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ON NILPOTENT ALGEBRAS

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Let G be a group with normal subgroup H and let H' be the derived group of H. P. Hall has shown that if H and G/H' are nilpotent, then G is nilpotent. This result has led to investigations in several directions. In particular, Robinson has provided a procedure which yields a variety of results of this type. On the other hand, Chao has found the Lie algebra analogue to Hall's result. It is the purpose of this note to consider Robinson's method in the area of nonassociative algebras. As consequences, generalizations of the result of Chao and of a result of Ravisankar on characteristically nilpotent algebras are obtained as well as analogues to some group theoretic results.

The construction will be developed for nonassociative algebras A over a commutative ring Φ with identity and the algebras will be unital Φ-modules. For most of the consequences, an algebra D of operators on A will be used and it will be necessary to restrict Φ to be a field. Furthermore, restrictions will be placed on the manner in which the operators act on A for the following reason. If A^n is the subalgebra of A generated by all elements of A with at least n factors (no matter how associated), then we want A^n to be D-invariant. The case when $D = A$ has been investigated by Zwier [15] where classes of algebras with this property are called 2-varieties. We shall require that the following identities be satisfied. There exist α_i, $\beta_j \in \Phi$ such that

$$(1)\qquad (ab)d = \alpha_1(da)b + \alpha_2(ad)b + \alpha_3 b(da) + \alpha_4 b(ad) + \alpha_5(db)a + \alpha_6(bd)a + \alpha_7 a(db) + \alpha_8 a(bd)$$

and

$$(2)\qquad d(ab) = \beta_1(da)b + \beta_2(ad)b + \beta_3 b(da) + \beta_4 b(ad) + \beta_5(db)a + \beta_6(bd)a + \beta_7 a(db) + \beta_8 a(bd).$$

for all $a, b \in A, d \in D$. If $D = A$ satisfies (1) and (2), then Anderson [2] has noted that A is in a 2-variety. Earlier, Albert [1] considered such algebras and called them almost alternative. They include Lie, associative and alternative algebras as well as (γ, δ)-algebras introduced by Albert and investigated many places (see [9]). We mention also that 2-varieties are considered in [3].

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1. The construction

Let Φ be a commutative ring with identity and let A be a unital Φ-algebra. Let $A^1 = A$ and let A^n be the set of all Φ-linear combinations of all elements of A which have at least n factors from A. Set $F_n = A^n/A^{n+1}$ for each n. For each positive integer n, let $X = \{x_1, \ldots, x_n\}$ and let $f = f(x_1, \ldots, x_n)$ be a multilinear monomial in $\mathfrak{F}(X)$, the free nonassociative algebra over Φ generated by X. Let $F = F_1$ and, for $i = 1, \ldots, n$, let $\bar{a}_i \in F$. The mapping from $F \times \cdots \times F$ into F_n given by $(\bar{a}_1, \ldots, \bar{a}_n) \to f(a_1, \ldots, a_n) + A^{n+1}$ is clearly well defined. Since f is multilinear, there exists a Φ-homomorphism $\pi(f)$ from $\otimes^n F$ into F_n given by

$$\bar{a}_1 \otimes \cdots \otimes \bar{a}_n \to f(a_1, \ldots, a_n) + A^{n+1}.$$

For fixed n, A^n is the sum of a finite number of homomorphic images of $\otimes^n F$. For this we let f vary over the possible multilinear monomials (and to obtain a smaller number of summands we can demand that in each monomial, x_i comes before x_{i+1}, $i = 1, \ldots, n-1$). Consequently we have:

LEMMA 1. *Let Λ be a class of Φ-modules such that if $B, C \in \Lambda$, then every homomorphic image of $B \otimes C$ is in Λ and Λ is closed under forming extensions. If A is a nonassociative algebra over Φ, then $A/A^2 \in \Lambda$ implies that $A^i/A^{i+1} \in \Lambda$ for each i. If, in addition, A is nilpotent, then $A \in \Lambda$.*

Proof. Each $F_i = A^i/A^{i+1}$ is a finite sum of homomorphic images of $\otimes^i F$, hence is an extension of elements in Λ and $F_i \in \Lambda$. If A is nilpotent, then $A \in \Lambda$.

Example. Let A be a nilpotent ring and suppose that A/A^2 satisfies the descending chain condition on additive subgroups. This is equivalent to A/A^2 being a finite direct sum of subgroups of type $Z(p^i)$, $i = 1, \ldots, \infty$, where the p's can vary [6, p. 110]. The tensor product of groups of this type remains of this type and the descending chain condition is closed under extensions. Hence, Lemma 1 applies and A satisfies the descending chain condition for subgroups. Furthermore, if A/A^2 satisfies the descending chain condition for left (right, 2-sided) ideals, then A/A^2 satisfies the descending chain condition for subgroups. It follows that A satisfies the descending chain condition for left (right, 2-sided) ideals.

Example. Let A be a nilpotent ring such that A/A^2 satisfies the ascending chain condition for additive subgroups. This is equivalent to A/A^2 being finitely generated, a condition which meets the requirements in the lemma. It now follows that A also satisfies this chain condition. As in the last example, the chain condition on subgroups can be replaced by one on left (right, 2-sided) ideals. This is the analogue of a result of Baer [4].

2. Applications

Throughout this section Φ will be a field. Let A be a Φ-algebra and D be a Φ-algebra of operators on A such that the identities (1) and (2) are satisfied.

These conditions on A and D will be assumed throughout the remainder of the paper.

Example. Let A be any algebra and let D be an algebra of derivations (acting on the right) of A. Let D act trivially on the left of A. Then A and D satisfy (1) and (2).

Example. Let $A = D$ be any Lie, alternative or (γ, δ)-algebra. Then conditions (1) and (2) are satisfied.

Let $f = f(x_1, \ldots, x_n)$ be as in the first section and let $\pi(f)$ be the induced map from $\otimes^n F$ into F_n. Since each $d \in D$ satisfies (1) and (2), d can be made to act in the natural way (on both sides) of $\otimes^n F$ so that d commutes with $\pi(f)$. This action depends on (1), (2) and the multilinear monomial f. We now reconsider the definition of Λ given in Lemma 1 to incorporate the action of D. We note that by a module M we mean a D-module where D is a Φ-algebra and by submodule or homomorphism we mean D-submodule or D-homomorphism. We assume that each class of modules is closed under isomorphic images and contains a zero module.

DEFINITION. Let $\mathfrak{X}$ be a class of Φ-modules. Then $M \in P\mathfrak{X}$ if M has a series of submodules of finite length whose factors belong to $\mathfrak{X}$.

DEFINITION. A class $\mathfrak{T}$ of Φ-modules is called tensorial if given D-modules A and B in $\mathfrak{T}$, then every D-homomorphic image of $A \otimes B$ is in $\mathfrak{T}$ for every D-module $A \otimes B$ defined by (1) and (2) for all choices of α_i, β_j in these identities.

Example. The class of trivial modules and the class of one dimensional modules are tensorial.

DEFINITION. A class of Φ-modules $\mathfrak{X}$ is called persistent if for every algebra A which admits D as an algebra of operators with action prescribed by an identity of type (1) and one of type (2) the following holds: If $A/A^2 \in \mathfrak{X}$, then $A^i/A^{i+1} \in \mathfrak{X}$ for all finite i.

Note that by (1) and (2), each A^i is D-invariant.

LEMMA 2. *If $\mathfrak{T}$ is tensorial and $A/A^2 \in \mathfrak{T}$, then $A^i/A^{i+1} \in P\mathfrak{T}$ for all finite i. If, in addition, A is nilpotent, then $A \in P\mathfrak{T}$.*

Proof. This follows by the argument in Section 1.

Robinson [13] has shown that in group theory tensorial classes of modules are persistent. This does not seem to be the case here, but it suffices for the applications to show that if $\mathfrak{T}$ is tensorial, then $P\mathfrak{T}$ is persistent. We now turn to this result, the key being the following lemma whose proof is the same as in [13].

LEMMA 3. *If $\mathfrak{T}$ is tensorial, then $P\mathfrak{T}$ is tensorial.*

Proof. Let $\mathfrak{T}$ be tensorial, A and B be D-modules and $A, B \in P\mathfrak{T}$. Let $T = A \otimes B$ and let D act on $A \otimes B$ according to (1) and (2). There exist series of D-submodules

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_m = A \quad \text{and} \quad 0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n = B$$

where each factor of successive terms is in $\mathfrak{T}$. Let T_i be the subspace of T generated by all $a \otimes b$ where $a \in A_j$, $b \in B_k$ and $j + k \leq i$. Each T_i is clearly a D-submodule of T and $0 = T_0 = T_1 \subseteq \cdots \subseteq T_{n+m} = T$. The map

$$(a + A_j, b + B_k) \to a \otimes b + T_i, \quad a \in A_{j+1}, b \in B_{k+1}, j + k = i - 1$$

is well defined and bilinear, hence there exists a Φ-homomorphism from $A_{j+1}/A_j \otimes B_{k+1}/B_k$ into T_{i+1}/T_i and this mapping is a D-homomorphism and by assumption, the image of this mapping is in $\mathfrak{T}$. Since T_{i+1}/T_i is a finite sum of these images, $T_{i+1}/T_i \in P\mathfrak{T}$. Hence $T \in P\mathfrak{T}$ and any homomorphic image of T is in $P\mathfrak{T}$.

THEOREM 1. *If $\mathfrak{T}$ is tensorial, then $P\mathfrak{T}$ is persistent.*

Proof. By Lemma 3, $P\mathfrak{T}$ is tensorial. If $A/A^2 \in P\mathfrak{T}$, then by Lemma 2, $A^i/A^{i+1} \in P(P\mathfrak{T})$ for each i. Clearly $P(P\mathfrak{T}) = P\mathfrak{T}$ and $P\mathfrak{T}$ is persistent.

COROLLARY. *If $\mathfrak{T}$ is tensorial and A is nilpotent, then $A/A^2 \in P\mathfrak{T}$ implies that $A \in P\mathfrak{T}$.*

Example. Let D be a Φ-algebra and let $\mathfrak{X}$ be the class of all trivial D-modules. If M is a D-module in $P\mathfrak{X}$, then D acts nilpotently on M. Let A be a nilpotent algebra on which D acts with action satisfying (1) and (2). If $A/A^2 \in P\mathfrak{X}$, then $A \in P\mathfrak{X}$ by the corollary. In particular, if D is an algebra of derivations of A, then the result holds. Suppose further that A is a characteristic ideal in B and that D is a collection of derivations of B which act nilpotently on the vector space B/A^2. Then D acts nilpotently on A. In particular we have the following generalization of a result of Ravisankar [12]. Recall that A is characteristically nilpotent if $A \in P\mathfrak{X}$ where $\mathfrak{X}$ is the class of all trivial D-modules where D is the derivation algebra of A.

COROLLARY. *Let B be a nonassociative algebra with nilpotent characteristic ideal A. Suppose that the derivation algebra of B acts nilpotently on the vector space B/A^2. Then B is characteristically nilpotent.*

If we let $\mathfrak{X}$ be the class of all one dimensional D-modules, then we obtain a similar result about characteristically supersolvable algebras.

DEFINITION. Let $\mathfrak{X}$ be a class of Φ-modules. The class of algebras $\mathfrak{D}\mathfrak{X}$ is defined to consist of all algebras A with a normal series of finite length which when regarded as A-modules belong to $\mathfrak{X}$.

THEOREM 2. *Let $\mathfrak{X}$ be a class of modules and let H be a nilpotent ideal in the algebra A where the identities* (1) *and* (2) *hold in A. Assume that $A/H^2 \in \mathfrak{D}\mathfrak{X}$. Then $A \in \mathfrak{D}\mathfrak{X}$ provided that:*

(1) *$\mathfrak{X}$ is tensorial.*

(2) *If B is a D-module belonging to $\mathfrak{X}$ and B_0 is a D-submodule of B, then B_0 belongs to $\mathfrak{X}$.*

Proof. Assume that $A/H^2 \in \mathfrak{D}\mathfrak{X}$. Then the A-module $H/H^2 \in P\mathfrak{X}$. Hence $H \in P\mathfrak{X}$ by the corollary to Theorem 1. Hence $A \in \mathfrak{D}\mathfrak{X}$.

Let $\mathfrak{X}$ be the class of all trivial modules. Then $\mathfrak{D}\mathfrak{X}$ is the class of all nilpotent algebras. When (1) and (2) are the identities obtained from the Jacobi identity, the theorem gives (in a slightly generalized form) Chao's theorem for Lie algebras [5]. We can also obtain a version for alternative algebras, (γ, δ)-algebras and for almost alternative algebras by varying (1) and (2). For example we have:

COROLLARY. *Let A be an almost alternative algebra and H be a nilpotent ideal of A. If A/H^2 is nilpotent, then A is nilpotent.*

Let $\mathfrak{X}$ be the class of one dimensional modules. Then $\mathfrak{D}\mathfrak{X}$ is the class of supersolvable algebras. Then Theorem 2 yields the following.

COROLLARY. *Let A be an almost alternative algebra and H be a nilpotent ideal of A. If A/H^2 is supersolvable, then A is supersolvable.*

3. Local classes

Just as in group theory, there are some results on local classes and the development parallels that of Robinson [13]. Consequently full proofs of the following are not necessary. We begin by mentioning some results on finitely presented algebras. For results on free algebras see [11, Chapter 1]. Let Φ be a commutative ring with 1: all algebras will be unital Φ-modules. Let A be a Φ-algebra. Then A is the homomorphic image of a free algebra F. If A is finitely generated, then F may be assumed to have a finite number of generators. Hence we have an exact sequence

$$0 \to R \to F \xrightarrow{\theta} A \to 0.$$

This sequence is called a finite presentation of A if F is finitely generated by $\{a_1, \ldots, a_n\}$ and if there exist a finite collection $\{\rho_1, \ldots, \rho_m\}$ of elements in F such that the kernel of θ is the smallest ideal in F which contains $\{\rho_1, \ldots, \rho_m\}$. Then A is said to be finitely presented by the generators $\{a_1, \ldots, a_n\}$ subject to the relations $\{\rho_1, \ldots, \rho_m\}$. The following results parallel the group case (see [14, p. 31–33]). The group theoretic analogue of the first theorem was obtained by B. H. Neumann in [10]. The next two theorems are analogous to group theoretic results shown by P. Hall in [7].

THEOREM A. *If A is finitely presented, then for any finite collection of generators there exist a finite number of relations.*

THEOREM B. *If A is finitely generated and N is an ideal in A such that A/N is finitely presented, then N is finitely generated as an A-module.*

THEOREM C. *The class of finitely presented algebras is closed with respect to forming extensions.*

COROLLARY D. *Finitely generated nilpotent algebras and supersolvable algebras are finitely presented.*

DEFINITION. If $\mathfrak{X}$ is a class of modules, then the class $L\mathfrak{X}$ is defined as follows. Let M be a D-module. $M \in L\mathfrak{X}$ if given finite subsets $F \subseteq M$ and $G \subseteq D$, there exists a subalgebra $D_0 \subseteq D$ and D_0-module M_0 contained in M such that $F \subseteq M_0$, $G \subseteq D_0$, and $M_0 \in \mathfrak{X}$.

LEMMA 4. *Let $\mathfrak{X}$ be a class of Φ-modules where Φ is a field. Then $L\mathfrak{X}$ is tensorial if the following conditions are satisfied:*

(1) *$\mathfrak{X}$ is tensorial.*
(2) *If M is a D-module belonging to $\mathfrak{X}$ and D_0 is a subalgebra of D, then M regarded as a D_0-module belongs to $\mathfrak{X}$.*

Proof. See [13, p. 229].

THEOREM 3. *Let $\mathfrak{X}$ be a class of Φ-modules where Φ is a field. Let H be a nilpotent ideal in the algebra A where* (1) *and* (2) *are satisfied in* A. *Assume that $A/H^2 \in L(\mathfrak{D}\mathfrak{X})$. Then $A \in L(\mathfrak{D}\mathfrak{X})$ if the following are satisfied:*

(1) *$\mathfrak{X}$ is tensorial.*
(2) *If B is a D-module belonging to $\mathfrak{X}$, D_0 is a subalgebra of D and B_0 is a D_0-module contained in B, then $B_0 \in \mathfrak{X}$.*
(3) *Finitely generated algebras in $\mathfrak{D}\mathfrak{X}$ are finitely presented.*

Proof. Let $A/H^2 \in L(\mathfrak{D}\mathfrak{X})$. Then H/H^2 regarded as an A-module is in $L(P\mathfrak{X})$. Now $L(P\mathfrak{X})$ is tensorial by Lemma 3 and Lemma 4. Hence $H \in P(L(P\mathfrak{X}))$ by Lemma 2. Hence there exists a series $H = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n = 0$ of A-modules such that $H_i/H_{i+1} \in L(P\mathfrak{X})$. The remainder of the proof follows the proof of [13, Theorem 3] using this series in place of the lower central series of H.

If $\mathfrak{X}$ is the class of trivial modules, then $L(\mathfrak{D}\mathfrak{X})$ is the class of locally nilpotent algebras. Hence:

COROLLARY. *If A is almost alternative and if H is a nilpotent ideal of A such that A/H^2 is locally nilpotent, then A is locally nilpotent.*

If $\mathfrak{X}$ is the class of 1-dimensional modules, then $L(\mathfrak{D}\mathfrak{X})$ is the class of locally supersolvable algebras. Hence there is a locally supersolvable analogue to the above corollary.

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