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LOCALLY ANALYTICALLY CONNECTED TOPOLOGIES

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We shall prove the existence of two natural analytic topologies which refine the given topology on a space with a specified collection of continuous complex-valued functions. Whereas one would expect [2] the two constructions always to yield the same analytic refinement, we show by example that is not the case.

This work was motivated by the search for analytic structure in the maximal ideal space of a function algebra, in particular the existence of nontrivial holomorphic mappings of analytic varieties into the maximal ideal space. (For restrictions on what type of analytic varieties need be considered, see Section 4.)

1. A smallest locally analytically connected topology

We will use a construction similar to the case of associating a smallest locally connected or locally arc connected refinement to a given topology on a space [2].

Let $\mathscr{V}$ and $\mathscr{X}$ be two subcategories of topological spaces, $\mathscr{F}$ a collection of continuous maps from the members of $\mathscr{V}$ to the members of $\mathscr{X}$, such that the composition of a morphism in $\mathscr{X}$ with a map in $\mathscr{F}$ again belongs to $\mathscr{F}$. Then replace the topology of a member of $\mathscr{X}$ with the finest topology which preserves the continuity of maps in $\mathscr{F}$.

Now let $\mathscr{V}$ be all connected analytic subvarieties of open domains in $\mathbf{C}^n$, let $\mathscr{X}$ consist of topological spaces with a specified collection of continuous complex-valued functions defined on the space. A member of $\mathscr{X}$ will be denoted $(X, \mathscr{T}, \mathscr{G})$, where $\mathscr{T}$ is the topology on X and $\mathscr{G} \subset \mathscr{C}(X, \mathscr{T})$. Let $\mathscr{F}$ consist of all continuous maps F from $V \in \mathscr{V}$ to $(X, \mathscr{T}, \mathscr{G}) \in \mathscr{X}$ such that $g \circ F \in \mathscr{H}(V)$ for all $g \in \mathscr{G}$. We will call such an F a holomorphic mapping of V into $(X, \mathscr{T}, \mathscr{G})$. Let the morphisms in $\mathscr{X}$ be those

$$f\colon (X_1, \mathscr{T}_1, \mathscr{G}_1) \to (X_2, \mathscr{T}_2, \mathscr{G}_2)$$

with $f\colon (X_1, \mathscr{T}_1) \to (X_2, \mathscr{T}_2)$ continuous and $g_2 \circ f \circ F \in \mathscr{H}(V)$ for all $g_2 \in \mathscr{G}_2$ whenever $F \in \mathscr{F}$ and $F\colon V \to (X_1, \mathscr{T}_1, \mathscr{G}_1)$.

Let $(X, \mathscr{T}, \mathscr{G}) \in \mathscr{X}$. For any subset $S \subset X$, $s, t \in S$, we say $s \sim t$ if there exists $s = s_0, s_1, \dots, s_n = t \in S$, $V_j \in \mathscr{V}$ and $F_j \in \mathscr{F}$ with $s_{j-1}, s_j \in F_j(V_j) \subset S$ for $j = 1, \dots, n$. The resulting equivalence classes will be called the analytic components of S.

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LEMMA 1. *For* $(X, \mathcal{T}, \mathcal{G}) \in \mathcal{X}$, *let* $\mathcal{B}(\mathcal{T})$ *be the set of all analytic components of members of* $\mathcal{T}$. *Then* $\mathcal{B}(\mathcal{T})$ *is a base for a topology* $\mathcal{T}'$ *on* X *with the following properties.*

(i) *Let* $(X, \mathcal{T}, \mathcal{G}) \in \mathcal{X}$, $F \in \mathcal{F}$. *If* $F: V \to (X, \mathcal{T})$ *is continuous then*

$$F: V \to (X, \mathcal{T}')$$

is continuous.

(ii) *If* $f: (X_1, \mathcal{T}_1, \mathcal{G}_1) \to (X_2, \mathcal{T}_2, \mathcal{G}_2)$ *is a morphism in* $\mathcal{X}$ *then*

$$f: (X_1, \mathcal{T}'_1) \to (X_2, \mathcal{T}'_2)$$

is continuous.

(iii) $(X, \mathcal{T}, \mathcal{G})$ *is locally analytically connected, that is every* $x \in X$ *has arbitrarily small analytically connected neighborhoods in* $\mathcal{T}$, *if and only if* $\mathcal{T}' = \mathcal{T}$.

(iv) *Let* $(X, \mathcal{T}, \mathcal{G}) \in \mathcal{X}$. *Then* $(X, \mathcal{T}', \mathcal{G})$ *is locally analytically connected.*

Proof. Part (i) follows since each $V \in \mathcal{V}$ is locally connected. Part (ii) consists of verifying that $f^{-1}(B) \in \mathcal{T}'_1$ for all $B \in \mathcal{B}(\mathcal{T}_2)$. Part (iii) follows from the altèrnate characterization of locally analytically connected as analytic components of open sets are open, i.e., $(X, \mathcal{T}, \mathcal{G})$ is locally analytically connected if and only if $\mathcal{B}(\mathcal{T}) \subset \mathcal{T}$. To prove part (iv), note that by definition of $\mathcal{T}'$ each point in X has arbitrarily small $\mathcal{T}'$-neighborhoods which are analytically connected in $(X, \mathcal{T}, \mathcal{G})$, hence by part (i), also analytically connected in $(X, \mathcal{T}', \mathcal{G})$.

THEOREM 2. *For* $(X, \mathcal{T}, \mathcal{G}) \in \mathcal{X}$, $\mathcal{T}'$ *is the smallest topology containing* $\mathcal{T}$ *for which* $(X, \mathcal{T}', \mathcal{G})$ *is locally analytically connected.*

Proof. If $(X, \mathcal{T}_0, \mathcal{G})$ is locally analytically connected with $\mathcal{T} \subset \mathcal{T}_0$, then applying part (ii) of Lemma 1 to the identity map on X, we have $\mathcal{T}' \subset \mathcal{T}'_0$, but by part (iii) of Lemma 1, $\mathcal{T}'_0 = \mathcal{T}_0$.

THEOREM 3. *If* $(X_1, \mathcal{T}_1, \mathcal{G}_1)$ *is locally analytically connected and*

$$f: (X_1, \mathcal{T}_1, \mathcal{G}_1) \to (X_2, \mathcal{T}_2, \mathcal{G}_2)$$

is a morphism in $\mathcal{X}$, *then*

$$f: (X_1, \mathcal{T}_1, \mathcal{G}_1) \to (X_2, \mathcal{T}'_2, \mathcal{G}_2)$$

is a morphism in $\mathcal{X}$.

In other words a morphism from a locally analytically connected space to any member of $\mathcal{X}$ *factors continuously through the associated smallest locally analytically connected topology.*

Proof. By parts (ii) and (iii) of Lemma 1, $\mathcal{T}'_1 = \mathcal{T}_1$ and

$$f: (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}'_2)$$

is continuous.

We now state another characterization of the smallest locally analytically connected topology $\mathscr{T}'$.

THEOREM 4. *Let* $(X, \mathscr{T}, \mathscr{G}) \in \mathscr{X}$ *and* $\mathscr{T}'$ *be the smallest locally analytically connected refinement of* $\mathscr{T}$. *Then* $\mathscr{T}'$ *is the largest topology on* X *with the following property: if* $(X_1, \mathscr{T}_1, \mathscr{G}_1)$ *is any analytically connected and locally analytically connected member of* $\mathscr{X}$ *and* $f\colon (X_1, \mathscr{T}_1, \mathscr{G}_1) \to (X, \mathscr{T}, \mathscr{G})$ *a morphism in* $\mathscr{X}$, *then* $f\colon (X_1, \mathscr{T}_1, \mathscr{G}_1) \to (X, \mathscr{T}', \mathscr{G})$ *is still a morphism in* $\mathscr{X}$.

Proof. We show that $\mathscr{T}'$ is the largest topology on X which preserves the continuity of functions mapping into $(X, \mathscr{T})$ with domain an analytically connected and locally analytically connected member of $\mathscr{X}$. Theorem 3 states that $\mathscr{T}'$ has the required property. Suppose $\mathscr{T}_0$ is another topology on X which also has the required property. If $(X, \mathscr{T}', \mathscr{G})$ is analytically connected, then the identity map from $(X, \mathscr{T}', \mathscr{G})$ to $(X, \mathscr{T}_0, \mathscr{G})$ must be a morphism, i.e., $\mathscr{T}_0 \subset \mathscr{T}'$. If $(X, \mathscr{T}', \mathscr{G})$ is not analytically connected, apply the preceding argument to each analytic component of $(X, \mathscr{T}')$; each component of $(X, \mathscr{T}')$ is open by the local connectedness of $(X, \mathscr{T}')$.

2. Another natural locally analytically connected topology

The characterization of $\mathscr{T}'$ in Theorem 4 requires that $\mathscr{T}'$ be the largest topology which preserves continuity of maps from analytically connected and locally analytically connected spaces. One expects $\mathscr{T}'$ to be equivalently characterized as the largest topology which preserves continuity of maps from members of $\mathscr{V}$. Surprisingly these are not equivalent; an example is given in Section 3. (Compare to the case of the smallest locally arc connected topology on a space being characterized as the largest topology preserving continuity of all maps from arc connected locally arc connected spaces, or simply preserving continuity of all maps from [0, 1]; see [2].)

For $(X, \mathscr{T}, \mathscr{G}) \in \mathscr{X}$ let $\mathscr{T}^*$ be the largest topology on X such that

$$F\colon V \to (X, \mathscr{T}^*)$$

is continuous whenever $F\colon V \to (X, \mathscr{T})$ is continuous for $F \in \mathscr{F}$, $V \in \mathscr{V}$;

$$\mathscr{T}^* = \{T^* \subset X\colon F^{-1}(T^*) \text{ is open in } V \text{ for all } F \in \mathscr{F} \text{ with } F\colon V \to (X, \mathscr{T}) \text{ continuous}\}.$$

LEMMA 5. $\mathscr{T}^*$ *is locally analytically connected and* $\mathscr{T}^* \supset \mathscr{T}'$.

Proof. By part (i) of Lemma 1 whenever $F\colon V \to (X, \mathscr{T}^*)$ is continuous then

$$F\colon V \to (X, (\mathscr{T}^*)')$$

is continuous, hence $(\mathscr{T}^*)' = \mathscr{T}^*$, and $\mathscr{T}^*$ is locally analytically connected by part (iii) of Lemma 1, with $\mathscr{T}^* \supset \mathscr{T}'$.

Note that the $\mathscr{T}^*$-components (or $\mathscr{T}'$-components) of a member S of $\mathscr{T}^*$ (or of $\mathscr{T}'$) are just the analytic components of S in $\mathscr{T}$, since both $\mathscr{T}^*$ and $\mathscr{T}'$ preserve continuity of maps from $V \in \mathscr{V}$, and both $\mathscr{T}^*$ and $\mathscr{T}'$ are locally analytically connected.

We also mention another method of constructing a locally analytically connected refinement of a given topology, due to G. S. Young [4]. For $(X, \mathscr{T}, \mathscr{G}) \in \mathscr{X}$ let G be the set of all analytically connected subsets of X. Then Young's G-topology is the same as $\mathscr{T}'$. First the G-topology is locally analytically connected and contains $\mathscr{T}$, hence contains $\mathscr{T}'$. On the other hand suppose $x \in X$ is a $\mathscr{T}'$-limit point of $S \subset X$. If N is a $\mathscr{T}$-neighborhood of x, let B be the analytic component of N containing x. Then $B \in \mathscr{T}' \cap G$ contains an element of $S - \{x\}$, hence x is a G-limit point of S.

3. Non-equivalence of the topologies $\mathscr{T}'$ and $\mathscr{T}^*$

We have given two constructions for associating a locally analytically connected refinement to a given topology. Retaining the same notation as the previous two sections, we note that $(\mathscr{T}')' = \mathscr{T}'$ and $(\mathscr{T}^*)' = \mathscr{T}^*$, that is both $\mathscr{T}'$ and $\mathscr{T}^*$ remain fixed under the construction of Section 1. We now show that $\mathscr{T}^*$ may strictly contain $\mathscr{T}'$; our example takes place in the context of function algebras.

Let $\mathscr{A}$ be a function algebra and $\mathscr{M}_{\mathscr{A}}$ its maximal ideal space (all multiplicative homomorphisms of $\mathscr{A}$ onto **C**) with the weak* topology $\mathscr{T}$ induced by $\mathscr{A}$. Apply the previous two constructions to $(\mathscr{M}_{\mathscr{A}}, \mathscr{T}, \mathscr{A})$. Note that $\mathscr{T}'$ (hence $\mathscr{T}^*$) is interesting only when some analytic structure is already present in $(\mathscr{M}_{\mathscr{A}}, \mathscr{T}, \mathscr{A})$; otherwise $\mathscr{T}'$ is discrete.

A stronger topology than $\mathscr{T}$ on $\mathscr{M}_{\mathscr{A}}$ is the metric topology inherited from $\mathscr{A}^*$. By a generalization of the Schwarz lemma to Banach-space-valued holomorphic functions on analytic varieties, any holomorphic $F \in \mathscr{F}$ from V to $(\mathscr{M}_{\mathscr{A}}, \mathscr{T}, \mathscr{A})$ must be continuous with respect to the metric topology on $\mathscr{M}_{\mathscr{A}}$, therefore $\mathscr{T}^*$ always contains the metric topology. We give an example of a function algebra $\mathscr{A}$ where $\mathscr{T}'$ does not contain the metric topology, hence where $\mathscr{T}^* \supset_{\neq} \mathscr{T}'$.

Example. Let $\mathscr{A}_i \subset \mathscr{C}(\overline{D_i})$ be the usual disk algebra $\mathscr{A}_i = A(\bar{D})$ of all functions holomorphic in the open unit disk $D = D_i$ and continuous on the closed disk $\bar{D}$, let $X = \prod_{i=1}^{\infty} \overline{D_i}$ and let $\mathscr{A} \subset \mathscr{C}(X)$ be the function algebra generated by $\{\mathscr{A}_i\}_{i=1}^{\infty}$. $\mathscr{A} \subset \mathscr{C}(X)$ is the smallest closed subalgebra of $\mathscr{C}(X)$ containing all functions of the form $a_i \circ \pi_i$ where $a_i \in \mathscr{A}_i$ and $\pi_i \colon X \to \overline{D_i}$ is the projection of X onto $\overline{D_i}$. The weak* topology on $\mathscr{M}_{\mathscr{A}} = X$ is the usual compact product topology on $\prod \overline{D_i}$.

Let $\phi = \langle \phi_i \rangle \in \mathscr{M}_{\mathscr{A}}$ with $\sup_i |\phi_i| < 1$ and let

$$W = \{\psi \in \mathscr{M}_{\mathscr{A}} \colon \|\psi - \phi\| < \varepsilon\}$$

be a metric neighborhood of ϕ with $\varepsilon < 2$. For $\psi \in W$, $\|\psi_i - \phi_i\| \leq \|\psi - \phi\| < \varepsilon$ for all i, hence $\pi_i(W) \subset_{\neq} D$ for all i. Suppose ϕ had a $\mathscr{T}'$-neighborhood $B \subset W$. Then we may assume B is the analytic component containing ϕ of a basic weak* neighborhood $\prod N_i$ of ϕ, where $N_i = \bar{D}$ for all but finitely many i. Let j be a coordinate index with $N_j = \bar{D}$, choose $\psi_j \in D \backslash \pi_j(W)$, and let

$$\psi = \langle \phi_1, \ldots, \phi_{j-1}, \psi_j, \phi_{j+1}, \ldots \rangle.$$

Then the holomorphic map $F: D \to X, z \mapsto \langle \phi_1, \ldots, \phi_{j-1}, z, \phi_{j+1}, \ldots \rangle$ has ϕ, ψ in its image $F(D) \subset \prod N_i$, therefore $\psi \in B \backslash W$. Hence the metric neighborhood W of ϕ contains no $\mathscr{T}'$-neighborhood of ϕ, and $W \in \mathscr{T}^* \backslash \mathscr{T}'$.

We note that $\prod D_i$ is not analytically connected; the analytic component of $\langle 0 \rangle \in \mathscr{M}_{\mathscr{A}}$ consists of $\{\phi \in \mathscr{M}_{\mathscr{A}}: \sup_i |\phi_i| < 1\}$. (For the Gleason part structure of this example see [1].)[1] In such cases one might wish to include infinite-dimensional analytic varieties when defining analytic structure [3].

4. Normal analytic structure

In this section we show that it is necessary to define analytic structure on the maximal ideal space of function algebras in terms of varieties and not just polydisks. Given a function algebra $\mathscr{A}$ with nontrivial analytic structure in $\mathscr{M}_{\mathscr{A}}$, let

$$\text{Holo}\,(\mathscr{A}) = \{f \in \mathscr{C}(\mathscr{M}_{\mathscr{A}}): f \circ F \in \mathscr{H}(V) \text{ for all } F\text{: analytic variety } V \to \mathscr{M}_{\mathscr{A}} \text{ with } a \circ F \in \mathscr{H}(V) \text{ for all } a \in \mathscr{A}\},$$

let

$$\text{Holo}_D\,(\mathscr{A}) = \{f \in \mathscr{C}(\mathscr{M}_{\mathscr{A}}): f \circ F \in \mathscr{H}(D) \text{ for all } F\text{: open unit disk } D \to \mathscr{M}_{\mathscr{A}} \text{ with } a \circ F \in \mathscr{H}(D) \text{ for all } a \in \mathscr{A}\}.$$

Then $\text{Holo}\,(\mathscr{A}) \subset \text{Holo}_D\,(\mathscr{A})$, and $\text{Holo}_D\,(\mathscr{A})$ is unchanged if we instead require $f \circ F \in \mathscr{H}(V)$ for all holomorphic maps F from normal analytic varieties V into $\mathscr{M}_{\mathscr{A}}$.

We will look at proper closed subalgebras $\mathscr{B}$ of the disk algebra $A(\bar{D})$ to obtain examples where $\text{Holo}\,(\mathscr{B}) \neq \text{Holo}_D\,(\mathscr{B})$.

For each integer $n \geq 1$ let $\mathscr{B}_n$ be the closure on $\bar{D}$ of the ring of polynomials in z^2, z^{2n+1}; then $\mathscr{B}_n = \{f \in A(\bar{D}): f'(0) = f'''(0) = \cdots = f^{(2n-1)}(0) = 0\}$. We have $\mathscr{M}_{\mathscr{B}_n} = \bar{D}$, since $\phi \in \mathscr{M}_{\mathscr{B}_n}$ is evaluation at the point $\phi(z^{2n+1})/\phi(z^{2n})$ in $\bar{D}$ (or at 0 if $\phi(z^2) = 0$). Also $\partial_{\mathscr{B}_n} = \bar{D} \backslash D$; in fact every $\lambda \in \bar{D} \backslash D$ is a peak point for $\mathscr{B}_n$, as we can see by taking $b = z^{2n+1} + \lambda z^{2n}$,

$$|b(z)| = |z^{2n}|\,|z + \lambda| \leq 2 \quad \text{for all } z \in \bar{D},$$

and $|b(z)| = 2$ if and only if $z = \lambda$.

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[1] The author wishes to thank the referee for pointing out this reference.

To find $\mathrm{Holo}_D(\mathscr{B}_n)$ we first note that since $G_1 = \text{identity}: D \to \bar{D} = \mathscr{M}_{\mathscr{B}_n}$ satisfies $b \circ G_1 \in \mathscr{H}(D)$ for all $b \in \mathscr{B}_n$ then $\mathrm{Holo}_D(\mathscr{B}_n) \subset A(\bar{D})$. Now suppose $G: D \to \bar{D}$ with $b \circ G \in \mathscr{H}(D)$ for all $b \in \mathscr{B}_n$; in particular $z \mapsto G(z)^{2n}$ and $z \mapsto G(z)^{2n+1}$ are holomorphic on D. On $D\backslash(\text{zero set of } G^{2n})$,

$$G(z)^{2n+1}/G(z)^{2n} = G(z)$$

is holomorphic. Since G^{2n} is holomorphic it has isolated zeros (or $G \equiv 0$); G is bounded on D since $G(D) \subset \bar{D}$, hence $G|_{D\backslash(\text{zero set of } G^{2n})}$ extends to a holomorphic function on D, which by continuity is G itself. Therefore if $g(z) = z$ then $g \circ G \in \mathscr{H}(D)$, and $A(\bar{D}) = \mathrm{Holo}_D(\mathscr{B}_n)$.

Let V_n be the analytic subvariety contained in $D \times D \subset \mathbf{C}^2$ defined by

$$V_n = \{(z, w) \in D \times D: z^2 = w^{2n+1}\},$$

and let $F_n: V_n \to \mathbf{C}$ be defined by $F_n(z, w) = z/w$ if $w \neq 0$ and $F_n(0, 0) = 0$. Then $F_n^2(z, w) = w^{2n-1}$ so $F_n(V_n) \subset \bar{D} = \mathscr{M}_{\mathscr{B}_n}$. (In fact $F_n(V_n) = D$: for each $p \in D$ choose a $q \in D$ with $q^{2n-1} = p$; then $(q^{2n+1}, q^2) \in V_n$ with $F(q^{2n+1}, q^2) = p$.) Also $F^{2n+1}(z, w) = z^{2n-1}$, hence if $b_1(z) = z^2$, $b_2(z) = z^{2n+1}$ we have $b_1 \circ F_n$, $b_2 \circ F_n \in \mathscr{H}(V_n)$. However if $g(z) = z$ then $g \circ F_n = F_n \notin \mathscr{H}(V_n)$; although F_n is holomorphic on $V_n\backslash(0, 0)$ and F_n is bounded and continuous on V_n, F_n does not extend to a holomorphic function at $(0, 0) \in V_n$ (V_n is not normal at $(0, 0)$). Hence $g \in A(\bar{D})\backslash\mathrm{Holo}(\mathscr{B}_n)$, and $\mathrm{Holo}_D(\mathscr{B}_n) \neq \mathrm{Holo}(\mathscr{B}_n)$.

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