ILLINOIS JOURNAL OF MATHEMATICS
Volume 21, Number 4, December 1977

ONE POINT REGULARITY PROPERTIES OF MULTIPLE FOURIER SERIES WITH GAPS

BY

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1. Introduction

We will extend results of Hsieh Xie-Fan [2], G. Freud [1], and M. Izumi, S. Izumi and J.-P. Kahane [3] to several variables and obtain some refinements in one variable. The main result in one variable is that if a Fourier series is lacunary in a certain sense and satisfies a Lipschitz condition of a certain order at some point then the series is in the Lipschitz class of that order.

To establish this and related results, two steps are needed. The first is to use lacunarity at a point to estimate the Fourier coefficients. This is accomplished by a technique due to Noble [7] who worked on a related problem. We modify the technique somewhat and where a special trigonometric polynomial was used, we use a summable function on R^n whose Fourier transform is C^∞ and has compact support. This is a technical improvement especially in several variables. It also makes it possible to investigate two point or n point regularity problems.

The second step is to go from the coefficient estimate to membership in a Lipschitz class, a problem first investigated by Lorentz. We will use a result of Pesek. (See Pesek [8] and [9] for proofs and references.) From these two results and a counting argument we obtain various one point regularity results for multiple Fourier series with gaps. These results apply to certain partial differential equations with constant coefficients. In the last section we give some counterexamples that show our previous results are in some instances best possible.

We wish to point out the possibility of posing analogous problems. Instead of assuming regularity at a point, assume it on larger sets such as neighborhoods, sets of positive measure, or submanifolds of the torus. Then ask what lacunarity conditions will guarantee regularity on the whole torus. For a neighborhood results are known. See Kahane [4]. In this case regularity is expressed in terms of Sobolev spaces.

2. Preliminaries

We summarize the conventions, definitions, and results that we shall need.

Let T^n be the n-dimensional torus. Let R^n be n-dimensional Euclidean space and Z^n be the lattice points of R^n with integer coordinates. By the identification

Received May 28, 1976.

[1] This paper is part of the author's Ph.D. dissertation written under the direction of Professor P. L. Duren at the University of Michigan.

$T^n \simeq R^n/(2\pi Z)^n$ we can and usually will identify functions on T^n as functions on R^n which are 2π periodic in each variable.

If $x = (x_1, \ldots, x_n) \in R^n$ and $t = (t_1, \ldots, t_n) \in R^n$ then $xt = \sum x_j t_j$ and $|x| = (\sum x_j^2)^{1/2}$.

If α is a multi-order, then $|\alpha| = \sum \alpha_j$. It will be clear from context which of these conventions is being used.

If $f \in L^1(R^n)$, we define the Fourier transform of f by

$$\hat{f}(x) = \int_{R^n} f(t)e^{-ixt}\, dt$$

$C_0^\infty(R^n)$ is the class of infinitely differentiable functions on R^n with compact support. $\mathscr{S}$ is the Schwartz class of rapidly decreasing functions.

If $f \in L^1(T^n)$ and $m \in Z^n$, the Fourier coefficient of f at m is given by

$$\hat{f}(m) = (2\pi)^{-n} \int_{[0,2\pi]^n} f(t)e^{-imt}\, dt$$

$\sum \hat{f}(m)e^{imx}$ is called the Fourier series of f. If this series is absolutely convergent, it converges to f.

If $f \in L^1(T^n)$ and $g \in L^1(R^n)$, their convolution $f * g$ is in $L^1(T^n)$ and is defined by

$$f * g(x) = \int_{R^n} f(x - y)g(y)\, dy.$$

Then

$$\| f * g \|_1 \le \| f \|_1 \| g \|_1.$$

We have

$$(f * g)^\wedge(m) = \hat{f}(m)\hat{g}(m) \tag{1}$$

where $\hat{f}(m)$ is the Fourier coefficient of f at m and $\hat{g}(m)$ is the Fourier transform of g evaluated at m.

We now define the Lipschitz class $\Lambda_\alpha(T^n)$ for $\alpha > 0$.

DEFINITION 1. Let k be the greatest integer less than α. We say that $f \in \Lambda_\alpha$ if f has partial derivatives of all orders less than or equal to k and if $\alpha - k < 1$ then

$$\| D^\beta f(x + t) - D^\beta f(x) \|_\infty = O(|t|^{\alpha - k})$$

for all β of order k, but if $\alpha - k = 1$ then

$$\| D^\beta f(x + t) - 2D^\beta f(x) + D^\beta f(x - t) \|_\infty = O(|t|)$$

for all β of order k.

Note that if α is an integer, we use Zygmund's Λ^* classes instead of the classical Lipschitz spaces.

We will not use this definition directly. We will use the following theorem from Pesek [8] and [9].

THEOREM A. *Let* $f \in L^1(T^n)$, $\alpha > 0$, $0 \le \tau \le n$, $q > 1$, *and* $E \subseteq Z^n$. *Assume*

$$|\hat{f}(m)| \le C|m|^{-(\alpha+\tau)}. \tag{2}$$

Then $\operatorname{supp} \hat{f} \subseteq E$ *implies* $f \in \Lambda_\alpha$ *if and only if*

$$\operatorname{card}\,(E \cap \{x \mid r \le |x| \le qr\}) \le C(q, E)r^\tau \tag{3}$$

where card *means set cardinality.*

This result will allow us to go from coefficient estimates to membership in a Lipschitz class.

Finally we define what it means to be Lipschitz of order α at a point.

DEFINITION 2. Let k be the greatest integer less than α; then f is Lipschitz of order α at x_0 if there is a number $\delta > 0$ and a polynomial $P(x)$ of degree k such that

$$|f(x) - P(x)| \le B|x - x_0|^\alpha \quad \text{if } |x - x_0| < \delta \tag{4}$$

where B is a constant not depending on x.

3. The coefficient estimate

In this section we will prove the following theorem.

THEOREM 1. *Let* $f \in L^1(T^n)$, $\alpha > 0$, $E \subseteq Z^n$, $\gamma > 0$, $0 < \delta < 1$, $B > 0$, *and* $0 < \theta \le 1$. *Assume*

(5) $\operatorname{supp} \hat{f} \subseteq E$,
(6) $\operatorname{dist}\,(m, E\backslash\{m\}) > \gamma|m|^\theta$ *for* $m \in E$,
(7) f *is Lipschitz of order* α *at some* $x_0 \in T^n$.

Then
(8) $|\hat{f}(m)| \le C|m|^{-\theta\alpha}$ *where* C *does not depend on* m.

The expression dist $(m, E\backslash\{m\})$ is the distance from m to its nearest neighbor in E. (5) and (6) give the lacunarity condition. The nonvanishing coefficients are separated by a distance proportional to a power of their distance from the origin.

For the case $n = 1$ this result is due to Hsieh Xie-Fan [2] and M. Izumi, S. Izumi, and J.-P. Kahane [3].

The proof of Theorem 1 depends on Lemmas 1 and 2. We shall state these lemmas and use them to prove Theorem 1. Then we will prove the lemmas.

LEMMA 1. *Let* $\{a_\beta\}_{0 \le |\beta| \le k}$ *be complex numbers indexed by multi-orders of total order at most* k. *Then there is a trigonometric polynomial* T *of degree at* *most nk (where n is the dimension) such that*

$$D^{\beta}T(0) = a_{\beta}, \quad 0 \le |\beta| \le k. \tag{9}$$

LEMMA 2. *Let* $f \in L^1(T^n)$, $\alpha > 0$, $B > 0$, $0 < \delta < 1$, $N > 0$, *and* $m \in Z^n$. *Assume:*

(10) *if* $p \in Z^n$ *and* $0 < |p - m| < N$, *then* $\hat{f}(p) = 0$,
(11) $|f(x)| \le B|x|^{\alpha}$ *for* $|x| < \delta$,
(12) $N > \delta^{-1/2}$.

Then

(13) $|\hat{f}(m)| \le C_0 N^{-\alpha}$ *where* C_0 *depends on* n, f, B, α, *and* δ *but not on* N *and* m.

The work of proving Theorem 1 is actually done in Lemma 2. The idea of such a "local" lemma is found in Katznelson [5, p. 105].

Proof of Theorem 1. Without loss of generality we assume $x_0 = 0$. By Lemma 1 there is a trigonometric polynomial $T(x)$ with total degree at most nk such that

$$D^{\beta}T(0) = D^{\beta}P(0) \quad \text{for } 0 \le |\beta| \le k. \tag{14}$$

Thus

$$\begin{aligned} |f(x) - T(x)| &\le |f(x) - P(x)| + |P(x) - T(x)| \\ &\le B|x|^{\alpha} + A|x|^{k+1} \\ &\le B'|x|^{\alpha} \quad \text{if } |x| < \delta. \end{aligned} \tag{15}$$

We estimate the first term by (7) and the second by (14) and Taylor's theorem. Recall that k is the greatest integer less than α.

Define $F(x) = f(x) - T(x)$. Then $\operatorname{supp} \hat{F} \subseteq E'$ where $E \subseteq E'$ and card $(E' \backslash E) < \infty$.

Thus E' satisfies (6) with γ replaced by some $\gamma' > 0$. So

$$\operatorname{dist}(m, E' \backslash \{m\}) > \gamma' |m|^{\theta} \quad \text{for } m \in E'.$$

By (15), $|F(x)| \le B'|x|^{\alpha}$ for $|x| < \delta$.

If we let $m \in E'$, F be f, B' be B, δ be δ, α be α, and $N = \gamma'|m|^{\theta}$, Lemma 2 applies when $\gamma'|m|^{\theta} > \delta^{-1/2}$ and therefore

$$|\hat{F}(m)| \le C_0(\gamma'|m|^{\theta})^{-\alpha} = C_0(\gamma')^{-\alpha}|m|^{-\theta\alpha}$$

for all but a finite number of the elements of E' by (13).

Since $\hat{F}(m) = \hat{f}(m)$ for all but a finite number of $m \in E$,

$$|\hat{f}(m)| \le C|m|^{-\theta\alpha} \quad \text{for } m \in E$$

where the constant C has been adjusted to cover the finite number of exceptions. If $m \notin E$, the inequality is still true since $\hat{f}(m) = 0$. Thus we have (8) which proves Theorem 1.

Now we must prove the lemmas.

Proof of Lemma 1. First we restrict our attention to the case $n = 1$. We must show that there is a trigonometric polynomial $T(x) = \sum_{j=0}^{k} b_j e^{ijx}$ such that $D^v T(0) = a_v$ for $0 \le v \le k$.

We see by computation that this is equivalent to $\sum (ij)^v b_j = a_v$ for $0 \le v \le n$. The matrix of this linear system is a Vandermonde matrix. The standard formula for the determinant of a Vandermonde matrix shows that this one is nonsingular. Thus this system has a solution. The lemma has been proven for $n = 1$.

From this result we see that there are trigonometric polynomials $T_{kv}(x)$ where $0 \le v \le k$ and the degree of T_{kv} is at most k such that

$$D^j T_{kv}(0) = \delta_{jv} \quad \text{for } 0 \le j \le k,\ 0 \le v \le k. \tag{16}$$

(δ_{jv} is the kronecker delta.)

Now we consider the general case. Let $T(x) = \sum_{0 \le |\beta| \le k} a_\beta \prod_{j=1}^{n} T_{k,\beta_j}(x_j)$. Then

$$\begin{aligned} D^\gamma T(0) &= \sum_{0 \le |\beta| \le k} a_\beta D^\gamma \prod_{j=1}^{n} T_{k,\beta_j}(x_j)\Big|_0 \\ &= \sum_{0 \le |\beta| \le k} a_\beta \prod_{j=1}^{n} D^{\gamma_j} T_{k,\beta_j}(x_j)\Big|_0 \\ &= \sum_{0 \le |\beta| \le k} a_\beta \prod_{j=1}^{n} \delta_{\gamma_j \beta_j} \quad \text{by (16)} \\ &= a_\gamma \quad \text{if } 0 \le |\gamma| \le k. \end{aligned}$$

This gives (9) and Lemma 1 is proven.

Proof of Lemma 2. Let g be a summable function on R^n such that:

(17) $\hat{g} \in C_0^\infty(R^n)$,
(18) $\hat{g}(x) = 0$ if $|x| > 1$,
(19) $\hat{g}(0) = 1$,
(20) g is even.

Since $\hat{g} \in C_0^\infty(R^n)$, g is in the Schwartz class $\mathscr{S}$. We claim

$$\hat{f}(m) = \int_{R^n} f(x) e^{-imx} N^n g(Nx)\, dx \tag{21}$$

Let us prove (21). Let $f \in L^1(T^n)$, $\phi \in \mathscr{S}$. The vth Fourier coefficient of $f * \phi$ is $\hat{f}(v)\hat{\phi}(v)$ by (1). Since $\hat{\phi}$ is rapidly decreasing, $f * \phi$ has an absolutely convergent Fourier series. Thus

$$f * \phi(0) = \sum \hat{f}(v)\hat{\phi}(v) = \int_{R^n} f(x)\phi(-x)\, dx.$$

Since $g \in \mathscr{S}$, $N^n g(Nx) \in \mathscr{S}$. $(N^n g(Nx))^\wedge = \hat{g}(x/N)$ by a change of variables. By the above and (20),

$$\begin{aligned}\int f(x)e^{-imx}N^n g(Nx)\,dx &= \int f(x)e^{-imx}N^n g(-Nx)\,dx\\ &= \sum \hat{f}(v+m)\hat{g}(v/N)\\ &= \sum_{|k|\le N} \hat{f}(v+m)\hat{g}(v/N) \quad \text{(by (18))}\\ &= \hat{f}(m)\hat{g}(0)\\ &= \hat{f}(m) \quad \text{(by (11) and (19)).}\end{aligned}$$

The claim (21) has been proven.

Now we have the estimate

$$\begin{aligned}|\hat{f}(m)| &\le \int_{R^n} |f(x)|\,|N^n g(Nx)|\,dx\text{ [2]}\\ &= \left(\int_{|x|<N^{-1}} + \int_{N^{-1}\le|x|\le N^{-1/2}} + \int_{N^{-1/2}\le|x|\le 1} + \int_{|x|\ge 1}\right)\\ &\quad\times |f(x)|\,|N^n g(Nx)|\,dx.\end{aligned}$$

Denote these pieces I, II, III, and IV respectively.

We will prove Lemma 2 by getting a satisfactory estimate on each of the four pieces I, II, III, and IV.

$$\begin{aligned}\mathrm{I} &= \int_{|x|\le N^{-1}} |f(x)|\,N^n\,|g(Nx)|\,dx\\ &\le B\int_{|x|\le N^{-1}} |x|^\alpha N^n\,|g(Nx)|\,dx \quad \text{(by (11))}\\ &\le BN^{-\alpha}\int_{R^n} |N^n g(Nx)|\,dx = B\,\|g\|_1 N^{-\alpha}\end{aligned}$$

This takes care of I.

$$\begin{aligned}\mathrm{II} &= \int_{N^{-1}\le|x|\le N^{-1/2}} |f(x)|\,N^n\,|g(Nx)|\,dx\\ &\le B\int_{N^{-1}\le|x|\le N^{-1/2}} |x|^\alpha\,|N^n g(Nx)|\,dx \quad \text{(by (12 and 13)).}\end{aligned}$$

[2] This method of estimating $\hat{f}(m)$ is originally due to Noble [7].

Since $g \in \mathscr{S}$ there exists c_1 such that $|g(x)| \leq c_1 |x|^{-(n+\alpha+1)}$. Thus

$$\mathrm{II} \leq c_1 B \int_{N^{-1} \leq |x| \leq N^{-1/2}} |x|^{\alpha} N^n (N|x|)^{-(n+\alpha+1)}\, dx$$

$$= c_1 B N^{-\alpha-1} \int_{N^{-1} \leq |x| \leq N^{-1/2}} |x|^{-(n+1)}\, dx.$$

By changing to spherical coordinates this equals $c_1 B\omega_{n-1} N^{-\alpha-1} \int_{N^{-1}}^{N^{-1/2}} r^{-2}\, dr$ where ω_{n-1} is the area of the $(n-1)$-sphere. This last equals

$$c_1 B\omega_{n-1} N^{-\alpha-1}(N - N^{1/2}) \leq c_1 B\omega_{n-1} N^{-\alpha}.$$

This takes care of II.

$$\mathrm{III} = \int_{N^{-1/2} \leq |x| \leq 1} |f(x)| N^n |g(Nx)|\, dx.$$

Since $g \in \mathscr{S}$, there exists c_2 such that $|g(x)| \leq c_2 |x|^{-2(n+\alpha)}$. Thus

$$\mathrm{III} \leq c_2 \int_{N^{-1/2} \leq |x| \leq 1} |f(x)| N^n (N|x|)^{-2(n+\alpha)}\, dx$$

$$\leq c_2 \int_{N^{-1/2} \leq |x| \leq 1} |f(x)| N^n (NN^{-1/2})^{-2(n+\alpha)}\, dx$$

$$= c_2 N^{-\alpha} \int_{N^{-1/2} \leq |x| \leq 1} |f(x)|\, dx$$

$$\leq c_1 \| f \|_1 N^{-\alpha}$$

This takes care of III.

Before we estimate IV, let us note that there is a constant c_3 such that

$$\int_{|x| \leq R} |f(x)|\, dx \leq c_3 R^n \tag{22}$$

since $f \in L^1(T^n)$. Since $g \in \mathscr{S}$, there exists a constant c_4 such that $|g(x)| \leq c_4 |x|^{-(n+\alpha)}$. Thus

$$\mathrm{IV} \leq c_4 \int_{|x| > 1} |f(x)| N^n (N|x|)^{-(n+\alpha)}\, dx$$

$$= c_4 N^{-\alpha} \int_{|x| > 1} |f(x)|\, |x|^{-(n+\alpha)}\, dx.$$

It will suffice to show that $\int_{|x|>1} |f(x)|\,|x|^{-(n+\alpha)}\,dx < \infty$. We have

$$\begin{aligned}\int_{|x|>1} |f(x)|\,|x|^{-(n+\alpha)}\,dx &= \sum_{j=0}^{\infty} \int_{2^j \le |x| \le 2^{j+1}} |f(x)|\,|x|^{-(n+\alpha)}\,dx \\ &\le \sum_{j=0}^{\infty} 2^{-j(n+\alpha)} \int_{|x|\le 2^{j+1}} |f(x)|\,dx \\ &\le c_3 \sum_{j=0}^{\infty} 2^{-j(n+\alpha)} 2^{(j+1)n} \quad \text{(by (22))} \\ &= c_3 2^n \sum_{j=0}^{\infty} 2^{-j\alpha} \\ &< \infty.\end{aligned}$$

So each of the four pieces has an estimate of the desired type. So $|\hat{f}(m)| \le C_0 N^{-\alpha}$ and we have proved Lemma 2.

4. Applications and examples

By use of Theorem A from Pesek [8] (quoted in Section 2), Theorem 1 and a geometric lemma, we shall prove various one point regularity results.

THEOREM 2. *Let* $f \in L^1(T^n)$, $\alpha > 0$, $E \subseteq Z^n$, $\gamma > 0$, $0 < \delta < 1$, $B > 0$, $0 < \theta \le 1$. *Assume:*

(5) $\operatorname{supp} \hat{f} \subseteq E$;
(6) $\operatorname{dist}(m, E\backslash\{m\}) > \gamma |m|^{\theta}$ *for* $m \in E$;
(7) *there is a point* x_0 *such that* f *satisfies a Lipschitz condition of order* α *at* x_0.

Then $f \in \Lambda_{\theta\alpha + n(\theta-1)}$ *if* $\alpha > n(1-\theta)/\theta$.

We state the case $\theta = 1$ as a corollary.

COROLLARY 1. *Let* $f \in L^1(T^n)$, $\operatorname{supp} \hat{f} \subseteq E$. *Assume*

$$\operatorname{dist}(m, E\backslash\{m\}) > \gamma |m| \quad \textit{for } m \in E, \tag{23}$$

and f *satisfies a Lipschitz condition of order* α *at some point of* T^n. *Then* f *is in the Lipschitz class* Λ_α.

In the case $n = 1$, the corollary is the result of Hsieh Xie-Fan [2] and M. Izumi, S. Izumi, and J.-P. Kahane [3]. G. Freud [1] obtained a special case prior to their results.

It is a consequence of Lemma 3, to be stated below, and Lemma 2 of Pesek [8] that if $E \subseteq Z^n$ satisfies (23) then in fact E is contained in the finite union of Hadamard sets. (A Hadamard set F can be described as a sequence $\{\lambda_k\}_1^\infty$ such that $|\lambda_{k+1}| > q|\lambda_k|$ for all k and some $q > 1$.) Thus condition (23) is fairly stringent.

We need the following geometric lemma.

LEMMA 3. *Let* $0 < \theta \leq 1$, $\gamma > 0$, $q > 1$, *and* $E \subseteq Z^n$. *Assume that*

$$\text{(6)} \qquad \operatorname{dist}(m, E \backslash \{m\}) > \gamma |m|^\theta \quad \textit{for } m \in E.$$

Then card $(E \cap \{x \mid r \leq |x| \leq qr\}) \leq C(q, E) r^{n(1-\theta)}$.

Thus the lacunarity condition of Theorems 1 and 2 implies the lacunarity condition (3) of Theorem A with $\tau = n(1 - \theta)$.

Proof. We wish to estimate card $(E \cap \{x \mid r \leq |x| \leq qr\})$. Let

$$m_1, m_2 \in E \cap \{x \mid r \leq |x| \leq qr\}.$$

Then $\operatorname{dist}(m_1, m_2) > \gamma |m_1|^\theta \geq \gamma r^\theta$ by (6). We can conclude that card $(E \cap \{x \mid r \leq |x| \leq qr\})$ is no more than the largest number of points that can be placed in a cube of side $2qr$ so that the distance between any two of these points is at least γr^θ.

Since

$$([2qr/(\gamma r^\theta/2\sqrt{n})] + 1)(\gamma r^\theta/2\sqrt{n}) \geq 2qr,$$

we shall still be estimating card $(E \cap \{x \mid r \leq |x| \leq qr\})$ if we estimate the largest number of points separated pairwise by a distance of at least γr^θ in an n-cube whose side has this larger length. Subdivide this n-cube into n-cubes with side $\gamma r^\theta/2\sqrt{n}$. The diameter of each of these smaller n-cubes is $\gamma r^\theta/2$. At most one point of a set of points separated by γr^θ can lie in any of the small n-cubes. Therefore

$$\begin{aligned} \operatorname{card}(E \cap \{x \mid r \leq |x| \leq qr\}) &\leq \text{number of small cubes} \\ &= ([2qr/(\gamma r^\theta/2\sqrt{n})] + 1)^n \\ &= ([4\sqrt{n}\, qr^{1-\theta}/\gamma] + 1)^n \\ &\leq (2^n(4\sqrt{n}\, q/\gamma)^n) r^{n(1-\theta)} \end{aligned}$$

since $[x] + 1 \leq 2x$ if $x > 1$.

This gives the desired estimate for large r. Since $E \subseteq Z^n$, we can complete the proof by adjusting the constant for small r. Lemma 3 has been proven.

Proof of Theorem 2 *and Corollary* 1. Theorem 2 has the same hypotheses as Theorem 1. Thus we have $|\hat{f}(m)| \leq C|m|^{-\theta\alpha}$. Using this estimate, and Theorem A with $\theta\alpha - n(1 - \theta)$ as α and $n(1 - \theta)$ as τ (which we are allowed to do by Lemma 3), we obtain $f \in \Lambda_{\theta\alpha - n(1-\theta)}$ if $\theta\alpha - n(1 - \theta) > 0$ which it is if $\alpha > n(1 - \theta)/\theta$.

Corollary 1 is just the special case $\theta = 1$ of Theorem 2. These proofs are complete.

We can combine Theorem 1 and Theorem A to get a further result.

THEOREM 3. *Let* $f \in L^1(T^n)$, $E \subseteq Z^n$, $\gamma > 0$, $0 < \delta < 1$, $B > 0$, $0 < \theta \leq 1$, $0 \leq \tau \leq n$, *and* $q > 1$. *Assume:*

(5) $\operatorname{supp} \hat{f} \subseteq E$;
(6) $\operatorname{dist}(m, E\backslash\{m\}) > \gamma |m|^\theta$ *for* $m \in E$;
(7) *there is a point* x_0 *such that* f *satisfies a Lipschitz condition of order* α *at that point; also assume condition* (3) *of Theorem* A:

$$\operatorname{card}(E \cap \{x \mid r \leq |x| \leq qr\}) \leq C(q, E) r^\tau.$$

Then $f \in \Lambda_{\theta\alpha - \tau}$ *if* $\theta\alpha > \tau$.

Proof. Apply Theorem 1; then apply Theorem A with $\theta\alpha - \tau$ as α and τ as τ.

The next result is a consequence of Theorem 1.

THEOREM 4. *Let* $f \in L^1(T^n)$. *Suppose that* (6) *and* (7) *hold for* f. *Suppose also that at some point* f *has differentials of all orders. Then* f *is a* C^∞ *function.*

Proof. Since f has differentials of all orders at some point, it satisfies Lipschitz conditions of all orders at that point. By Theorem 1 we have $|\hat{f}(m)| \leq C_\alpha |m|^{-\theta\alpha}$ for $\alpha > 0$. Thus $|\hat{f}(m)| \leq C_k |m|^{-k}$ for all k. So $f \in C^\infty(T^n)$. Theorem 4 is proven.

When $n = 1$, this is due to M. Izumi, S. Izumi, and J.-P. Kahane [3].

We shall devote the rest of this section to examples.

Example 1. Let $n = 1$, and let $E = \{4^m\}_1^\infty \cup \{4^m + 2^m\}_1^\infty$. Then if f is Lipschitz of order $\alpha > 0$ at some point, $f \in \Lambda_{\alpha/2}$.

Example 2. Let $n = 1$ and let $E = \{27^m\} \cup \{27^m + 3^m\}$. Then if f is Lipschitz of order $\alpha > 0$ at some point, $f \in \Lambda_{\alpha/3}$.

Example 3. Let $n = 1$, and let $E = \{m^2\}$. Then if f is Lipschitz of order $\alpha > 1$ at some point, then $f \in \Lambda_{(\alpha-1)/2}$.

Example 4. Let $n = 3$ and $E = \{(m, m^2, 3^m)\}$. Then if f is Lipschitz of order $\alpha > 0$ at some point, $f \in \Lambda_\alpha$.

Example 5. Let $n = 3$ and $E = \{(m, m^2, m^3)\}$. Then if f is Lipschitz of order $\alpha > 1/2$ at some point, $f \in \Lambda_{(2/3)\alpha - 1/3}$.

The examples given above exhibit both polynomial and exponential growth.

We will conclude this section with two examples concerning partial differential equations.

Example 6. Consider the equation $u_{xx} = iu_t$ and assume $u \in L^1(T^2)$. Then

$$u(x, t) = \sum_{m \in Z} a_m e^{i(mx + m^2 t)} \quad \text{(formally)}$$

so supp $\hat{u} \subset \{(m, m^2)\}_1^\infty$ and our results apply.

(1) If u has differentials of all orders at some point, then $u \in C^\infty(T^2)$.

(2) If u satisfies a Lipschitz condition of order $\alpha > 1$ at some point, then $f \in \Lambda_{(\alpha - 1)/2}$.

Next we give an example of a partial differential equation with a stronger property.

Example 7. Consider the equation $u_{xx} - 2u_{tt} + u = 0$ with $u \in L^1(T^2)$. Then

$$\operatorname{supp} \hat{u} \subseteq \{(m_1, m_2) \mid m_1^2 - 2m_2^2 = 1\}.$$

We claim that if u satisfies a Lipschitz condition of order α at some point then u is in the Lipschitz class Λ_α. By Corollary 1 it suffices to show E satisfies (23). E is the set of solutions to Pell's equation with $d = 2$. See, for instance, Nagell [6, p. 195]. The solutions with positive entries are given by $(a_k, b_k)_{k=1}^\infty$ where $a_k + b_k\sqrt{2} = (3 + 2\sqrt{2})^k$. (We equate rational and irrational parts.) We also have the point (1, 0). The other solutions are given by changes of sign. We note that the positive solutions grow geometrically. With a little care we see that solutions in different quadrants are sufficiently far apart. Thus (23) is satisfied, and our claim is valid.

There are, of course, many more partial differential equations to which our results apply. But given the role of Diophantine equations it would be very difficult and probably impossible to characterize this class of equations.

5. Some counterexamples

In this section we will construct counterexamples that show that the lacunarity condition (6) of Theorem 1 cannot be weakened if $\alpha \leq 1$.

THEOREM 6. *Let* $0 < \alpha \leq 1$, $E \subseteq Z^n$, $0 < \theta \leq 1$. *Assume that E does not satisfy* (6). *Then there exists an* $f \in L^1(T^n)$ *such that* supp $\hat{f} \subseteq E$, *f satisfies a Lipschitz condition of order* α *at* 0, *but* $|\hat{f}(m)| \neq O(|m|^{-\alpha\theta})$.

To prove this, we need a lemma.

LEMMA 4. *Let* $a_m > 0$, $b_m > 0$, *and* $b_m = o(a_m)$. *Then there exists* $c_m \geq 0$ *such that* $\limsup_m c_m a_m = \infty$ *and* $\sum c_m b_m < \infty$.

Proof. There exists $c'_m > 0$ such that $\lim c'_m a_m = \infty$ and $\lim c'_m b_m = 0$. Indeed let $c'_m = 1/\sqrt{a_m b_m}$. Since $\lim c'_m b_m = 0$, for every k there exists j_k such that $c'_{j_k} b_{j_k} \leq 2^{-k}$. Let $c_m = c'_{j_k}$ if $m = j_k$ for some k. Let $c_m = 0$ otherwise. Then

$\sum c_m b_m \leq \sum 2^{-k} < \infty$ and $\limsup c_m a_m = \infty$. The proof of Lemma 4 is complete.

Proof of Theorem 6. Assume dist $(m, E\backslash\{m\}) > \gamma|m|^\theta$ fails for all $\gamma > 0$. Then there exists $\{m_k\}_0^\infty \subseteq E$ such that $\lim_{k\to\infty} |m_{2k}|^\theta / |m_{2k} - m_{2k+1}| = \infty$. So $|m_{2k} - m_{2k+1}|^\alpha = o(|m_{2k}|^{\theta\alpha})$.

We apply Lemma 4 to $|m_{2k}|^{\theta\alpha}$ and $|m_{2k} - m_{2k+1}|^\alpha$. There exists $C_k \geq 0$ such that $\sum C_k |m_{2k} - m_{2k+1}|^\alpha < \infty$ and $\limsup_m C_k |m_{2k}|^{\theta\alpha} = \infty$. Then $\sum C_k |m_{2k} - m_{2k+1}|^\alpha < \infty$ implies $\sum C_k < \infty$. Since the Fourier series $\sum C_k(e^{im_{2k}x} - e^{im_{2k+1}x})$ is absolutely convergent, it defines an L^1 function f. We will show f has the desired properties. We have that $\limsup_k C_k |m_{2k}|^{\theta\alpha} = \infty$ implies $\hat{f}(m) \neq O(|m|^{-\theta\alpha})$. And we have

$$\begin{aligned} |f(x) - f(0)| &= |\sum C_k(e^{im_{2k}x} - e^{im_{2k+1}x}) - 0| \\ &\leq \sum C_k \min(|m_{2k} - m_{2k+1}|\,|x|, 2) \\ &\leq 2\sum C_k |m_{2k} - m_{2k+1}|^\alpha |x|^\alpha \quad (\text{since } \alpha \leq 1) \\ &\leq B|x|^\alpha \end{aligned}$$

where $B = 2\sum C_k |m_{2k} - m_{2k+1}|^\alpha$ is finite. Theorem 6 has been proved.

We would like to extend the argument to larger α but have not succeeded. In the case where $n = 1$ and $\theta = 1$, this argument can be generalized by taking primitives. This technique is awkward in higher dimensions.

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