PEAK SETS AND SUBNORMAL OPERATORS[1]

BY

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1. Preliminaries

If T is a bounded operator on a Hilbert space $\mathfrak{H}$, its spectrum and point spectrum will be denoted by $\sigma(T)$ and $\sigma_p(T)$ respectively. An operator T on $\mathfrak{H}$ is said to be subnormal if it has a normal extension on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$. Concerning such operators, see, for example, Halmos [8]. We recall some properties. If N is the minimal normal extension of T then $\sigma(N) \subset \sigma(T)$ (P. R. Halmos). In fact, $\sigma(T)$ consists of $\sigma(N)$ together with some of the bounded components of the complement of $\sigma(N)$ (J. Bram). A subnormal T on $\mathfrak{H}$ will be called completely subnormal if there exists no nontrivial subspace of $\mathfrak{H}$ which reduces T and on which T is normal. In particular, if T is completely subnormal, its point spectrum is empty.

For a compact set X of the complex plane, let $C(X)$ denote the continuous functions on X and $R(X)$ denote the functions on X which are uniformly approximable on X by rational functions with poles off X. Recall that a compact S is a spectral set for an operator T if X contains $\sigma(T)$ and if $\|f(T)\| \leq \sup_f \{|f(z)|: z \in X\}$, where f ranges over the rational functions with poles off X. It is clear that the spectrum of a subnormal operator is a spectral set for that operator. According to a result of von Neumann [12], if X is a spectral set for any operator T and if $f \in R(X)$, then $f(T)$ is defined and $f(X)$ is a spectral set for $f(T)$. Further, if $\sigma(T)$ is a spectral set for any operator T and if $R(\sigma(T)) = C(\sigma(T))$, then T must be normal. (See also Lebow [11].) A local, generalized version of this last theorem for subnormal operators, and also for operators T for which $\sigma(T)$ is a spectral set, was given by Clancey and Putnam [3], [4].

Clearly, if T is subnormal on $\mathfrak{H}$ and if $f \in R(\sigma(T))$, then $f(T)$ is subnormal; in fact, if N is a normal extension on $\mathfrak{K}$ of T then $f(N)$ is a normal extension on $\mathfrak{K}$ of $f(T)$. It may be noted that if N is the minimal extension of T then $f(N)$ is the minimum extension of $f(T)$ provided f satisfies certain natural necessary conditions; see Olin [13], also Conway and Olin [5]. If also $R(\sigma(T)) = C(\sigma(T))$ then T is normal.

A closed subset Q of a compact set X is said to be a peak set of $R(X)$ if there exists a function (peak function) f in $R(X)$ such that $f(z) = 1$ for $z \in Q$ and $|f(z)| < 1$ for $z \in X - Q$. See Gamelin [7, p. 56]. A peak set consisting of a single point is called a peak point.

The main result (Theorem 1 of section 2) establishes a connection between peak sets of $R(\sigma(T))$ and reducing subspaces of T when T is subnormal. This

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result, and the Corollaries 1–4 (Section 3) and Theorem 2 (Section 4), imply that if T is completely subnormal and if $N = \int z\, dE_z$ is its minimal normal extension, then $E(Q) = 0$ for certain "small" Borel sets Q of the complex plane, an absolute continuity property of N. Several related open problems are noted in Section 5.

2. The main result

THEOREM 1. *Let T be a subnormal operator on $\mathfrak{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $\mathfrak{K} \supset \mathfrak{H}$. Suppose that*

(i) *Q is a proper peak set of $R(\sigma(T))$ and $E(Q) \neq 0$.*

Then $E(Q)\mathfrak{H}$ is a subspace of $\mathfrak{H}$, $\neq 0$ and $\mathfrak{H}$, which reduces T; $T \mid E(Q)\mathfrak{H}$ is subnormal with the minimal normal extension $E(Q)N$ on $E(Q)\mathfrak{K}$ (the range of $E(Q)$); and

(2.1) $$\sigma(T \mid E(Q)\mathfrak{H}) \subset Q.$$

If, in addition to (i), *it is also assumed that*

(ii) $R(Q) = C(Q)$,

then

(2.2) $$T \mid E(Q)\mathfrak{H} \text{ is normal.}$$

Proof. Let P denote the (orthogonal) projection $P\colon \mathfrak{K} \to \mathfrak{H}$. For $f \in R(\sigma(T))$ and $x \in \mathfrak{H}$, we have $f(T)x = f(N)x$ and hence $f^n(T)x = f^n(N)x$ for $n = 1, 2, \ldots$. If f is a peak function for Q then, on letting $n \to \infty$, we obtain $f^n(T)x \to E(Q)x$, so that $E(Q)\mathfrak{H} \subset \mathfrak{H}$ and $E(Q)P = PE(Q)P$. Thus, $E(Q)P = PE(Q)P$ is an orthogonal projection. Since, for $x \in \mathfrak{H}$, $TE(Q)x = \lim Tf^n(T)x = \lim f^n(T)Tx = E(Q)Tx$ (strong limits), we see that T (hence T^*) commutes with $PE(Q)P$, and hence $E(Q)\mathfrak{H}$ reduces T. This argument is similar to one used by Sz.-Nagy and Foiaş [20, pp. 253–254].

Next, we show that $E(Q)\mathfrak{H} \neq 0$. Suppose the contrary, so that $E(Q)P = 0$, and hence $0 = N^{*k}E(Q)P = E(Q)N^{*k}P$ for $k = 0, 1, 2, \ldots$. Since N is the minimal normal extension of T, $\mathfrak{K}$ is the space spanned by the vectors $\{N^{*k}x\colon x \in \mathfrak{H}, k = 0, 1, 2, \ldots\}$, and so $E(Q) = 0$, in contradiction to (i). Since Q is a proper subset of $\sigma(T)$, the relation $E(Q)\mathfrak{H} \neq \mathfrak{H}$ will follow from (2.1).

That $N_1 = E(Q)N$ on $E(Q)\mathfrak{K}$ is a normal extension of $T \mid E(Q)\mathfrak{H}$ is clear. Since $N_1^{*k} = E(Q)N^{*k}$ for $k = 0, 1, 2, \ldots$, then any reducing space of N_1 between $E(Q)\mathfrak{H}$ and $E(Q)\mathfrak{K}$ contains the space spanned by the vectors $\{E(Q)N^{*k}x\colon x \in \mathfrak{H}, k = 0, 1, 2, \ldots\}$, that is, $E(Q)\mathfrak{K}$, and hence N_1 is minimal.

In order to prove (2.1) we first show that

(2.3) $$\partial\sigma(T \mid E(Q)\mathfrak{H}) \subset Q.$$

Let $z \in \partial\sigma(T \mid E(Q)\mathfrak{H})$. Then there exists a sequence of unit vectors $x_n = E(Q)x_n \in \mathfrak{H}$ such that $(T - z)x_n \to 0$. Thus,

$$\|(T - z)z_n\| = \|(N - z)E(Q)x_n\| \geq \text{dist}\,(z, Q) \to 0,$$

so that $z \in Q$, that is, (2.3) holds.

Since, by (2.3), relation (2.1) holds if $\sigma(T \mid E(Q)\mathfrak{H})$ has no interior, it may be supposed that

$$R = \text{int}\,\sigma(T \mid E(Q)\mathfrak{H}) \neq \emptyset.$$

Clearly, $\sigma(T \mid E(Q)\mathfrak{H}) \subset \sigma(T)$ and, by (2.3), $\partial R \subset Q$. Since Q is a peak set of $R(\sigma(T))$, there exists an $f \in R(\sigma(T))$ such that $f = 1$ on Q and $|f| < 1$ on $\sigma(T) - Q$. In particular, f is analytic on R, continuous on $R \cup \partial R$, and $f = 1$ on ∂R. Hence, $f \equiv 1$ on R, so that $R \subset Q$, and (2.1) follows.

If (ii) holds, then (2.2) follows either from von Neumann's result mentioned above or from [3].

3. Peak sets

A number of consequences of Theorem 1 can be deduced by noting some special peak sets.

COROLLARY 1. *Let T be a completely subnormal contraction* ($\|T\| \leq 1$) *on $\mathfrak{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $\mathfrak{K} \supset \Omega$. If Z is any Borel set on $|z| = 1$ of arc length measure* 0, *then* $E(Z) = 0$.

Proof. Clearly, it is sufficient to prove the corollary when Z is closed and is contained in $\sigma(T)$. It was shown by F. and M. Riesz [18, pp. 36–37], using a result of Fatou, that there exists a function $f(z)$ continuous on $|z| \leq 1$, analytic on $|z| < 1$, and such that $f = 1$ on Z and $|f| < 1$ otherwise. By Mergelyan's theorem (cf. [7, p. 48]), $f(z)$ is the uniform limit on $|z| \leq 1$ of polynomials and hence, since $\sigma(T) \subset \{z: |z| \leq 1\}$, $f \in R(\sigma(T))$. Thus, Z is a peak set of $R(\sigma(T))$. Since Z has planar measure 0, it follows from the Hartogs-Rosenthal theorem (cf. [7, p. 47]) that $R(Z) = C(Z)$. Since T is completely subnormal, Theorem 1 implies that $E(Z) = 0$.

Remarks. The above corollary was proved by Olin [13] using other methods; see also Conway and Olin [5, Corollary 7.11, p. 63]. The use of the Fatou-Riesz result above is based on a similar argument in Sz.-Nagy and Foias [20, p. 253].

COROLLARY 2. *Let T be completely subnormal on $\mathfrak{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $K \supset H$. Let C be any rectifiable simple closed curve satisfying*

(3.1) *either* $\sigma(T) \subset (C \cup \text{int}\, C)$ *or* $\sigma(T) \subset (C \cup \text{ext}\, C)$.

If Z is any Borel subset of C having arc length measure 0 *then* $E(Z) = 0$.

Proof. Suppose first that $\sigma(T) \subset (C \cup \operatorname{int} C)$. In view of the Riemann mapping theorem there exists a function $w = f(z)$ which maps $C \cup \operatorname{int} C$ homeomorphically onto $|w| \leq 1$ and which is conformal in $\operatorname{int} C$; cf., e.g., Rudin [19, p. 282]. As in the proof of Corollary 1, it follows from Mergelyan's theorem that $f(z)$ is the uniform limit on $C \cup \operatorname{int} C$ of polynomials in z, so that, in particular, $f \in R(\sigma(T))$. Since T is subnormal, then, as noted in Section 1, $f(T)$ is subnormal on $\mathfrak{H}$ with $f(N)$ on $\mathfrak{K}$ as a normal extension. Further, $f(N)$ is the minimal normal extension of $f(T)$. For, if M is the minimal extension of $f(T)$ then $f(T) \subset M \subset f(N)$ and if $z = g(w)$ is the inverse mapping of $w = f(z)$, then $T = g(f(T)) \subset g(M) \subset g(f(N)) = N$. Since N is the minimal extension of T, it follows that $g(M) = N$, and hence $M = f(N)$. In this connection, see Putnam [14]. A similar argument shows that, since T is completely subnormal, so also is $f(T)$. According to a theorem of F. and M. Riesz, the function f is absolutely continuous on C, so that $W = f(Z)$ has arc length measure 0 on $|w| = 1$; cf., e.g., Tsuji [21, p. 318]. Clearly, if $f(N)$ has the spectral resolution $f(N) = \int w\, dF_W$ $(= \int f(z)\, dE_z)$, then $E(Z) = F(W)$. Thus, the assertion of Corollary 2 follows from Corollary 1 in case the first condition of (3.1) is assumed. The proof of Corollary 2 when the second condition of (3.1) is assumed can be reduced to the proof in the first case by mapping $\operatorname{ext} C$ conformally onto $\operatorname{int} C$.

For any closed set α of the complex plane, let cap (α) denote its logarithmic capacity. For this concept, see Hille [9, pp. 280–289], Tsuji [21, pp. 55 ff.], or, for a concise summary, Zalcman [22, pp. 132–136].

COROLLARY 3. *Let T be completely subnormal on $\mathfrak{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $\mathfrak{K} \supset \mathfrak{H}$. Let C be any simple closed curve (not necessarily rectifiable) satisfying* (3.1). *If Z is a closed subset of $\sigma(T) \cap C$ for which* cap $(Z) = 0$, *then $E(Z) = 0$.*

Proof. As in the proof of Corollary 2 it may be assumed that the first condition of (3.1) holds. Since C is now not necessarily rectifiable the F. and M. Riesz theorem is not applicable. However, since cap $(Z) = 0$, then also cap $(f(Z)) = 0$ (Tsuji [21, p. 347]) and hence $f(Z)$ has arc length measure 0 on $|w| = 1$. The desired result now follows from Corollary 1 by the same argument as that used at the end of the proof of Corollary 2. (Of course, if C is rectifiable, Corollary 3 is already implied by Corollary 2.)

COROLLARY 4. *Let T be completely subnormal on $\mathfrak{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $\mathfrak{K} \supset \mathfrak{H}$. Let C be a rectifiable simple closed curve having an arc length parametrization of class C^2 (or, more generally, C may consist of a finite number of C^2-smooth arcs) and satisfying*

$$\text{(3.2)} \qquad \operatorname{meas}_1 (C \cap \sigma(T)) = 0,$$

where meas_1 *refers to arc length measure on C. Then $E(C) = 0$.*

Remark. Note that in Corollary 4, the sets C, int C and ext C all may contain portions of $\sigma(T)$, unlike the situation in Corollaries 2 or 3, where it was assumed that one of the sets int C or ext C was free of points of $\sigma(T)$.

Proof. It can be supposed that int $C \cap \sigma(T) \neq \emptyset$ and ext $C \cap \sigma(T) \neq \emptyset$, since, otherwise, Corollary 4 is a special case of Corollary 2. It was shown by Lautzenheiser [10, Chapter 4, proof of Theorem 4.6], using a result of Davie and Øksendal [6], that the set $Q = (\text{ext } C)^{-} \cap \sigma(T)$ is a peak set of $R(\sigma(T))$. (See also Putnam [15, pp. 269–270] and the remarks of [16, Section 7].) Clearly, relation (i) of Theorem 1 is satisfied and so $E(Q)N$ is the minimal normal extension of $T \mid E(Q)\mathfrak{H}$. Relation (3.2) and an application of Corollary 2 to $T \mid E(Q)\mathfrak{H}$ then imply that $E(C) = 0$.

4. Peak points

It follows from Theorem 1 that if T is subnormal on $\mathfrak{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $\mathfrak{K} \supset \mathfrak{H}$ and if z is a peak point of $R(\sigma(T))$ which is not an eigenvalue of T then $E(\{z\}) = 0$. Equivalently, if Ω denotes the set of peak points of $R(\sigma(T))$, then $(\sigma_p(N) \cap \Omega) \subset \sigma_p(T)$. Since $\sigma_p(T) \subset \sigma_p(N)$, we have then the following:

THEOREM 2. *If T is a subnormal operator on $\mathfrak{H}$ with the minimal normal extension $N = \int z\, dE_z$ on $\mathfrak{K} \supset \mathfrak{H}$, then*

$$\sigma_p(T) \cap \Omega = \sigma_p(N) \cap \Omega. \tag{4.1}$$

It was recently proved by Radjabalipour [17, p. 388], that if T is subnormal then

$$(\sigma_p(T^*))^* \cap \Omega = \sigma_p(T) \cap \Omega, \tag{4.2}$$

where $(\sigma_p(T^*))^* = \{\bar{z} : z \in \sigma_p(T^*)\}$. As Radjabalipour has noted, in view of Melnikov's peak point criterion (cf., e.g., [7, p. 205], or [22, p. 45]), any boundary point, z, of a component of the complement of $\sigma(T)$ is a peak point of $R(\sigma(T))$, and hence, if T is completely subnormal, relation (4.2) implies that $\bar{z}$ cannot belong to $\sigma_p(T^*)$. It follows in a similar way from (4.1) of Theorem 2 that if T is completely subnormal then $E(\{z\}) = 0$. This result was obtained by a different method in Olin [13], and also in Conway and Olin [5, Corollary 7.10, p. 63]. Incidentally, it is possible for T to be completely subnormal even though $\sigma_p(T^*)$ is empty; such an example was given by Clancey and Morrel [2] using a result of Brennan [1].

5. Remarks

We do not know whether Corollary 4 remains true if C is not assumed to be smooth but is supposed only to be rectifiable, thus yielding a generalization of Corollary 2. Correspondingly, the problem is open as to whether a modification of Corollary 4, obtained by omitting completely the hypothesis of rectifiability and replacing (3.2) by the weaker condition

$$\operatorname{cap}(C \cap \sigma(T)) = 0, \tag{3.3}$$

also remains true. Such a hypothetical result would of course generalize Corollary 3. Finally, we note that it is also unknown whether the minimal normal extension of a completely subnormal T can have an eigenvalue lying in the boundary of $\sigma(T)$. As is seen from Theorem 2, a necessary condition is that such a value not be a peak point of $R(\sigma(T))$.

Added in proof. The above question as to whether the minimal normal extension of a completely subnormal T may have an eigenvalue in the boundary of $\sigma(T)$ has since been answered affirmatively (and independently) in the preprints *A class of subnormal operators* by R. F. Olin and *Eigenvalues of minimal normal extensions* by M. Radjabalipour.

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