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SOLUBLE GROUPS WITH EVERY PROPER QUOTIENT POLYCYCLIC

BY

J. R. J. GROVES

1. In this note, we investigate soluble groups which are not polycyclic but in which every proper quotient is polycyclic (hereafter referred to as JNP groups). We observe that such a group has the maximal condition on normal subgroups and so, by a result of Hall [1], is finitely generated. The main result is the following description of metanilpotent JNP groups.

THEOREM. *Let G be a metanilpotent JNP group. Then G is a subgroup of a split extension $G^* = A^* \cdot \bar{G}$ where:*

(i) *$\bar{G}$ is a finitely generated, nilpotent, centre by finite group.*

(ii) *A^* is a faithful, irreducible, finite dimensional $K\bar{G}$-module and K is either the field of rationals or the field of rational functions in one indeterminate over a finite prime field.*

(iii) *G is a supplement of A^* in G^*.*

(iv) *In the matrix representation of $\bar{G}$ defined by A^*, at least one eigenvalue is not an algebraic integer over K.*

Conversely, if H is a finitely generated supplement of A^* in G^*, then H is either a JNP group or is isomorphic to $\bar{G}$.

(For the present purposes, an algebraic integer over a field of rational functions is defined to be an element integral over the corresponding ring of polynomials.)

In particular, every metanilpotent JNP group is a finite extension of a metabelian group; the proof of the theorem, in fact, shows more.

COROLLARY 1. *A metanilpotent JNP group is a finite extension of a metabelian group and every proper quotient is centre by finite.*

Thus we have determined some groups which are not nilpotent by finite but in which every proper quotient is nilpotent by finite. Using arguments (which we omit) very similar to the proof of the theorem, we can in fact show the following.

COROLLARY 2. *Let G be a metanilpotent group which is not nilpotent by finite but in which every proper quotient is nilpotent by finite. Then G satisfies the conclusion of the theorem if* (iv) *is replaced by:*

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(iv)′ *In the matrix representation of $\bar{G}$ defined by A^*, at least one eigenvalue is not a root of unity.*

An analogous extension of this corollary also describes all infinite soluble groups with every proper quotient finite—the restriction to metanilpotent being unnecessary in this case—cf. McCarthy [3] and [4]. We observe that soluble groups which are just not supersoluble have been classified by R. Bentley in an unpublished dissertation at the University of Illinois.

The theorem clearly implies that every metanilpotent JNP group has a natural matrix representation. Metabelian JNP groups, in fact, have a particularly simple description.

COROLLARY 3. *A metabelian JNP group has a representation as a group of matrices of the form*

$$\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$$

over some finite extension of K.

(The detailed description of the theorem and the analogous result of Corollary 2 can also be applied here.)

The theorem will be proved in Section 2. The proofs of Corollaries 1 and 3 and some remarks occur in Section 3. We remark that, because we have to consider abelian normal subgroups both as subgroups and modules, we shall, according to the context, use both multiplicative and additive notation for their group operation.

I wish to acknowledge indebtedness to Dr. J. S. Wilson for some useful conversations prior to the commencement of this work.

2. Throughout this section, G will denote a metanilpotent JNP group. We begin with some straightforward observations about G. Let A denote the Fitting subgroup of G.

LEMMA 1. *The group A is abelian and is either* (i) *torsion-free or* (ii) *of prime exponent p.*

Proof. Since G is metanilpotent, A is nilpotent. If A is not abelian, then the derived group A' of A is a nontrivial normal subgroup of G and so G/A' is polycyclic. Hence A/A' is finitely generated and so A is finitely generated (Theorem 2.26 of Robinson [5]). Hence, as G/A is polycyclic, so is G—a contradiction. Thus A is abelian.

As we observed above, G has the maximal condition on normal subgroups. Hence the torsion subgroup T of A has finite exponent, k say, and so, if A^k is the subgroup of all kth powers in A, $T \cap A^k = \{1\}$. But T and A^k are normal in G and G clearly cannot be a finite nontrivial subdirect product. Thus $T = \{1\}$, giving case (i), or $A^k = \{1\}$. In the latter case it is easily seen that k must be prime, giving case (ii).

We now amalgamate the two cases of Lemma 1. Let R denote

(i) the ring of integers Z, if A is torsion-free,

(ii) the polynomial ring $GF(p)[t]$ over the field with p elements, if A has exponent p;

and let K be the ring of quotients of R. Let $\bar{G}$ denote G/A. In case (i), A is clearly an $R\bar{G}$-module. In case (ii), we use a technique of Hall [2] and turn A into an $R\bar{G}$-module by choosing an element z in the center of $\bar{G}$ and defining the action of t on A to be that given by conjugation by z. Since $\bar{G}$ is clearly infinite, we can choose z to be of infinite order (Theorem 2.24 of Robinson [5]).

LEMMA 2. *As an R-module, A is torsion-free of finite rank.*

Proof. We first show that A is a torsion-free R-module. This is clear in case (i). In case (ii), an argument similar to that of Lemma 1 shows that, if A is not torsion-free, then A has a nonzero annihilator in R. But then A can be annihilated by some element of the form $t^n - 1$ $(n \neq 0)$ and so the element z^n centralizes A. But A is the Fitting subgroup and so self centralizing in G. Since z^n is nontrivial in $\bar{G}$ (z was chosen to be an element of infinite order), we have a contradiction. Hence A is a torsion-free R-module.

By Lemma 5.2 and Lemma 6 of Hall [2], A has a free R-submodule B such that A/B is π-torsion, where π is some finite set of primes in R. Choose q to be a prime in R which is not in π. Then $A \cdot q$ is nontrivial and is a normal subgroup of G. Thus $A/A \cdot q$ is finitely generated as abelian group. Also, since $q \notin \pi$ and A/B is π-torsion, $A \cdot q \cap B = B \cdot q$. Hence $A/A \cdot q \cong B/B \cdot q$ and so $B/B \cdot q$ is finitely generated as abelian group. But B is a free R-module and so must be itself finitely generated. Hence as A/B is a torsion R-module, A has finite rank as R-module.

We now embed G in a larger group which is somewhat easier to handle. Let A^* denote $A \otimes_R K$. Then A^* is naturally a K-vector space (of finite dimension by the previous lemma) and so a KG-module; also A embeds in A^* by $\eta\colon a \to a \otimes 1$.

We now define an extension of A^* by $\bar{G}$, by taking the pushout of the diagram

$$\begin{array}{ccccc} 0 & A & G & \bar{G} & 1 \\ & {\scriptstyle\eta}\downarrow & \downarrow & \| & \\ 0 & A^* & G^* & \bar{G} & 1. \end{array}$$

Equivalently, η induces a map $\eta_*\colon H^2(\bar{G}, A) \to H^2(\bar{G}, A^*)$ of second cohomology groups and, identifying elements of these groups with equivalence classes of extensions, we define G^* to be the image of G under η_*.

LEMMA 3. *The group G^* defined above contains a subgroup isomorphic to G (which we identify with G) and $GA^* = G^*$, $G \cap A^* = A$. Also A^* is the Fitting subgroup of G^*.*

We now proceed to show that A^* is an irreducible $K\bar{G}$-module, acquiring some extra information in the course of the proof.

LEMMA 4. *The group $\bar{G}$ is center by finite.*

Proof. Let $\{0\} = A_0 < A_1 < \cdots < A_n = A^*$ be a $K\bar{G}$ composition series of A^*. By a theorem of Suprunenko (Theorem 3.13 of Wehrfritz [8]), a nilpotent group with a faithful irreducible representation of finite degree has center of finite index depending only on its class and the degree of the representation. Hence we can find a subgroup $\bar{H}$ of finite index in $\bar{G}$ such that $[\bar{H}, \bar{G}]$ stabilizes the composition series.

Thus we can find a normal subgroup H of finite index in G such that $[H, G^*]$ stabilizes the composition series of A^*. But then $[H, G^*]$ is nilpotent and so lies in the Fitting subgroup A^* of G^*. Hence HA^*/A^* lies in the center of G^*/A^* and has finite index in G^*/A^*.

This shows that $[\bar{H}, \bar{G}] = \{1\}$ and so $\bar{G}$ is center by finite.

LEMMA 5. *Suppose I is the annihilator of A^* in $K\bar{G}$ and W is the center of $\bar{G}$. If $\alpha, \beta \in KW$, then $\alpha\beta \in I$ implies $\alpha \in I$ or $\beta \in I$.*

Proof. There exist $r, s \in R$ such that $r\alpha, s\beta \in R\bar{G}$. Suppose $\alpha \notin I$. Then $r\alpha \notin I$ and so $A \cdot r\alpha \neq \{0\}$. Since α is central in $K\bar{G}$, $A \cdot r\alpha$ is a G-normal subgroup of A and so, as G is a JNP group, $A/A \cdot r\alpha$ is finitely Z-generated, by $a_1, \ldots, a_n$ say. Hence $A \cdot s\beta$ is finitely Z-generated by $a_1 s\beta, \ldots, a_n s\beta$. But $A \cdot s\beta$ is a G-normal subgroup of A and G can clearly have no nontrivial polycyclic normal subgroups. Thus $A \cdot s\beta = \{0\}$ and so $\beta \in I$.

LEMMA 7. *As $K\bar{G}$-module, A^* is irreducible.*

Proof. By Lemma 6, $KW/KW \cap I$ has no divisors of zero; it is also a finite dimensional commutative K-algebra and so must be a field. Hence A^* is a completely reducible KW-module.

Suppose K has finite characteristic p. If $\bar{G}/W$ contains an element of order p, then so does W (Theorem 2.25 of Robinson [5]). But this gives rise to a nilpotent normal subgroup of G^* properly containing the Fitting subgroup A^*. Hence the order of $\bar{G}/W$ is not divisible by the characteristic of K and so the analogue of Maschke's theorem for infinite groups (Theorem 1.5 of Wehrfritz [8]) applies to show that A^* is completely reducible as $K\bar{G}$-module.

It follows, from the fact that A is finitely subdirectly irreducible as $R\bar{G}$-module, that A^* is finitely subdirectly irreducible as $K\bar{G}$-module. Hence, since A^* is completely reducible, it is irreducible.

Proof of the theorem. To prove the first part of the theorem, it only remains to show that the extension of G^* by A^* splits. But this is a direct consequence of results of Robinson (see, for example, Corollary AB(ii) of [7]).

We now turn to the converse; we retain the notation used previously. Firstly, let C be any nontrivial $R\bar{G}$-submodule of A^*. Then, as A^* is an irreducible $K\bar{G}$-module, $C \otimes_R K = A^*$ and so A^*/C is a torsion R-module.

Now let H be any finitely generated supplement of A^* in G^*. If $H \cap A^* = \{1\}$, then $H \cong \bar{G}$. So suppose $H \cap A^* = B > \{1\}$. Let C be any H-normal subgroup of B; by the previous comment, B/C is a torsion R-module. But, because H is a finitely generated abelian by nilpotent group, B/C is finitely generated as RH-module, and so as $R\bar{G}$-module. Thus B/C has a nonzero annihilator in R. Finally, B/C has finite rank as R-module. It is now straightforward to show that B/C is finite.

We have shown that, if C is any nontrivial normal subgroup of H lying in A^*, then H/C is polycyclic. We now show that if D is a normal subgroup of H with $D \cap B = \{1\}$, then $D = \{1\}$. But $D \cap B = \{1\}$ implies D centralizes B and so D centralizes A^* (since $B \otimes_R K = A^*$). Hence, as A^* is the Fitting subgroup of G^* and so self-centralizing, $D \leq A^*$. But then $D \leq H \cap A^* = B$ and so $D = \{1\}$. Thus any proper quotient of H is polycyclic.

It remains to show that B is not polycyclic. If B is polycyclic and so finitely generated, then, for each $g \in \bar{G}$ and $b \in B$, $\{b \cdot g^i \colon i = 0, 1, 2, \ldots\}$ is finitely generated and so there exists a monic polynomial $f_b(g)$ over R such that $b \cdot f_b(g) = 0$. Again since B is finitely generated, we can choose $f_b(g)$ independent of b. But this implies that the matrix representing g over K has a monic minimum polynomial over R. Thus all the eigenvalues are algebraic integers, contrary to the assumption of the theorem.

Thus B, and so also H, is not polycyclic and the proof of the theorem is complete.

3. *Proof of Corollary* 1. Let H be a metanilpotent JNP group and A its Fitting subgroup. It follows from the description given in the theorem that, if N is a nontrivial normal subgroup of H, then $A/A \cap N$ is finite. Thus $H/A \cap N$ is finite by center by finite and so has finite derived group. Hence it is a group with finite conjugacy classes and so, being finitely generated, is center by finite.

Proof of Corollary 3. Since G is metabelian, $\bar{G}$ is abelian and so, because A^* is a faithful irreducible $K\bar{G}/I$ module, the latter is a field, F say. Clearly F is a finite extension of K. Also the representation of $\bar{G}$ on A^* is just that of a subgroup of the multiplicative group of F on the additive group of F, by multiplication. But it is well known, and easily proved, that the corresponding split extension has a representation by matrices of the type described.

Remark 1. Every split extension $A^* \cdot \bar{G}$ of the type we describe can be "induced" in a natural way from a split extension corresponding to a metabelian JNP group. More precisely, let W be the center of $\bar{G}$ and let A_1^* be an irreducible KW-submodule of A^* (all such submodules are isomorphic). Then the split extension $A_1^* \cdot W$ is one corresponding to a metabelian JNP group. If we now form the induced module $A_1^* \otimes_{KW} K\bar{G}$, then A^* occurs as a component of this module and so $A^* \cdot \bar{G}$ occurs as a quotient of the "induced" group $(A_1^* \otimes_{KW} KG) \cdot \bar{G}$.

Remark 2. Although a JNP group is not, in general, a split extension of the Fitting subgroup, it lies "sandwiched" between two split extensions. More precisely, it follows from Theorem D of Robinson [7], that the cohomology group $H^2(\bar{G}, A)$ is bounded as an R-module (a straightforward extension of Theorem D is required when R is $GF(p)[t]$). But now, if $H^2(\bar{G}, A) \cdot r = \{0\}$ for some $0 \neq r \in R$, then the map $\bar{r}$: $A \rightarrow A^*$ defined by $a \rightarrow ar$ extends to a monomorphism ρ: $G \rightarrow A \cdot \bar{G}$ (where $A \cdot \bar{G}$ denotes the split extension of A by $\bar{G}$). Also $(G\rho) \cap (Ar \cdot \bar{G}) = (G\rho \cap G) \cdot Ar$ has finite index in $G\rho$ and, as A is R-torsion free, $Ar \cong A$. Hence G contains a subgroup of finite index which is a split extension of A by a subgroup of $\bar{G}$ (cf. Lemma 10 of Robinson [6]).

Thus a JNP group both embeds in a most "natural" split extension and contains a subgroup of finite index which is a split extension.

(I am grateful to the referee for drawing my attention to these facts.)

Remark 3. We comment briefly on the question of soluble JNP groups in general; clearly all such groups are abelian by polycyclic. If the Fitting subgroup is also torsion-free, then the proof given here can be adjusted to yield results similar to those given here (the main change being that $\bar{G}$ will be abelian by finite but not necessarily center by finite). If, however, the Fitting subgroup has finite exponent, then the technique of using $GF(p)[t]$ is no longer available and the problem appears to enter a new area of complexity.

REFERENCES

1. P. HALL, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. (3), vol. 4 (1954), pp. 419–436.
2. ———, *On the finiteness of certain soluble groups*, Proc. London Math. Soc. (3), vol. 9 (1959), pp. 595–622.
3. D. MCCARTHY, *Infinite groups whose proper quotients are finite I*, Comm. Pure Appl. Math., vol. 21 (1968), pp. 545–562.
4. ———, *Infinite groups whose proper quotients are finite II*, Comm. Pure Appl. Math., vol. 23 (1970), pp. 767–789.
5. D. J. S. ROBINSON, *Finiteness conditions and generalized soluble groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vols. 62 and 63, Springer-Verlag, Berlin, 1972.
6. ———, *Splitting theorems for infinite groups*, Symposia Math., vol. 17 (1976), pp. 441–470.
7. ———, *The vanishing of certain homology and cohomology groups*, J. Pure Appl. Algebra, vol. 7 (1976), pp. 145–167.
8. B. A. F. WEHRFRITZ, *Infinite linear groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 76, Springer-Verlag, Berlin, 1973.

UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA, AUSTRALIA