA) = \mathfrak{A}(BA, \mathfrak{A}(B))$, so we can apply 3.4 with $\mathfrak{A}(A)$, $\mathfrak{A}(B)$ and $\mathfrak{A}(BA)$ in place of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{F}$ to prove (c).

(c) $\Rightarrow$ (a) follows from 3.3.

3.7 THEOREM. *Among the following three conditions, the implications*

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) *always hold. If $\mathfrak{A}(G) = \mathbf{C}$ or if $\mathscr{B} \subseteq \mathscr{K}$ then the three conditions are equivalent.*

(a) $\mathfrak{A}(A, \mathscr{B}) = \mathbf{C}$
(b) $\mathscr{A} \perp \mathscr{B}$
(c) $BA = G$

Remark. $\mathscr{K}$ is defined after 2.4 above.

Proof. (a) $\Rightarrow$ (b) Since $\mathfrak{A}(A, \mathscr{B}) = \mathbf{C}$ we know that if p, $q \in M$ then $q \in (A \mid \mathscr{B}) \circ p$. Thus there are nets $\alpha_n \in A$ and $t_n \in T$ with $t_n \to p$ and $\alpha_n t_n \mid \mathscr{B} \to p \mid \mathscr{B}$. Then

$$\alpha_n t_n \mid \mathscr{A} \to p \mid \mathscr{A} \quad \text{and} \quad \alpha_n t_n \mid \mathscr{B} \to q \mid \mathscr{B}.$$

This proves $\mathscr{A} \perp \mathscr{B}$.

(b) $\Rightarrow$ (c) follows from 3.3.

(c) $\Rightarrow$ (a) If $\mathfrak{A}(G) = \mathbf{C}$, and $BA = G$, then BA is a group, so by 3.6 we have $\mathfrak{G}(\mathfrak{A}(A, \mathscr{B})) = G$, whence $\mathfrak{A}(A, \mathscr{B}) = \mathbf{C}$. On the other hand, suppose $\mathscr{B} \subseteq \mathscr{K}$. Then $BA = G$ implies $A \mid \mathscr{B} = G \mid \mathscr{B}$, and $|\mathscr{B}| = G \mid \mathscr{B}$ since $\mathscr{B} \subseteq \mathscr{K}$. Therefore

$$|\mathfrak{A}(A, \mathscr{B})| = \{(G \mid \mathscr{B}) \circ p : p \in M\} = \{|\mathscr{B}| \circ p : p \in M\},$$

and this last set contains one point. Hence $\mathfrak{A}(A, \mathscr{B}) = \mathbf{C}$.

4. Flow disjointness

We will now discuss the implications of these results for flows.

In certain cases, e.g., when T is abelian, an algebra $\mathscr{A}$ is disjoint from $\mathscr{E}$ (the algebra of equicontinuous functions) iff $\mathscr{A}$ is weakly mixing, i.e., iff $|\mathscr{A}| \times |\mathscr{A}|$ is ergodic. This is the relevance of the next theorem. See for example [5], [6], and [8].

4.1 THEOREM. *If $\mathscr{A}$ and $\mathscr{B}$ are disjoint from $\mathscr{E}$, $AB \supseteq G_\infty$, and $\mathscr{A}$, $\mathscr{B} \subseteq \mathscr{K}$, then $\mathscr{A} \vee \mathscr{B}$ is disjoint from $\mathscr{E}$.*

Proof. The hypotheses imply that $AE = BE = G$. By 2.5, $AG_1 = BG_1 = G$. Now apply 2.4 with G in place of F to conclude $(A \cap B)G_1 = G$. Since $G_1 \subseteq E$, $(A \cap B)E = G$. Then by 3.7, $(\mathscr{A} \vee \mathscr{B}) \perp \mathscr{E}$.

The following theorem and its corollaries generalize all results we know of concerning disjointness and no common factor, e.g., [6, 3.4] and [8, Theorems 12 and 14].

4.2 THEOREM. *Suppose $\mathscr{B} \subseteq \mathscr{K}$, $(\mathscr{A} \cap \mathscr{E}) \perp (\mathscr{B} \cap \mathscr{E})$ and $AB \supseteq G_\infty$. Then $\mathscr{A} \perp \mathscr{B}$.*

Proof. By [1.14.6], $\mathfrak{G}(\mathscr{A} \cap \mathscr{E}) = AE$ and $\mathfrak{G}(\mathscr{B} \cap \mathscr{E}) = BE$. So by 3.3, $AEBE = G$, hence $ABE = G$. By 2.6, $BE = BG_1$, whence $ABG_1 = G$. Now use 2.3 to conclude $AB = G$. Since $\mathscr{B} \subseteq \mathscr{K}$, we have $\mathscr{A} \perp \mathscr{B}$ by 3.7.

4.3 COROLLARY. *Suppose T is abelian and $AB \supseteq G_\infty$. Then $\mathscr{A} \perp \mathscr{B}$ iff $\mathscr{A} \cap \mathscr{B} = \mathbf{C}$.*

Proof. If $\mathscr{A} \cap \mathscr{B} = \mathbf{C}$ then $(\mathscr{A} \cap \mathscr{E}) \cap (\mathscr{B} \cap \mathscr{E}) = \mathbf{C}$, so $(\mathscr{A} \cap \mathscr{E}) \perp (\mathscr{B} \cap \mathscr{E})$ (see [1, 18.11.2]). Since T is abelian, both $\mathscr{A}$ and $\mathscr{B}$ are subalgebras of $\mathscr{K}$. Hence 4.2 can be applied, and $\mathscr{A} \perp \mathscr{B}$.

4.4 COROLLARY. *Suppose T is abelian, $\mathscr{A} \perp \mathscr{B}$, and $\mathscr{F} \supseteq \mathscr{A}$, $\mathscr{L} \supseteq \mathscr{B}$ are $\mathscr{P}\mathscr{I}$ extensions. Then $\mathscr{F} \perp \mathscr{L}$ iff $\mathscr{F} \cap \mathscr{L} = \mathbf{C}$.*

Proof. Since $\mathscr{F} \supseteq \mathscr{A}$, $\mathscr{L} \supseteq \mathscr{B}$ are $\mathscr{P}\mathscr{I}$ extensions, $F \supseteq A_\infty$ and $L \supseteq B_\infty$. Hence $FL \supseteq A_\infty B_\infty$. Since $AB = G$, $A_\infty B_\infty = G_\infty$ by 2.2. Hence $FL \supset G_\infty$. Now apply 4.3.

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