A NOTE ON IMMERSIONS UP TO COBORDISM

BY

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In Section 4 of the reference Liulevicius proved that if M^n is cobordant to RP^n for $n = 2^r - 2$, $r \geqq 4$, then M^n cannot be immersed in R^{n+1}. Of course if $r = 1, 2$, or 3 then such immersions of RP^n itself are well known. We use these results to observe that if M^n immerses in R^{n+1} then the cobordism class $[M^n]$ must be zero except for the three cases just mentioned. More precisely we have the following result.

THEOREM. *If M^n immerses in R^{n+1} (or embeds in R^{n+2}) then M^n is a boundary or is cobordant to RP^0, RP^2, or RP^6.*

Proof. Let M^n immerse in R^{n+1} (or embed in R^{n+2}). Then the total dual Stiefel-Whitney class of M^n reduces to $\bar{w} = 1 + \bar{w}_1$, and the only possibly nonzero Stiefel-Whitney number corresponds to $\bar{w}_1^n$. Assume M^n is not a boundary, so $\bar{w}_1^n \neq 0$, and, for convenience, denote $\bar{w}_1$ by α. The total Stiefel-Whitney class of M^n is $w = (1 + \alpha)^{-1}$ and hence the total Wu class is

$$v = Sq^{-1}w = Sq^{-1}(1 + \alpha)^{-1} = 1 + \alpha + \alpha^3 + \alpha^7 + \alpha^{15} + \cdots.$$

The last line can be seen as follows:

$$\begin{aligned} Sq\left(\sum_{i=0}^{\infty} \alpha^{2^i - 1}\right) &= \sum_{i=0}^{\infty} (Sq\alpha)^{2^i - 1} \\ &= \sum_{i=0}^{\infty} (\alpha + \alpha^2)^{2^i - 1} \\ &= \sum_{i=0}^{\infty} \alpha^{2^i - 1}(1 + \alpha)^{2^i}(1 + \alpha)^{-1} \\ &= (1 + \alpha)^{-1} \sum_{i=0}^{\infty} (\alpha^{2^i - 1} + \alpha^{2^{i+1} - 1}) \\ &= (1 + \alpha)^{-1}. \end{aligned}$$

About the Wu classes we know that for any n-manifold, $v_i = 0$ for $i > n/2$. In our case this cannot hold unless $n = 2^r - 2$ so that the highest nonzero v_i has $i = 2^{r-1} - 1$. Furthermore M^n then has the same Stiefel-Whitney numbers as RP^n, so M^n is cobordant to RP^n. Now the result of Liulevicius shows that $n = 0, 2$, or 6.

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COROLLARY. *In the notation of Liulevicius we have*

$$^{alg}F_1\pi_n(MO) = \begin{cases} Z_2 & \text{if } n = 2^r - 2,\ r \geqq 1, \\ 0 & \text{otherwise}; \end{cases}$$

$$^{geo}F_1\pi_n(MO) = \begin{cases} Z_2 & \text{if } n = 0,\ 2, \text{ or } 6, \\ 0 & \text{otherwise}. \end{cases}$$

REFERENCE

A. LIULEVICIUS, *Immersions up to cobordism*, Illinois J. Math., vol. 19 (1975), pp. 149–164.

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