FIBER BUNDLES WITH CROSS-SECTIONS AND NONCOLLAPSING SPECTRAL SEQUENCES

BY

DANIEL HENRY GOTTLIEB[1]

If a fibration has a cross section, does its Serre spectral sequence collapse? The answer is no and the counterexample is provided by a fibration $S^2 \times S^3 \to E \to S^2$ due to G. Hirsch (see [1, p. 45]). In this note we construct a large class of counterexamples by a very simple method. These examples show that fibrations with cross sections and noncollapsing spectral sequences are not an isolated phenomena, and they provide counterexamples for numerous variations of the question. In particular there is Koszul's problem [1, p. 46]: Does a fiber bundle with *structural group a connected Lie group* with a cross section have a collapsing spectral sequence? We answer that in the negative.

Let G be any connected topological group which is not a sphere or contractible. Let $G \to H \to \Sigma G$ be the Hopf fibration. (Think of it as the fibration given by the clutching map $\mu\colon G \times G \to G$, or equivalently the principal bundle induced by the pullback of the map $\Sigma G = \Sigma\Omega B_G \to B_G$ which is the adjoint of the identity $\Omega B_G \to \Omega B_G$.)

Now let G act on ΣG by $\hat{\omega}(g, \langle h, t\rangle) = \langle gh, t\rangle$.

PROPOSITION. *$\Sigma G \to H \times_G \Sigma G \to \Sigma G$ has a cross section and a noncollapsing spectral sequence.*

Proof. The cross section exists since the north pole is a fixed point under the action $\hat{\omega}$ of G on ΣG.

Let $E = H \times_G \Sigma G$. Consider the generalized Wang exact sequence as explained in Spanier [4, p. 455, Theorem 5],

$$\cdots \longrightarrow H^q(E) \xrightarrow{i^*} H^q(\Sigma G) \xrightarrow{\delta\hat{\omega}^*} H^{q+1}((CG, G) \times (\Sigma G)) \longrightarrow \cdots$$

where $\delta\colon H^q(G \times \Sigma G) \to H^{q+1}((CG, G) \times (\Sigma G))$. Note that δ is an isomorphism for $q > 0$. Assume the coefficients are a field. Now the spectral sequence will not collapse if i^* is not onto, and i^* is not onto if $\delta\hat{\omega}^*$ is not trivial. Now $\delta\hat{\omega}^*$ is not trivial if

$$\hat{\omega}^*(k) = (1 \times k) + (\omega^*(k) \times 1) + \Sigma(a_i \times b_i)$$

has nonzero cross-terms. Now by arguments similar to [3, Theorem 4], we see that if $\mu^*(x) = (1 \times x) + (x \times 1) + \Sigma(a_i \times b_i)$, then

$$\hat{\omega}^*(\sigma(x)) = (1 \times \sigma(x)) + \Sigma(a_i \times \sigma(b_i)) \quad \text{where } \sigma\colon H^*(X) \to H^*(\Sigma X)$$

Received October 6, 1975.

[1] This research was partially supported by the National Science Foundation.

is the suspension homomorphism. Since G is not contractible or a sphere, there is an element $x \in H^*(G)$ such that $\mu^*(x)$ has nonzero cross-terms. Hence $\hat{\omega}^*(\sigma(x))$ has nonzero cross terms and so the spectral sequence does not collapse.

Remarks. (1) One may construct examples of fibrations with cross sections and noncollapsing spectral sequences given actions $\hat{\omega}: G \times F \to F$ such that $\omega = \hat{\omega}(\ \ , *)$ is contractible and $\hat{\omega}^*$ has nonzero cross terms. These fibrations are of the form $F \to E \to \Sigma G$.

(2) One may construct examples where the base is not a suspension. For example $\Sigma G \to E_G \times_G (\Sigma G) \to B_G$ has a cross section but the spectral sequence cannot collapse since this bundle is universal with respect to the one constructed in the proposition.

(3) We may construct an example where the fiber is a manifold and the structure group is a Lie group G. We may imbed G equivalently in D^n, the closed unit disk with equivalent closed tubular neighborhood T. Then G acts on the subset W of $D^n \times I$ given by

$$W = D^n \times I - (D^n - T) \times (1/3, 2/3).$$

Now W is homotopy equivalent to ΣG and is a manifold with boundary and ω is homotopic to a constant. By doubling W we get a closed manifold M and an action $\omega: G \times M \to M$ such that ω is homotopic to a constant. Hence, by (1), we have the required example.

Bibliography

1. Armand Borel, *Topics in the homology of fibre bundles*, Lecture Notes in Mathematics, vol. 36, Springer Verlag, Berlin, 1967.
2. Morton Curtis, Alan Wiederheld, and Bruce Williams, *Normalizers of maximal tori: Localization in group theory and homotopy theory*, Lecture Notes in Mathematics, vol. 418, Springer Verlag, Berlin, 1974, pp. 31–47.
3. Daniel Henry Gottlieb, *The evaluation map and homology*, Michigan Math. J., vol. 19 (1972), pp. 289–297.
4. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

Purdue University
Lafayette, Indiana