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DISJOINT SEQUENCES, COMPACTNESS, AND SEMIREFLEXIVITY IN LOCALLY CONVEX RIESZ SPACES

BY

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1. Introduction

A number of recent results [6], [7], [11], [12] give characterizations of topological properties of Banach lattices in terms of disjoint sequences. The present work extends a number of these results to the setting of locally convex Riesz spaces. Our method is based largely upon techniques given by Fremlin [5] rather than the representation theorems of Kakutani for abstract L and M spaces, and the principal technical tools for the work are given in Proposition 2.1 and Proposition 2.2. The following notions are then characterized in terms of disjoint sequences: conditional weak sequential compactness (Proposition 2.3), weakly compact order intervals (Proposition 3.1), compact order intervals (Proposition 4.1), relative compactness (Proposition 4.3), semireflexivity (Proposition 5.4).

Our notation and terminology will be drawn from [5], [8] and results from these sources will occasionally be used without explicit reference. Throughout the paper L will denote an Archimedean Riesz space with order dual $L^{\sim}$. The band of normal integrals on L will be denoted by $L_n^{\sim}$. A locally convex Riesz space (L, T) is an Archimedean Riesz space equipped with a locally solid, locally convex Hausdorff topology T. We shall refer to a locally solid, locally convex topology T on L simply as a locally convex Riesz space topology on L. If $M \subset L^{\sim}$ is a separating ideal the locally convex Riesz space topology on L defined by the Riesz seminorms $x \mapsto |\phi|(|x|)$, $x \in L$, $\phi \in M$ will be denoted by $|\sigma|(L,M)$. A Riesz seminorm ρ on L is called a Fatou seminorm if $0 \leq x_\tau \uparrow_\tau x$ holds in L implies $\rho(x) = \sup_\tau \rho(x_\tau)$. We will use the following terminology of [5]. A locally convex Riesz space topology T on L is called (a) Fatou if T is defined by its continuous Fatou Riesz seminorms, (b) Levi if each T-bounded upwards directed system in L^+ has a least upper bound in L, (c) Lebesgue if $0 \leq x_\tau \downarrow_\tau 0$ holds in L implies $\{x_\tau\}$ is T-convergent to 0.

The well-known theorem of Nakano on completeness may now be stated in the following form (see [5], [2]):

NAKANO'S THEOREM. (a) *If L is Dedekind complete and if T is a Fatou locally convex Riesz space topology on L then each order interval of L is T-complete.*

(b) *If L is Dedekind complete and if T is a Levi Fatou locally convex Riesz space topology on L then L is T-complete.*

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One consequence of this theorem is that if (L, T) is a locally convex Riesz space with (topological) dual L', then each order interval of L' is $\beta(L', L)$ complete (see Proposition 4.1.8 of [13]).

Notation. If A is a subset of the Riesz space L, the order ideal generated in L by A will be denoted $\langle A \rangle$.

2. Disjoint sequences

In this section we wish to gather the basic technical tools for the work. The first result will be used repeatedly and is due to D. H. Fremlin [5] and P. Meyer-Nieberg [11] and is explicitly stated here for ease of reference.

PROPOSITION 2.1. *Let L be a Riesz space and T a locally convex Riesz space topology on L. The following statements are equivalent.*

(i) *Each order bounded disjoint sequence in L^+ is T-convergent to* 0.
(ii) *Each order bounded upwards directed system in L^+ is T-Cauchy.*

It should be pointed out that Fremlin proves this result under the assumption that T is a linear space topology on L for which order bounded sets are bounded. (See Lemma 24 H of [5].)

The following is a technical lemma prompted by several results of [1], [5], and [12].

PROPOSITION 2.2. *Let (L, T) be a locally convex Riesz space with the property that each monotone order-bounded T-Cauchy sequence is T-convergent. Let $A \subset L$ and $B \subset L'$ be solid bounded sets and assume*

$$\sup \{|\phi|(|x|): \phi \in B, x \in A\} < \infty.$$

The following statements are equivalent.

(i) $\sup \{|\phi_n|(|x|): x \in A\} \to 0$ *as* $n \to \infty$ *for each disjoint sequence* $\{\phi_n\} \subset B$.
(ii) $\sup \{|\phi|(|x_n|): \phi \in B\} \to 0$ *as* $n \to \infty$ *for each disjoint sequence* $\{x_n\} \subset A$.
(iii) (a) *For each $\varepsilon > 0$, there exists $\psi \in \langle B \rangle^+$ such that*

$$\sup [(|\phi| - |\phi| \wedge \psi)(|x|): \phi \in B, x \in A] < \varepsilon.$$

(b) *For each $\varepsilon > 0$, there exists $y \in \langle A \rangle^+$ such that*

$$\sup [|\phi|(|x| - |x| \wedge y): \phi \in B, x \in A] < \varepsilon.$$

Proof. (i) $\Rightarrow$ (ii). If (ii) does not hold, there exists $\varepsilon > 0$, a disjoint sequence $\{x_k\} \subset A^+$, and a sequence $\{\phi_k\} \subset B^+$ such that $\phi_k(x_k) \geq \varepsilon$ for each $k = 1, 2, \ldots$. Denote by ψ_k the component of ϕ_k in the carrier band in L' defined by the normal integral $x_k \in (L')_n^\sim$. For properties of the carrier see note VIII, pp. 106–119, note IX, pp. 312–376, of [8].

Observe that $\{\psi_k\} \subset B$ since B is solid, that $\{\psi_k\}$ is pairwise disjoint and that $\psi_k(x_k) \geq \varepsilon$ for every $k = 1, 2, \ldots$, which contradicts (i).

(ii) $\Rightarrow$ (i). It is an easy consequence of (ii) and the Riesz decomposition lemma that every disjoint sequence $\{y_n\} \subset \langle A \rangle$, which is order bounded in $\langle A \rangle$, converges to zero uniformly on B. By Proposition 2.1 and Corollary 3.9 of [1], it follows that each disjoint sequence in B converges to zero $|\sigma|(\langle B \rangle, \langle A \rangle)$. Now suppose that (i) is not satisfied. There exists $\varepsilon > 0$, a sequence $\{x_n\} \subset A^+$ and a disjoint sequence $\{\phi_n\} \subset B^+$ such that $\phi_n(x_n) \geq 2\varepsilon$. Passing to a subsequence if necessary, we may assume that $\phi_n(x_m) < \varepsilon/n2^n$ for $m < n$. Define

$$y_{n,k} = \left(x_n - 2^n \sum_{m<n} x_m - \sum_{k \geq m > n} x_m/2^m\right)^+$$

and note that $y_{n,k} \downarrow_k \geq 0$. If ρ is any continuous Riesz seminorm, then

$$\rho(y_{n,k} - y_{n,k'}) \leq \left(\sum_{k<m\leq k'} 1/2^m\right) \sup\{\rho(x) : x \in A\}$$

holds for $k' > k$. Let $y_n = T - \lim_k y_{n,k}$ for each n. It is easily verified that the sequence $\{y_n\} \subset A$ is pairwise disjoint and

$$\begin{aligned}\phi_n(y_n) &\geq \phi_n(x_n) - 2^n \sum_{m<n} \phi_n(x_m) - \sum_{m>n} \phi_n(x_m)/2^m \\ &\geq \varepsilon - (1/2^n) \sup\{|\phi|(|x|) : \phi \in B, x \in A\} \\ &\geq \varepsilon/2 \quad \text{for } n \text{ sufficiently large}\end{aligned}$$

and this is a contradiction.

We show next that (ii) $\Rightarrow$ (iii)(b). For simplicity, assume

$$\sup\{|\phi|(|x|) : x \in A, \phi \in B\} = 1.$$

Suppose (iii)(b) is not true. We may assume there exists $x \in A^+$, $\phi_1 \in B^+$ such that $\phi_1(x) \geq \varepsilon$. Let $z_{1,1} = x$ and suppose that for $1 \leq j \leq k$ have been defined elements $\phi_j \in B^+$ and subsets $S_j = \{z_{j,i} : 1 \leq i \leq j\} \subset A^+$ with the following properties:

(i) $S_j \subset A^+$ is pairwise disjoint for each j with $1 \leq j \leq k$;

(ii) $z_{j,i} \geq z_{j+1,i}$, $1 \leq i \leq j \leq k-1$;

(iii) $\rho(z_{j,i} - z_{j+1,i}) \leq \varepsilon/2^{j+1} \sup\{\rho(x) : x \in A\}$ for each continuous Riesz seminorm ρ on L, for $1 \leq i \leq j \leq k-1$,

(iv) $\phi_j(z_{j,j}) \geq \varepsilon$, $\phi_i(z_{j,i}) \geq \varepsilon(1 - 1/2^2 - \cdots - 1/2^j)$ for $1 \leq i < j \leq k$.

By assumption, there exists $y \in A^+$, $\phi_{k+1} \in B^+$, such that

$$\phi_{k+1}\left(\left(y - \frac{2^{k+1}}{\varepsilon} \sup\{z_{k,j} : 1 \leq j \leq k\}\right)^+\right) \geq \varepsilon.$$

Define

$$z_{k+1,k+1} = (y - (2^{k+1}/\varepsilon) \sup\{z_{k,j} : 1 \leq j \leq k\})^+$$

and

$$z_{k+1,i} = (z_{k,i} - \varepsilon/2^{k+1}y)^+ \quad \text{for } 1 \le i \le k.$$

Observe that $z_{k+1,k+1} \wedge z_{k+1,i} = 0$ for $1 \le i \le k$ and that $z_{k+1,i} \le z_{k,i}$ for $1 \le i \le k$. It follows from the induction hypothesis that

$$S_{k+1} = \{z_{k+1,i}: 1 \le i \le k+1\} \subset A^+$$

is pairwise disjoint. Further, if ρ is any continuous Riesz seminorm on L,

$$\begin{aligned}\rho(z_{k,i} - z_{k+1,i}) &= \rho\left(z_{k,i} - \left(z_{k,i} - \frac{\varepsilon}{2^{k+1}} y\right)^+\right) \\ &= \rho\left(z_{k,i} \wedge \frac{\varepsilon}{2^{k+1}} y\right) \\ &\le \frac{\varepsilon}{2^{k+1}} \sup\{\rho(x): x \in A\}.\end{aligned}$$

In particular, for $1 \le i < k+1$,

$$\begin{aligned}\phi_i(z_{k+1,i}) &= \phi_i(z_{k,i}) - \phi_i(z_{k,i} - z_{k+1,i}) \\ &\ge \varepsilon(1 - 1/2^2 - \cdots - 1/2^k - 1/2^{k+1}).\end{aligned}$$

Now observe that the partial sums of the series $\sum_{k>j}(z_{k,j} - z_{k+1,j})$, $j = 1, 2, \ldots$, are order bounded and that $\sum_{k>j}\rho(z_{k,j} - z_{k+1,j}) < \infty$ for each continuous Riesz seminorm ρ on L and each $j = 1, 2, \ldots$. Let $z_j = T-\lim_k z_{k,j}$ for $j = 1, 2, \ldots$. It is clear that the sequence $\{z_j\} \subset A^+$ is pairwise disjoint and that $\phi_j(z_j) \ge \varepsilon/2$ holds for $j = 1, 2, \ldots$, which is a contradiction to (ii).

To show that (i) $\Rightarrow$ (iii)(a) observe that the hypotheses of the proposition are satisfied if L is a Dedekind complete Riesz space with a separating family of normal integrals and T is the locally convex topology $|\sigma|(L, M)$ induced by the separating ideal $M \subset L_n^\sim$. Applying these remarks to L' and taking for M the ideal generated by L in $(L')_n^\sim$, the implication (i) $\Rightarrow$ (iii)(a) follows from the implication (ii) $\Rightarrow$ (iii)(b).

(iii) $\Rightarrow$ (ii). Let $\{x_n\} \subset A^+$ be disjoint and let $\varepsilon > 0$ be given. By assumption, there exists $\psi \in \langle B\rangle^+$ and $y \in \langle A\rangle^+$ such that

$$(|\phi| - |\phi| \wedge \psi)(|x|) < \varepsilon \quad \text{and} \quad |\phi|(|x| - |x| \wedge y) < \varepsilon$$

for every $\phi \in B$ and $x \in A$. Observe that $|\phi|(x_n \wedge y) \to 0$ for each $\phi \in L'$. Let $0 \le \phi \in B$. From

$$\begin{aligned}\phi(x_n \wedge y) &\le (\phi - \phi \wedge \psi)(x_n \wedge y) + \psi(x_n \wedge y) \\ &\le \sup\{(\phi - \phi \wedge \psi)(|x|): x \in A\} + \psi(x_n \wedge y) \\ &\le \varepsilon + \psi(x_n \wedge y)\end{aligned}$$

it follows that $\sup\{|\phi|(x_n \wedge y): \phi \in B\} \to 0$ as $n \to \infty$. Now

$$\begin{aligned}\sup\{|\phi|(x_n): \phi \in B\} &\leq \sup\{|\phi|(x_n - x_n \wedge y): \phi \in B\} \\ &\quad + \sup\{|\phi|(x_n \wedge y): \phi \in B\} \\ &\leq \varepsilon + \sup\{|\phi|(x_n \wedge y): \phi \in B\}\end{aligned}$$

from which (ii) follows and the proof is complete.

Remark 1. The proof given above of the equivalence of statements (i) and (ii) is due to D. H. Fremlin. The proposition contains (i) $\Leftrightarrow$ (ii) of Satz II.8 and Satz II.2 in [12]. We point out that the proof of the proposition does not invoke the representation theorems of Kakutani, nor does it use Grothendieck's classical criterion for weakly compact sets of Radon measures.

Remark 2. It is to be noted that if a solid set $A \subseteq L$ satisfies one of the equivalent conditions of Proposition 2.2 then the convex hull of A also satisfies the equivalent conditions of Proposition 2.2.

If L is an Archimedean Riesz space and $M \subset L^{\sim}$ is a separating ideal, a subset $A \subset L$ is called conditionally (relatively) sequentially $\sigma(L, M)$ compact if and only if each sequence in A contains a $\sigma(L, M)$ Cauchy (convergent) subsequence. We give one immediate application of Proposition 2.2 related to the authors' earlier work [3].

PROPOSITION 2.3. *Let (L, T) be a locally convex, locally solid Riesz space and assume that $L' \subset L_n^{\sim}$. If L has the countable sup property, the following statements are equivalent for a subset $A \subset L$.*

(i) *A is conditionally $\sigma(L, L')$ sequentially compact.*
(ii) *The solid hull of A is conditionally $\sigma(L, L')$ sequentially compact.*
(iii) $0 \leq \phi_n \downarrow 0$, $\{\phi_n\} \subset L'$ *implies* $\sup\{\phi_n(|x|): x \in A\} \to 0$ *as* $n \to \infty$.
(iv) *A is $|\sigma|(L, L')$ bounded and each order bounded disjoint sequence in L' converges to zero uniformly on the solid hull of A.*
(v) *A is $|\sigma|(L, L')$ bounded and each disjoint sequence in the solid hull of A is $|\sigma|(L, L')$ convergent to* 0.
(vi) *Given $\varepsilon > 0$ and $0 \leq \phi \in L'$, there exists an element $0 \leq y$ in the ideal generated by A such that $\phi(|x| - |x| \wedge y) < \varepsilon$ holds for each $x \in A$.*

Proof. Denote by D the Dedekind completion of L and by B the solid hull of A in D. By Theorem 32.7 of [8], L' may be identified with an ideal $M \subset D_n^{\sim}$. It follows readily that A satisfies condition (iv), (respectively (v), (vi)) if and only if B satisfies condition (iv), (respectively (v), (vi)) with L replaced by D and L' by M. Since the intervals of D are $|\sigma|(D, M)$ complete by Nakano's theorem, the equivalence of (iv), (v), (vi) follow from the above observation and Proposition 2.2. The remaining equivalences follow from Propositions 2.2, 2.3, 3.11 of [3].

Remark. The above Proposition 2.3 is also true with the hypotheses that L be T-complete and that L' has an (at most) countable order basis. (See [3].)

3. Weak compactness

The first result of this section extends results of [7] and characterizes the weakly compact order intervals of locally convex Riesz spaces in terms of disjoint sequences. As in [5], if A is a subset of the Riesz space L we will denote by $\mathscr{I}A$ (respectively $\mathscr{D}A$) the set of all elements $x \in L$ for which there exists a system $\{x_\tau\} \subset A$ with $x_\tau \uparrow_\tau x$ (respectively $x_\tau \downarrow_\tau x$) holding in L. If x, y are elements of the Riesz space L with $y \leq x$ we shall denote by $[y, x]$ the order interval $\{z \in L : y \leq z \leq x\}$.

In the following Proposition, the equivalence of statements (i)–(iv) is essentially known.

PROPOSITION 3.1. *Let (L, T) be a locally convex Riesz space. The following conditions are equivalent for $0 \leq x \in L$.*

(i) *$[0, x]$ is $\sigma(L, L')$ compact.*

(ii) *$[-x, x]$ is a solid subset of $(L')_n^\sim$.*

(iii) *$[-x, x]$ is a solid subset of the bidual L''.*

(iv) *The principal ideal $\langle x\rangle$ generated by x in L is Dedekind complete and the restriction of T to $\langle x\rangle$ is Lebesgue.*

(v) *Each disjoint sequence in $[0, x]$ is T-convergent to 0 and $[0, x]$ is T-complete.*

(vi) *Each disjoint sequence in $[0, x]$ is T-convergent tc 0 and each directed T-Cauchy system in $[0, x]$ is T-convergent.*

Proof. (i) $\Rightarrow$ (ii). The interval $[0, x] \subset L$ is $|\sigma|(L_n'^\sim, L')$ dense in

$$I = \{z \in (L')_n^\sim : 0 \leq z \leq x\}$$

and hence $[0, x]$ is $\sigma((L')_n^\sim, L')$ dense in I. However, since $[0, x]$ is $\sigma(L, L')$ compact it follows that $[0, x]$ coincides with I.

The equivalence of (ii) and (iii) is easily seen.

(ii) $\Rightarrow$ (iv). By assumption, the principal ideal generated by x in L coincides with the principal ideal generated by x in $(L')_n^\sim$ and so is Dedekind complete. If $\{x_\tau\} \subset \langle x\rangle$ satisfies $x_\tau \downarrow_\tau 0$ in L, then also $x_\tau \downarrow_\tau 0$ holds in $(L')_n^\sim$. Consequently $\{x_\tau\}$ converges to 0 pointwise on L' and hence uniformly on each equicontinuous subset of L' by the well known theorem of Dini.

(iv) $\Rightarrow$ (v). Since $\langle x\rangle$ is Dedekind complete and T induces a Lebesgue topology on $\langle x\rangle$, it follows from Nakano's theorem that each order interval of $\langle x\rangle$ is complete for the topology induced on $\langle x\rangle$ by T. If now $\{x_n\} \subset [0, x]$

is disjoint, it follows that the sequence of partial sums $\{\sum_{k=1}^{n} x_k\} \subset [0, x]$ is T-convergent and hence the sequence $\{x_n\}$ is T-convergent to 0.

The implication (v) $\Rightarrow$ (vi) is clear.

(vi) $\Rightarrow$ (ii). Denote by I the interval $[0, x]$ in L and by J the interval

$$\{z: z \in (L')_n^{\sim}, 0 \leq z \leq x\}.$$

I is $|\sigma|((L')_n^{\sim}, L')$ dense in J and the locally convex Riesz space topology $|\sigma|((L')_n^{\sim}, L')$ is Lebesgue. It follows from Lemma 23 H of [5] that the interval J is precisely $\mathscr{I}\mathscr{D}I$. It follows from Proposition 2.1 that each directed system in I is T-Cauchy and hence T-convergent. It follows immediately that I coincides with J.

The implication (ii) $\Rightarrow$ (i) is clear from the well-known fact that each interval of $(L')_n^{\sim}$ is $\sigma((L')_n^{\sim}, L')$ compact.

For the case that L is a Banach lattice, the following result is Satz II.6 in [12], where it is noted that the converse holds if L is assumed to be an abstract L-space.

PROPOSITION 3.2. *Let (L, T) be a locally convex Riesz space and assume that every directed T-Cauchy system is T-convergent. Let $A \subset L$ be solid and bounded. If every disjoint sequence in A is T-convergent to* 0, *then A is relatively $\sigma(L, L')$ compact.*

Proof. Denote by $\bar{A}$ the T-closure of A. We observe that every disjoint sequence in $\bar{A}$ is T-convergent to zero. In fact, if this is not true, there exists $\varepsilon > 0$, a disjoint sequence $\{y_n\} \subset \bar{A}^+$ and a continuous Riesz seminorm ρ such that $\rho(y_n) > \varepsilon$ for each $n = 1, 2, \ldots$. Now choose $\{z_n\} \subset A^+$ with $\rho(y_n - z_n) < 1/n$, for $n = 1, 2, \ldots$. It follows that

$$0 \leq y_n \leq (y_n - z_n)^+ + y_n \wedge z_n \leq |y_n - z_n| + y_n \wedge z_n$$

so $\rho(y_n) \leq \rho(y_n - z_n) + \rho(y_n \wedge z_n)$. Observe that the sequence $\{y_n \wedge z_n\} \subset A^+$ is pairwise disjoint and so $\lim_{n\to\infty} \rho(y_n \wedge z_n) = 0$. It follows that $\rho(y_n) \leq \varepsilon/2$ for all sufficiently large n, which is a contradiction. Thus, without loss of generality, we may assume that A is T-closed and by Remark 2 following Proposition 2.2 we may also assume that A is convex. It follows from Proposition 3.1 above that A is a solid subset of $(L')_n^{\sim}$. Moreover, by Propositions 2.1, 2.2 above and Corollary 2.14 of [2], it follows that A is relatively $\sigma((L')_n^{\sim}, L')$ compact. Suppose now that $0 \leq x_\tau \uparrow_\tau \subset A^+$ and that V is a solid neighborhood of 0 in L. Let $\varepsilon > 0$ be given. By Proposition 2.2, there exists $x_0 \in \langle A \rangle^+$ such that $|\phi|(x - x \wedge x_0) < \varepsilon/3$ for all $x \in A^+$ and all $\phi \in V^0$. By Proposition 2.1, there exists an index τ_0 such that $x_\tau, x_{\tau'} \geq x_{\tau_0}$ imply

$$|\phi|(|x_0 \wedge x_\tau - x_0 \wedge x_{\tau'}|) < \varepsilon/3 \quad \text{for all } \phi \in V^0.$$

It follows that, for all $\phi \in V^0$, and $x_\tau, x_{\tau'} \geq x_{\tau_0}$,

$$\begin{aligned}|\phi|(|x_{\tau'} - x_\tau|) &\leq |\phi|(x_{\tau'} - x_{\tau'} \wedge x_0) + |\phi|(x_\tau - x_\tau \wedge x_0) \\ &\quad + |\phi|(|x_{\tau'} \wedge x_0 - x_\tau \wedge x_0|) \\ &\leq \varepsilon.\end{aligned}$$

Thus the system $\{x_\tau\}$ is T-Cauchy and hence T-convergent by assumption. Since A is T-closed, it follows that $A \subset L_n'^{\sim}$ satisfies $\mathscr{I}A \subset A$ and so A is $|\sigma|((L')_n^{\sim}, L')$ closed by Lemma 23 L of [5]. Since A is convex, it follows that A is $\sigma((L')_n^{\sim}, L')$ closed and the proof is complete.

COROLLARY 3.3. *Let the locally convex Riesz space (L, T) be T-complete, L' be $\beta(L', L)$-complete, and let $B \subset L'$ be bounded and solid. If each T-bounded disjoint sequence in L converges uniformly to zero on B, then B is relatively $\sigma(L', L'')$ compact.*

PROPOSITION 3.4. *Let (L, T) be a locally convex Riesz space. If L has the countable sup property, then each solid relatively $\sigma(L, L')$ compact subset of L is relatively $\sigma(L, L')$ sequentially compact.*

Proof. Note that $A \subset (L')_n^{\sim}$ is relatively $\sigma((L')_n^{\sim}, L')$ compact and so it follows from Proposition 2.15 of [1] that each disjoint sequence in A is $|\sigma|(L, L')$ convergent to 0. Let $\{x_n\} \subset A$ and let I be the ideal in L generated by $\{x_n\}$. Denote by $[L']$ the set of restrictions of elements of L' to I. It follows from Proposition 3.1 that $[L'] \subset I_n^{\sim}$ and it is easily seen from Theorem 19.2 of [8] that $[L']$ is an ideal of $I_n^{\sim}$. Since I has the countable sup property, it follows from Proposition 2.3 that $\{x_n\}$ is conditionally $\sigma(I, [L'])$ sequentially compact. Thus the sequence $\{x_n\}$ contains a $\sigma(I, [L'])$-Cauchy subsequence $\{x_{n_k}\}$. It is clear that $\{x_{n_k}\}$ is $\sigma(L, L')$-Cauchy. Let x_0 be a $\sigma(L, L')$ accumulation point of $\{x_{n_k}\}$. It is clear that $\{x_{n_k}\}$ is $\sigma(L, L')$ convergent to x_0 and the proof is complete.

4. Compactness

The element $0 \neq x$ in the Riesz space L is called an *atom* whenever it follows from $0 \leq u \leq |x|$, $0 \leq v \leq |x|$ and $u \wedge v = 0$ that $u = 0$ or $v = 0$. It is a simple matter to show [10] that if L has the principal projection property, then the element $0 \neq x \in L$ is an atom if and only if it follows from $x = u + v$, $0 \leq u, v$, with $u \wedge v = 0$ that $u = 0$ or $v = 0$. The band generated in an Archimedean Riesz space L by the atoms of L will be denoted by $\mathscr{A}$. L will be called discrete if $L = \mathscr{A}$.

PROPOSITION 4.1. *Let (L, T) be a locally convex Riesz space. The following are equivalent for $0 \leq x \in L$.*

(i) $[0, x]$ *is $\sigma(L, L')$ compact and $x \in \mathscr{A}$.*
(ii) $[0, x]$ *is $|\sigma|(L, L')$ compact.*

(iii) $[0, x]$ *is* T*-compact.*

(iv) *Each weakly convergent sequence in* $[0, x]$ *is* T*-convergent and* $[0, x]$ *is* T*-complete.*

(v) *Each* $\sigma(L, L')$*-Cauchy net in* $[0, x]$ *is* T*-convergent.*

(vi) *The lattice operations in* L' *are* $\sigma(L', \langle x \rangle)$ *continuous on the equicontinuous subsets of* L', *and* $[0, x]$ *is* T*-complete.*

Proof. (i) $\Rightarrow$ (vi). Observe first that if $0 \leq y \in L$ is an atom, then $|\phi|(y) = |\phi(y)|$ for each $\phi \in L'$, so that the map $\phi \mapsto |\phi|$, $\phi \in L'$ is clearly continuous for the weak topology on L' defined by the linear span of the atoms in L. If $0 \leq z \in \langle x \rangle$, there exists $\{z_\tau\} \subset \langle x \rangle$ with $0 \leq z_\tau \uparrow z$ and each z_τ is a linear combination of atoms of L. By Proposition 3.1(iii), $\{z_\tau\}$ is T-convergent to z and so $z_\tau \to z$ uniformly on equicontinuous subsets of L' and it follows that the lattice operations of L' are $\sigma(L', \langle x \rangle)$ continuous on the equicontinuous subsets of L'. That $[0, x]$ is T-complete is a simple consequence of Nakano's theorem.

(vi) $\Rightarrow$ (v). Let $\{x_\alpha\} \subset [0, x]$ satisfy $x_\alpha \to x_0$, $\sigma(L, L')$. If $\{x_\alpha\}$ is not T-convergent to x_0, there exists an equicontinuous set $A \subset L'$, a positive number $\varepsilon > 0$, and $\{\phi_\alpha\} \subset A$ such that $|\phi_\alpha(x_\alpha - x_0)| > \varepsilon$ (on passing to a cofinal subnet if necessary). Let ϕ be a $\sigma(L', L)$ accumulation point of $\{\phi_\alpha\}$. There exists α_0 such that $\alpha \geq \alpha_0$ implies $|\phi(x_\alpha - x_0)| < \varepsilon/2$ so for $\alpha \geq \alpha_0$,

$$|(\phi_\alpha - \phi)(x_\alpha - x_0)| \geq \varepsilon/2$$

and it follows that

$$\varepsilon/4 \leq \tfrac{1}{2}|\phi_\alpha - \phi|(|x_\alpha - x_0|) \leq |\phi_\alpha - \phi|(x).$$

It follows that 0 is not a $\sigma(L', \langle x \rangle)$ accumulation point of $\{|\phi_\alpha - \phi|\}$ although 0 is a $\sigma(L', \langle x \rangle)$ accumulation point of the net $\{\phi_\alpha - \phi\}$. Hence the lattice operations are not continuous on A which contradicts (v). It now follows that each disjoint sequence in $[0, x]$ is T-convergent to zero. From Proposition 3.1 above it follows that the interval $[0, x]$ is $\sigma(L, L')$ compact hence $\sigma(L, L')$ complete and it follows that each weakly Cauchy net in $[0, x]$ is T-convergent.

The implication (v) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (iii). It follows easily that each disjoint sequence in $[0, x]$ is T-convergent to 0 and so $[0, x]$ is $\sigma(L, L')$ compact. We will show that $[0, x]$ is totally bounded for T. If not, there exists a closed convex solid neighborhood V of 0 and a sequence $\{x_n\} \subset [0, x]$ for which $|x_n - x_k| \notin V$ for $1 \leq k < n$. By Proposition 2.2, given $\varepsilon > 0$, there exists $0 \leq \phi_0 \in L'$ such that

$$(|\phi| - |\phi| \wedge \phi_0)(x) < \varepsilon \quad \text{for all } \phi \in V^0.$$

Now $[0, x]$ is a solid subset of $(L')_n^\sim$. Denote by P the band projection of $(L')_n^\sim$ onto the carrier band of ϕ_0 in $(L')_n^\sim$; then $[0, Px] \subset [0, x] \subset L$. By Proposition 4.7 of [1], $[0, Px]$ is $\sigma(L, L')$ sequentially compact. Therefore there exists a weakly convergent subsequence $\{Px_{n_k}\}$ which is by assumption T-convergent.

Choose n_0 such that $\phi_0(|Px_{n_k} - Px_{n_m}|) \leq \varepsilon$ for $k, m \geq n_0$. Then

$$\begin{aligned}|\phi|(|x_{n_k} - x_{n_m}|) &\leq (|\phi| - |\phi| \wedge \phi_0)|x_{n_k} - x_{n_m}| \\ &\quad + |\phi| \wedge |\phi_0|(|x_{n_k} - x_{n_m}|) \\ &\leq 2\varepsilon + |\phi_0|(|x_{n_k} - x_{n_m}|) \\ &= 2\varepsilon + |\phi_0|(|Px_{n_k} - Px_{n_m}|) \\ &\leq 3\varepsilon \quad \text{for all } \phi \in V^0 \text{ and for all } k, m \geq n_0.\end{aligned}$$

Thus $|x_{n_k} - x_{n_m}| \in V$ for all k, m sufficiently large, which is a contradiction It follows that $[0, x]$ is totally bounded for T and hence T-compact.

The implication (iii) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i). It is clear that $[0, x]$ is $\sigma(L, L')$ compact. Suppose that $x \notin \mathscr{A}$. Without loss of generality we may assume that $\langle x \rangle$ contains no atoms. Note that it follows from Proposition 3.1(iii) that $L' \subset \langle x \rangle_n^\sim$ and that $\langle x \rangle$ is Dedekind complete. Let $0 \leq \phi \in L'$ satisfy $\phi(x) > 0$. By passing to the carrier of ϕ in $\langle x \rangle$, we may assume that ϕ is strictly positive on $\langle x \rangle$. It follows by a standard argument that for each $0 \leq z \leq x$ and real number α with $0 < \alpha \leq \phi(z)$, that there exists z', $0 \leq z' \leq z$, $z' \wedge (z - z') = 0$ and $\phi(z') = \alpha$. For $k = 1, 2, \ldots$, define $0 \leq z_j^{(k)}$, $j = 1, 2, \ldots, 2^k$ such that for each $k = 1, 2, \ldots$, $z_j^{(k)} \quad z_{j'}^{(k)} = 0$ if $j \neq j'$, $\sum_j z_j^{(k)} = x$, $\phi(z_j^{(k)}) = (1/2^k)\phi(x)$ and $z_j^{(k)} = z_{2j}^{(k+1)} + z_{2j+1}^{(k+1)}$. For each $k = 1, 2, \ldots$, set $x_k = \sum_j (-1)^j z_j^{(k)}$ and observe that $\phi(|x_k - x_{k'}|) = \frac{1}{2}\phi(x)$. If $x \neq 0$, this contradicts the assumed $|\phi|(L, L')$ compactness of $[0, x]$ and it follows that $x \in \mathscr{A}$.

Remark. The implication (ii) $\Rightarrow$ (i) is proved in [15] using methods of spectral theory. We have preferred to indicate a direct proof using standard techniques. Other implications of the Proposition are prompted by results of [7].

An inspection of the proof of the above Proposition with reference to Proposition 2.2 yields the following strengthening.

COROLLARY 4.2. *Let (L, T) be a locally convex Riesz space. The following conditions are equivalent.*

(i) *L is discrete, Dedekind complete and T is Lebesgue.*

(ii) *Each order interval of L is $|\sigma|(L, L')$ compact.*

(iii) *Each order interval of L is T-compact.*

(iv) *Each order bounded weakly convergent sequence is T-convergent and each order bounded T-Cauchy directed system in L is T-convergent.*

(v) *Each order bounded weakly Cauchy net in L is T-convergent.*

(vi) *The lattice operations in L' are $\sigma(L', L)$ continuous on the equicontinuous subsets of L' and each order bounded T-Cauchy directed system in L is T-convergent.*

The corollary extends Proposition 2.1 of [7].

PROPOSITION 4.3. *Let the locally convex Riesz space (L, T) be T-complete and let $\mathscr{A}$ be the band generated by the atoms of L. The following are equivalent for a bounded solid subset $A \subset L$.*

(i) *A is relatively T-compact.*

(ii) *Each disjoint sequence in A is T-convergent to* 0 *and $A \subset \mathscr{A}$.*

(iii) *A is relatively $\sigma(L, L')$ compact and each weakly convergent sequence in A is T-convergent.*

Proof. (i) $\Rightarrow$ (ii). Let $\{x_n\} \subset A$ be a disjoint sequence and let V be a solid neighborhood of 0. Since $\{x_n\}$ is totally bounded, there exists a finite set $\{x_i\}_{i=1}^k$ such that $\{x_n\} \subset \bigcup_{i=1}^k (x_i + V)$. If $n > k$, then $|x_n - x_i| \in V$ for some $1 \leq i \leq k$ and by disjointness, it follows that $|x_n| \leq |x_n| + |x_i| = |x_n - x_i| \in V$. It follows that $\{x_n\}$ is T-convergent to zero, and since each interval of A is T-compact, it follows from Proposition 4.1 that $A \subset \mathscr{A}$.

(ii) $\Rightarrow$ (iii). Without loss of generality we may, as in the proof of Proposition 3.2, assume that A is closed. Suppose now that $\{x_n\} \subset A$ is a sequence and that $x_n \to x_0$ $\sigma(L, L')$. Let V be a closed convex solid neighborhood of 0. By Proposition 2.2, given $0 < \varepsilon$, there exists $\phi_0 \in L'$ such that

$$(|\phi| - |\phi| \wedge |\phi_0|)(|x|) < \varepsilon \quad \text{for all } \phi \in V^0 \text{ and all } x \in A.$$

It follows, by Theorem 4.7 of [4], that $|x_n - x_0| \to 0$ $\sigma(L, L')$. Choose n_0 such that $|\phi_0|(|x_n - x_0|) < \varepsilon$ for $n \geq n_0$. Then

$$|\phi|(|x_n - x_0|) \leq (|\phi| - |\phi| \wedge |\phi_0|)(|x_n - x_0|) + |\phi_0|(|x_n - x_0|) \leq 3\varepsilon$$

for all $\phi \in V^0$ and $n \geq n_0$. Hence $\{x_n\}$ is T-convergent to x_0. By Proposition 3.2 it follows directly that A is relatively $\sigma(L, L')$ compact.

(iii) $\Rightarrow$ (i). We will show that A is totally bounded for T. If not, there exists a closed convex solid neighborhood V of 0 and a sequence $\{x_n\} \subset A$ for which $|x_n - x_m| \notin V$ for $n \neq m$. By Proposition 2.2, given $\varepsilon > 0$, there exist $0 \leq \phi_0 \in L'$ such that

$$(|\phi| - |\phi| \wedge \phi_0)(|x|) < \varepsilon \quad \text{for all } \phi \in V \text{ and all } x \in A.$$

Denote by P the band projection of $(L')_n^\sim$ onto the carrier band of ϕ_0 in $(L')_n^\sim$. Then $P(A) \subset A$ since A is a solid subset of $(L')_n^\sim$. It follows from Proposition 4.7 of [1] that $P(A)$ is $\sigma((L')_n^\sim, L')$ sequentially compact. Without loss of generality, we may assume that the sequence $\{Px_n\}$ is $\sigma((L')_n^\sim, L')$-Cauchy so that $P(x_{n+1} - x_n) \to 0$ $\sigma(L, L')$ and hence $P(x_{n+1} - x_n)$ is T-convergent to 0. Choose n_0 such that $\phi_0(|P(x_{n+1} - x_n)|) \leq \varepsilon$ for $n \geq n_0$. Then

$$\begin{aligned} |\phi|(|x_{n+1} - x_n|) &\leq (|\phi| - |\phi| \wedge \phi_0)(|x_{n+1} - x_n|) + |\phi| \wedge \phi_0(|x_{n+1} - x_n|) \\ &\leq 2\varepsilon + \phi_0(|P(x_{n+1} - x_n|) \\ &\leq 3\varepsilon \end{aligned}$$

for all $\phi \in V^0$ and all $n \geq n_0$. Consequently $|x_{n+1} - x_n| \in V$ for all $n \geq n_0$ and from this contradiction it follows that A is T-compact which completes the proof.

Combining Propositions 3.4 and 4.3 yields the following.

PROPOSITION 4.4. *Let the locally convex Riesz space (L, T) be T-complete. If L has the countable sup property, the following are equivalent for a solid subset $A \subset L$.*

(i) *A is relatively T-compact.*
(ii) *A is relatively sequentially T-compact.*

5. Semireflexivity

We first state the following immediate Corollary to Proposition 3.1.

PROPOSITION 5.1. *The following statements are equivalent for the locally convex Riesz space (L, T).*

(i) *L is an ideal in the bidual L''.*
(ii) *Each order interval of L is $\sigma(L, L')$ compact.*
(iii) *L is Dedekind complete and T is Lebesgue.*
(iv) *Each disjoint order-bounded sequence in L is T-convergent to 0 and each directed order-bounded T-Cauchy system in L^+ is T-convergent.*

PROPOSITION 5.2. *The following statements are equivalent for the locally convex Riesz space (L, T).*

(i) *L is a band in L''.*
(ii) *T is Levi and Lebesgue.*
(iii) *T is complete and each disjoint sequence in L with T-bounded partial sums is T-convergent to 0.*
(iv) *Each upwards directed T-bounded system in L^+ is T-convergent.*
(v) *L is $|\sigma|(L, L')$ complete.*

Proof. (i) $\Rightarrow$ (ii). That T is Lebesgue follows from Proposition 5.1. That T is Levi follows from the simply observed fact that if $0 \leq x_\tau \uparrow_\tau \subset L$ is $\sigma(L, L')$ bounded then $\sup_\tau x_\tau$ exists in L''.

(ii) $\Rightarrow$ (iii). That T is complete is a consequence of Nakano's theorem. Let $\{x_n\} \subset L^+$ be a disjoint sequence whose sequence of partial sums $\{\sum_{j=1}^{n} x_j\}$ is T-bounded. Since T is Levi and Lebesgue, it is immediate that the sequence $z_n = \sum_{j=n}^{\infty} x_j$ is T-convergent to 0 and (iii) follows.

(iii) $\Rightarrow$ (iv). Let $A \subset L^+$ be upwards directed T-bounded, $B \subset L'$ solid and equicontinuous. Let $\varepsilon > 0$ be given. It follows from (iii) and Proposition 2.2 that there exists $x_0 \in L^+$ such that $|\phi|(x - x \wedge x_0) < \varepsilon$ for all $\phi \in B$ and all $x \in A$. From (iii) and Proposition 2.1, it follows that the upwards directed system $x_0 \wedge A$ is T-Cauchy so there exists $x'' \in A$ such that $x, x' \in A$, $x, x' \geq x''$ imply $|\phi|(|x \wedge x_0 - x' \wedge x_0|) < \varepsilon$ for each $\phi \in B$. Consequently, for each $\phi \in B$ and $x, x' \in A$ with $x, x' \geq x''$, it follows that

$$|\phi|(|x - x'|) \leq |\phi|(x - x \wedge x_0) + |\phi|(x' - x' \wedge x_0) + |\phi|(|x \wedge x_0 - x' \wedge x_0|) \leq 3\varepsilon.$$

It follows that the system A is T-Cauchy and hence T-convergent since L is assumed T-complete.

The remaining implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) are easily established via Proposition 5.1 and the proofs are omitted.

PROPOSITION 5.3. *The following statements are equivalent for the locally convex Riesz space (L, T).*

(i) *L'' is the band generated by L in L''.*
(ii) *$\beta(L', L)$ is Lebesgue.*
(iii) *Each order bounded disjoint sequence in L' is $\beta(L', L)$ convergent to* 0.

Proof. (i) $\Rightarrow$ (ii). It follows from (i) that $L'' \subset (L')_n^\sim$. The proof of the implication (i) $\Rightarrow$ (ii) is then a routine modification of Lemma 22.6 of [8] and is accordingly omitted.

(ii) $\Rightarrow$ (iii). This implication follows directly from the fact that L' is Dedekind complete.

(iii) $\Rightarrow$ (ii). Since each order interval of L' is $\beta(L', L)$ complete, the implication follows from Proposition 5.1.

(ii) $\Rightarrow$ (i). Since $\beta(L', L)$ is Lebesgue, it follows that $L'' \subset (L')_n^\sim$ and the implication follows from the fact that the band generated by L in $(L')_n^\sim$ is $(L')_n^\sim$.

A locally convex topological vector space L is called semireflexive if L coincides with its bidual L''. The reader is referred to [14] for basic properties of semireflexive spaces. Semireflexive locally convex Riesz spaces have been characterized in [16]. The following result characterizes semireflexive locally convex Riesz spaces in terms of disjoint sequences.

PROPOSITION 5.4. *The following statements are equivalent for a locally convex Riesz space (L, T).*

(i) *L is semireflexive.*
(ii) *T is Levi and Lebesgue and $\beta(L', L)$ is Lebesgue.*
(iii) *L is T-complete, each equicontinuous disjoint sequence in L'^+ is $\sigma(L', L'')$ convergent to* 0 *and each T-bounded disjoint sequence in L^+ is $\sigma(L, L')$ convergent to* 0.
(iv) *L is T-complete, each disjoint sequence in L^+, whose sequence of partial sums is T-bounded, is T-convergent to* 0, *and each T-bounded disjoint sequence in L^+ is $\sigma(L, L')$ convergent to* 0.

Proof. Observe first that if L is semireflexive, it is immediate that T is Levi and from Proposition 5.1, it follows that T is Lebesgue. By Nakano's theorem, L is T-complete. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) follow from this remark and Propositions 5.2, 5.3, and 2.2.

(iii) $\Rightarrow$ (iv). If each equicontinuous disjoint sequence in L'^+ is $\sigma(L', L'')$ convergent to 0, then each order bounded disjoint sequence in L'' converges to 0 uniformly on equicontinuous subsets of L' by Proposition 2.2 since the order intervals of L' are $\beta(L', L)$ complete. Since each upwards directed T-bounded system in L^+ is order bounded in L'', it follows that each disjoint sequence in L^+ whose partial sums are T-bounded is T-convergent to 0.

(i) $\Rightarrow$ (iii). If L is semireflexive, then each order bounded disjoint sequence in L^+ is T-convergent to 0 by Proposition 5.1. Since L is T-complete, each equicontinuous disjoint sequence in L'^+ is $\sigma(L', L)$ convergent to 0. That each T-bounded disjoint sequence in L^+ is $\sigma(L, L')$ convergent to 0 is a consequence of Propositions 2.2 and 5.3.

A corollary of the preceding Proposition 5.4 is the following result given in [6].

COROLLARY 5.5. *The following properties of a Banach lattice L are equivalent.*

(i) *L is reflexive.*
(ii) *No closed sublattice of L or L' is order isomorphic to c_0.*
(iii) *No closed sublattice of L is order isomorphic to c_0 or l^1.*
(iv) *No closed sublattice of L or L' is order isomorphic to l^1.*

Remark. It is clear that a number of variants of Proposition 5.4 above may be obtained by various combinations of the conditions given in Propositions 5.2 and 5.3. The following is one such variant.

PROPOSITION 5.6. *The following statements are equivalent for the locally convex Riesz space (L, T).*

(i) *L is semireflexive.*
(ii) *L is $|\sigma|(L, L')$ complete and each T-bounded disjoint sequence in L^+ is $\sigma(L, L')$ convergent to 0.*

Recall that the Riesz space L with $L_n^\sim$ separating is called perfect if and only if $L = (L_n^\sim)_n^\sim$, see pp. 87–90 of [5].

COROLLARY 5.7. *Let L be an Archimedean Riesz space with $L_n^\sim$ separating on L. If T is any locally convex Riesz space topology on L consistent with the duality $(L, L_n^\sim)$, the following statements are equivalent.*

(i) *(L, T) is semireflexive.*
(ii) *L is perfect and each $\sigma(L, L_n^\sim)$ bounded disjoint sequence in L^+ is $\sigma(L, L_n^\sim)$ convergent to 0.*

COROLLARY 5.8. *Let L be a Banach lattice. If L' is separable, then L is reflexive if and only if L is perfect.*

Proof. Assume that L' is separable. It follows since L' is Dedekind complete that each order bounded disjoint sequence in L' is norm convergent to 0. By Proposition 2.1, $L' = L_n^\sim$ and by Proposition 2.2, each norm bounded disjoint sequence in L^+ is $\sigma(L, L')$ convergent to 0 and the result follows from Corollary 5.7.

LEMMA 5.9. *Let (L, T) be a locally convex Riesz space. If L'' is $\beta(L'', L')$ separable, then each interval in L' is $\beta(L', L)$ separable.*

Proof. Let $\{x_n''\}$ be a sequence which is $\beta(L'', L')$ dense in L''. Let $0 \leq \phi \in L'$, $0 \leq \psi \leq \phi$, and $x'' \in L''^+$. Without loss of generality, we may assume that $\psi(x'') \neq 0$. Let $\varepsilon > 0$ be given. Choose an integer m such that

$$m \geq \max\,(2\psi(x'')/\phi(x''),\ 1/\phi(x''))$$

and choose n such that $\phi(|x'' - x_n''|) < \varepsilon/2m$. In particular,

$$|\phi(x_n'')| \geq \phi(x'') - \varepsilon/2m \geq \phi(x'')/2$$

so that $|\phi(x_n'')|/\psi(x'') \geq \phi(x'')/2\psi(x'')$. Choose a rational number λ such that

$$\lambda \leq \frac{2\psi(x'')}{\phi(x'')} \quad \text{and} \quad \left|1 - \lambda\,\frac{\phi(x_n'')}{\psi(x'')}\right| \leq \frac{\varepsilon}{2\psi(x'')}.$$

Then

$$\begin{aligned} |\psi(x'') - \lambda\phi(x'')| &\leq |\psi(x'') - \lambda\phi(x_n'')| + \lambda\phi(|x_n'' - x''|) \\ &\leq \varepsilon/2 + \lambda\varepsilon/2m \\ &\leq \varepsilon. \end{aligned}$$

It follows that rational multiples of ϕ are $\sigma(L', L'')$ dense in $[0, \phi]$ and hence $\beta(L', L)$ dense in $[0, \phi]$.

PROPOSITION 5.10. *Let (L, T) be a T-complete locally convex Riesz space. If L'' is $\beta(L'', L')$ separable, then L is semireflexive.*

Proof. By Lemma 5.7, each order interval of L' is $\beta(L', L)$ separable. From the fact that L' is Dedekind complete, it follows that each order bounded disjoint sequence in L'^+ is $\beta(L', L)$ convergent to 0. Since L'' is Dedekind complete and $\beta(L'', L')$ separable by assumption, it follows similarly that each order bounded disjoint sequence in L''^+ is $\beta(L'', L')$ convergent to 0. Since each upwards directed T-bounded system in L^+ is order bounded in L'', it follows that each disjoint sequence in L^+ with T-bounded partial sums converges to 0 uniformly on each $\beta(L', L)$ bounded subset of L' and hence uniformly on each equicontinuous subset of L'. That L is semireflexive follows from Propositions 5.3 and 5.2.

For the case of Banach lattices, the above Proposition 5.8 is proved in [9].

PROPOSITION 5.11. *The following statements are equivalent for the locally convex Riesz space* (L, T).

(i) (L, T) *is reflexive.*

(ii) T *is the Mackey topology,* T *is complete, and* L' *is a band in* $L^\sim$. *Each* T*-bounded disjoint sequence in* L^+ *is* $\sigma(L, L')$ *convergent to* 0, *and each strongly bounded disjoint sequence in* L'^+ *is* $\sigma(L, L'')$ *convergent to* 0.

Proof. (i) $\Rightarrow$ (ii). Follows immediately from Proposition 5.4.

(ii) $\Rightarrow$ (i). Consider a disjoint sequence $\{x_n\} \subset L^+$ such that $\{\sum_1^n x_k\}$ is T-bounded. Then $x = \sup_n (\sum_1^n x_k)$ exist in L''. It then follows from Proposition 2.2 that $x_n \to 0$ $\beta(L, L')$. Thus (L, T) is semireflexive by Proposition 5.4 and hence $L' \subset L_n^\sim$. Since L' is a band in $L^\sim$ and also L' is $\sigma(L_n^\sim, L)$ dense in $L_n^\sim$, it follows that $L' = L_n^\sim$. Hence each bounded subset of $L' = L_n^\sim$ is relatively $\sigma(L', L)$ compact by Proposition 2.15 of [1]. Thus (L, T) is a barrelled space and hence (L, T) is reflexive.

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