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GLOBAL DIMENSION OF SPECIAL ENDOMORPHISM RINGS OVER ARTIN ALGEBRAS[1]

BY

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Introduction

In this paper we deal with the following theorem proved by M. Auslander in [2], [3]. If Λ is an Artin algebra with the Loewy length of Λ, $\mathscr{L}(\Lambda) = n$, $M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$ where $r_\Lambda = \operatorname{Rad} \Lambda$ and $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$, then the global dimension of Γ, gl. dim $\Gamma \leq n$. The main goal of this paper is to prove that this inequality is optimal. In fact we prove the following. Given $n \geq m \geq 2$ there exists an Artin algebra Λ with $\mathscr{L}(\Lambda) = n$ and such that gl. dim $\operatorname{End}(M)^{\mathrm{op}} = m$ where $M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$ with

$$r_\Lambda = \operatorname{Rad} \Lambda.$$

We then, as an easy corollary, get that the inequality in the theorem referred to above is optimal. In the proof of this result we will use a description of the module category of the ring

$$T_2(\Lambda) = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$$

for a ring Λ in terms of the Λ-modules, taken from [4]. The ring

$$\begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$$

is the natural one we get by usual matrix operation. We will after this description give some properties of Λ which are invariant under T_2, and at last show that a full lower triangular matrix ring over a field satisfies these properties. Before we start with the lemmas which lead to the main result, we are going to refer in Section 1 some results from [2] and [3] and give some of the notation used in the rest of the paper. The reader is referred to [1] and [5] for general background in ring theory and homological algebra.

Section 2 is devoted to proving our main result.

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1. Preliminaries

We shall here recall some of the results from [2], [3] that we shall need in this paper.

Let Λ be an Artin algebra, M a finitely generated Λ-module and $\Gamma = \text{End}\,(M)^{\text{op}}$. Now $\text{Hom}_\Lambda(M, -) = (M, -)$ is a functor from mod (Λ), the category of finitely generated Λ-modules, to mod (Γ). Let add M be the full subcategory of mod (Λ) consisting of the modules A which are summands of finite sums of copies of M. Then $(M, -)$ induces an equivalence between add M and the full subcategory of mod (Γ) consisting of the projective modules. Now let Λ be an Artin algebra with the Loewy length of Λ, $\mathscr{L}(\Lambda) = n$,

$$M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$$

where $r_\Lambda = \text{Rad}\,\Lambda$ and $\Gamma = \text{End}\,(M)^{\text{op}}$. In [6] is given a complete description of what the Γ look like, but here we only need the following property. The indecomposable projective Γ-modules $P_1, \ldots, P_k$ which contain no proper projective submodule have a unique composition series with nonisomorphic composition factors. Further, if P' is an indecomposable projective Γ-module there is a unique P_i, $i \in \{1, \ldots, k\}$, such that $P'/r_\Gamma P'$ is a composition factor of P_i, where $r_\Gamma = \text{Rad}\,\Gamma$. For the sake of completeness we sketch the proof here. The P_i' in the proposition are exactly the projective Γ-modules $(M, Q_i/r_\Lambda Q_i)$ where Q_i, $i = 1, \ldots, k$ are the nonisomorphic indecomposable projective Λ-modules. If we now look at one particular such, we get the following exact sequences:

$$0 \to r_\Lambda Q_i \to Q_i \xrightarrow{p_{\mathscr{L}(Q_i)-1}} Q_i/r_\Lambda Q_i \to 0,$$

$$0 \to r_\Lambda Q_i/r_\Lambda^{\mathscr{L}(Q_i)-1} Q_i \to Q_i/r_\Lambda^{\mathscr{L}(Q_i)-1} Q_i \xrightarrow{p_{\mathscr{L}(Q_i)-2}} Q_i/r_\Lambda Q_i \to 0,$$

$$\vdots$$

$$0 \to r_\Lambda Q_i/r_\Lambda^2 Q_i \to Q_i/r_\Lambda^2 Q_i \xrightarrow{p_1} Q/r_\Lambda Q_i \to 0,$$

$$0 \to Q_i/r_\Lambda Q_i \to Q_i/r_\Lambda Q_i \to 0.$$

From this we get $\text{Im}\,(M, p_j) \subseteq (M, Q_i/r_\Lambda Q_i)$, $j = 1, \ldots, \mathscr{L}(Q_i) - 1$, which forms the unique composition series for $(M, Q_i/r_\Lambda Q_i)$, i.e., for every i we have the following situation

$$\begin{array}{ccccc} (M, Q_i) & & & (M, Q_i/r_\Lambda^2 Q_i) & \\ \downarrow & & & \downarrow & \\ r_\Gamma^{\mathscr{L}(Q_i)-1}(M, Q_i/r_\Lambda Q_i) & \subseteq & \cdots \subseteq & r_\Gamma(M, Q_i/r_\Lambda Q_i) & \subseteq (M, Q_i/r_\Lambda Q_i) \end{array}$$

where $r_\Gamma = \text{Rad}\,\Gamma$, each map downwards is an epimorphism and the $(M, Q_i/r_\Lambda^j Q_i)$ are indecomposable projectives.

Let Λ be a ring. Then we will denote the radical of Λ by $\text{Rad}\,\Lambda = r_\Lambda$ and if A is a Λ-module we will denote the socle of A which is the unique largest semisimple submodule of A by Soc A.

2. The main result

We start this section with the following preliminary result.

LEMMA 1. *Let Λ be an Artin algebra with Loewy length $\mathscr{L}(\Lambda) = n$, $M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$, $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$ and Q an indecomposable, not simple projective Λ-module. Then the sequence*

$$0 \to (M, r_\Lambda Q/r_\Lambda^j Q) \to (M, r_\Lambda Q/r_\Lambda^{j-1} Q \amalg Q/r_\Lambda^j Q) \to r_\Gamma(M, Q/r_\Lambda^{j-1} Q) \to 0$$

of Γ-modules will be exact where $2 \leq j \leq \mathscr{L}(Q)$ and the maps are the natural ones.

Proof. From the preliminaries we get the exact sequence

$$0 \to (M, r_\Lambda Q/r_\Lambda^{j-1} Q) \to r_\Gamma(M, Q/r_\Lambda^{j-1} Q) \to r_\Gamma^{j-1}(M, Q/r_\Lambda Q) \to 0$$

and the commuting diagram

$$\begin{tikzcd} (M, Q/r_\Lambda^j Q) \arrow[d] \arrow[dr] & \\ r_\Gamma(M, Q/r_\Lambda^{j-1} Q) \arrow[r] & r_\Gamma^{j-1}(M, Q/r_\Lambda Q) \end{tikzcd}$$

where both maps into $r_\Gamma^{j-1}(M, Q/r_\Lambda Q)$ are epic. From these we get

$$\begin{tikzcd} & & & (M, Q/r_\Lambda^j Q) \arrow[d] & \\ 0 \arrow[r] & (M, r_\Lambda Q/r_\Lambda^{j-1} Q) \arrow[r] & r_\Gamma(M, Q/r_\Lambda^{j-1} Q) \arrow[r] & r_\Gamma^{j-1}(M, Q/r_\Lambda Q) \arrow[r] & 0 \end{tikzcd}$$

where $(M, Q/r_\Lambda^j Q)$ is a projective Γ-module. It is now clear that the sequence $(M, r_\Lambda Q/r_\Lambda^{j-1} Q \amalg Q/r_\Lambda^j Q) \to r_\Gamma(M, Q/r_\Lambda^{j-1} Q) \to 0$ is exact with the map the natural one. The kernel of this map is isomorphic to $(M, r_\Lambda Q/r_\Lambda^j Q)$ where the isomorphism is given by $f \to (p \circ f, -f)$ with p the natural epimorphism from $r_\Lambda Q/r_\Lambda^j Q$ to $r_\Lambda Q/r_\Lambda^{j-1} Q$.

From this lemma we are able to prove the following.

PROPOSITION 2. *Let Λ be an Artin algebra with $\mathscr{L}(\Lambda) = n \geq 2$,*

$$M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$$

and $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$. Then $\operatorname{gl.dim} \Gamma = 2$ if and only if r_Λ is in $\operatorname{add} M$.

Proof. Assume first that $\operatorname{gl.dim} \Gamma = 2$. Then we will for every indecomposable projective Λ-module Q have, in mod Γ, the exact sequence

$$0 \to (M, r_\Lambda Q) \to (M, Q) \to (M, Q/r_\Lambda Q) \to A \to 0$$

with (M, Q) and $(M, Q/r_\Lambda Q)$ projective. Now it follows from the preliminaries that if $r_\Lambda Q \neq 0$ then $A \neq 0$ so $(M, r_\Lambda Q)$ must be projective or (0) for every indecomposable projective Λ-module Q, which is equivalent to saying that $r_\Lambda Q$ is in add M. So r_Λ is in add M.

Now assume to the contrary that r_Λ is in add M. It then follows that r_Λ/r_Λ^j is in add M for $n \geq j \geq 1$. Now let S be a simple Γ-module. From the preliminaries and Lemma 1 there exist an indecomposable projective Λ-module Q and an exact sequence

$$0 \to (M, r_\Lambda Q/r_\Lambda^{j+1}Q) \to (M, r_\Lambda Q/r_\Lambda^j Q \amalg Q/r_\Lambda^{j+1}Q) \to (M, Q/r_\Lambda^j Q) \to S \to 0$$

if $\mathscr{L}(Q) > j \geq 1$ and an exact sequence

$$0 \to (M, r_\Lambda Q) \to (M, Q) \to S \to 0$$

if $\mathscr{L}(Q) = j \geq 1$. In both cases pd $S \leq 2$. Since $n \geq 2$ it follows that Γ is not hereditary. So gl. dim $\Gamma = 2$.

We now give a lemma which under certain assumptions reduces the number of Γ-modules of which we have to find projective dimension when we shall calculate the global dimension of Γ.

LEMMA 3. *Let Λ be an Artin algebra with $\mathscr{L}(\Lambda) = n \geq 2$,*

$$M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$$

and $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$. Now, suppose that pd $(M, r_\Lambda Q) \geq$ pd $(M, r_\Lambda Q/r_\Lambda^j Q)$ *as Γ-modules for every indecomposable projective Λ-module Q where $2 \leq j \leq \mathscr{L}(Q)$. Then*

$$\text{gl. dim } \Gamma = \max \{\text{pd } (M, r_\Lambda Q) \mid Q \textit{ indecomposable projective } \Lambda\textit{-module}\} + 2.$$

Proof. Now $(M, r_\Lambda Q) = r_\Gamma(M, Q)$ for every indecomposable projective Λ-module Q. From the preliminaries we have the exact sequence

$$0 \to (M, r_\Lambda Q) \to (M, Q) \to (M, Q/r_\Lambda Q) \to A \to 0,$$

with (M, Q) and $(M, Q/r_\Lambda Q)$ projective Γ-modules. If now $r_\Lambda Q \neq 0$ then $A \neq 0$ and in this case pd $(A) =$ pd $(M, r_\Lambda Q) + 2$. Now

$$\begin{aligned} \text{gl. dim } \Gamma \geq \max \{&\text{pd } (A) \mid A = (M, Q/r_\Lambda Q)/\operatorname{Im}(M, p_{\mathscr{L}(Q)-1}), \\ &Q \text{ indecomposable nonsimple projective } \Lambda\text{-module}\} \\ = \max \{&\text{pd } (M, r_\Lambda Q) \mid Q \text{ indecomposable nonsimple projective } \Lambda\text{-module}\} + 2 \\ = \max \{&\text{pd } (M, r_\Lambda Q) \mid Q \text{ indecomposable projective } \Lambda\text{-module}\} + 2. \end{aligned}$$

For the converse inequality let S be a simple Γ-module. From the preliminaries there exists a unique indecomposable projective Λ-module Q such that S is a composition factor of $(M, Q/r_\Lambda Q)$. We can assume that S is not projective, i.e.,

Q is not simple. From Lemma 1 we have associated with Q the exact sequences

$$0 \to 0 \to (M, r_\Lambda Q) \to r_\Gamma(M, Q) \to 0,$$
$$0 \to (M, r_\Lambda Q) \to (M, r_\Lambda Q/r_\Lambda^j Q \amalg Q) \to r_\Gamma(M, Q/r_\Lambda^j Q) \to 0,$$
$$0 \to (M, r_\Lambda Q/r_\Lambda^j Q) \to (M, r_\Lambda Q/r_\Lambda^{j-1} Q \amalg Q/r_\Lambda^j Q) \to r_\Lambda(M, Q/r_\Lambda^{j-1} Q) \to 0,$$
$$\vdots$$
$$0 \to (M, r_\Lambda Q/r_\Lambda^2 Q) \to (M, Q/r_\Lambda^2 Q) \to r_\Gamma(M, Q/r_\Lambda Q) \to 0$$

where $\mathscr{L}(Q) = j + 1$. Since the modules $(M, Q/r_\Lambda^i Q)$ are all projective, it follows that the projective dimension of the Γ-module in the middle of one sequence is the same as the projective dimension of the module at the left end of the sequence below. From the construction of the composition series of $(M, Q/r_\Lambda Q)$ it now follows that $\operatorname{pd} S = \operatorname{pd} r_\Gamma(M, Q/r^i Q) + 1$ for some i, $1 \leq i \leq j$. Further we have that

$$\begin{aligned}\operatorname{pd} r_\Gamma(M, Q/r_\Lambda^i Q) &\leq \max\{\operatorname{pd}(M, r_\Lambda Q/r_\Lambda^{i+1} Q), \operatorname{pd}(M, r_\Lambda Q/r_\Lambda^i Q)\} + 1 \\ &\leq \operatorname{pd}(M, r_\Lambda Q) + 1\end{aligned}$$

by use of the property of Ext. The last inequality follows from the assumption. Totally we have that $\operatorname{pd} S \leq \operatorname{pd}(M, r_\Lambda Q) + 2$, which gives the lemma.

Before we go on we will describe the module category of the matrix ring

$$T_2(\Lambda) = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$$

for a ring Λ in terms of the module category of Λ. The modules in mod $T_2(\Lambda)$ can be described as triples (A, B, f) where A and B are Λ-modules and f: $A \to B$ a Λ-homomorphism. A $T_2(\Lambda)$-homomorphism ϕ: $(A, B, f) \to (A', B', f')$ is a couple (α, β) of Λ-homomorphism where α: $A \to A'$ and β: $B \to B'$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[\beta]{} & B' \end{array}$$

commutes. In this paper we are particularly interested in what the indecomposable projective $T_2(\Lambda)$-modules look like in this description and which pairs of Λ-homomorphisms (α, β) are $T_2(\Lambda)$-monomorphisms and which are $T_2(\Lambda)$-epimorphisms. From [4] we have that the indecomposable projective $T_2(\Lambda)$-modules are precisely the ones of type (Q, Q, id) and $(0, Q, 0)$ for indecomposable projective Λ-modules Q. Further a $T_2(\Lambda)$-homomorphism $\phi = (\alpha, \beta)$ is a $T_2(\Lambda)$-monomorphism (resp. epimorphism) if and only if both α and β are Λ-monomorphisms (resp. epimorphisms). Besides, a sequence

$$0 \to (A', B', f') \xrightarrow{(\alpha', \beta')} (A, B, f) \xrightarrow{(\alpha, \beta)} (A'', B'', f'') \to 0$$

is exact if and only if both the sequences

$$0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A'' \to 0 \quad \text{and} \quad 0 \to B' \xrightarrow{\beta'} B \xrightarrow{\beta} B'' \to 0$$

are exact. If in addition Λ is an Artin ring with $\mathscr{L}(\Lambda) = n$ then $\mathscr{L}(T_2(\Lambda)) = n + 1$.

To avoid confusion when talking about an homomorphism

$$\phi\colon (A, B, f) \amalg (A', B', f') \to (A'', B'', f'')$$

we will use $(\ ,\ )$ to denote one single map and $[(\ ,\ ), (\ ,\ )]$ to denote a homomorphism written as a matrix.

Example.

$$\phi = [(\alpha, \beta), (\alpha', \beta')] = [\phi_1, \phi_2]\colon (A, B, f) \amalg (A', B', f') \to (A'', B'', f'')$$

We are now able to continue.

LEMMA 4. *Let Λ be an Artin algebra with $\mathscr{L}(\Lambda) = n$, $M_0 = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$ and assume that for every indecomposable Λ-module N_0 in* add M_0, Soc $(N_0) = r_\Lambda^k N_0$ *for a k in* $\mathbf{N}$. *Then* add M_1 *will have the same property, where*

$$M_1 = T_2(\Lambda) \amalg T_2(\Lambda)/r_{T_2(\Lambda)}^n \amalg \cdots \amalg T_2(\Lambda)/r_{T_2(\Lambda)}.$$

Proof. From the description of $T_2(\Lambda)$ we get that the indecomposable modules N_1 in add M_1 are precisely of the following types: (Q, Q, id), $(Q/r_\Lambda^i Q, Q/r_\Lambda^{i-1} Q, p)$ and $(0, Q/r_\Lambda^i Q, 0)$, with $1 \le i \le \mathscr{L}(Q)$ and p the natural epimorphisms. Easy computation now gives

$$\begin{aligned} \operatorname{Soc}(Q, Q, \mathrm{id}) &= (0, \operatorname{Soc} Q, 0) = (0, r_\Lambda^k Q, 0) \\ &= r_{T_2(\Lambda)}^{k+1}(Q, Q, \mathrm{id}) \\ \operatorname{Soc}(Q/r_\Lambda^i Q, Q/r_\Lambda^{i-1}, p) &= (r_\Lambda^{i-1} Q/r_\Lambda^i Q, 0, 0) \amalg (0, r_\Lambda^{i-2} Q/r_\Lambda^{i-1} Q, 0) \\ &= r_{T_2(\Lambda)}^{i-1}(Q/r_\Lambda^i Q, Q/r_\Lambda^{i-1} Q, p) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Soc}(0, Q/r_\Lambda^i Q, 0) &= (0, \operatorname{Soc} Q/r_\Lambda^i Q, 0) = (0, r_\Lambda^{i-1} Q/r_\Lambda^i Q, 0) \\ &= r_{T_2(\Lambda)}^{i-1}(0, Q/r_\Lambda^i Q, 0), \end{aligned}$$

i.e., add M_1 satisfies the claimed property.

We now give a definition which will be useful in the rest of this paper.

DEFINITION. Let Λ be an Artin algebra with $\mathscr{L}(\Lambda) = n$, $M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$ and X a finitely generated Λ-module. An exact sequence

$$0 \to N_k \xrightarrow{f_k} N_{k-1} \to \cdots \to N_1 \xrightarrow{f_1} N_0 \xrightarrow{f_0} X \to 0$$

with N_i in add M such that f_k is not a split homomorphism and such that the sequence remains exact when $(M, -)$ is applied, will be called a minimal resolution of X in add M.

Observe that such a minimal resolution exists for every Λ-module X since M is a generator, gl. dim $\Gamma \leq n$ where $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$ and add M is equivalent to the category of finitely generated projective Γ-modules.

LEMMA 5. *Let Λ be an Artin algebra with $\mathscr{L}(\Lambda) = n$, $M_0 = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$ and suppose that*

$$0 \to Q_m \xrightarrow{f_m} Q_{m-1} \to \cdots \to Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} r_\Lambda Q \to 0$$

is a minimal resolution of $r_\Lambda Q$ in add M_0 *for an indecomposable nonsimple projective Λ-module Q where each Q_i, $i = 0, \ldots, m$, is projective.*

$$\begin{aligned} &0 \to (0, Q_m, 0) \\ &\xrightarrow{\left[\begin{smallmatrix}(0,-f_m)\\(0,\mathrm{id})\end{smallmatrix}\right]} (0, Q_{m-1}, 0) \amalg (Q_m, Q_m, \mathrm{id}) \to \cdots \\ &\to (0, Q_0, 0) \amalg (Q_1, Q_1, \mathrm{id}) \\ &\xrightarrow{\left[\begin{smallmatrix}(0,-f_0) & 0\\(0,\mathrm{id})(f_1,f_1)\end{smallmatrix}\right]} (0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \\ &\xrightarrow{[(0,\mathrm{id})(f_0,f_0)]} (r_\Lambda Q, Q, i) = r_{T_2(\Lambda)}(Q, Q, \mathrm{id}) \to 0 \end{aligned}$$

is then a minimal resolution of $r_{T_2(\Lambda)}(Q, Q, \mathrm{id})$ in add M_1, *where*

$$M_1 = T_2(\Lambda) \amalg T_2(\Lambda)/r^n_{T_2(\Lambda)} \amalg \cdots \amalg T_2(\Lambda)/r_{T_2(\Lambda)}.$$

Proof. From the description of exact sequences in mod $T_2(\Lambda)$ it follows at once that the sequences described above are exact. Since $-f_m$ does not split and there is no nonzero map $(Q_m, Q_m, \mathrm{id}) \to (0, Q_m, 0)$, the last $T_2(\Lambda)$-homomorphism is not split. It remains to show that the sequence is still exact after the action of $(M_1, -)$. We will start by proving that $(M_1, -)$ preserves the exactness of

$$(0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \xrightarrow{[(0,\mathrm{id})(f_0,f_0)]} (r_\Lambda Q, Q, i) \to 0.$$

Since $(M_1, -)$ is a natural equivalence from the category add M_1 to the full category of finitely generated projective Γ_1-modules where $\Gamma_1 = \operatorname{End}(M_1)^{\mathrm{op}}$, it suffices to show that every homomorphism ϕ: $N_1 \to (r_\Lambda Q, Q, i)$ where N_1 is an indecomposable module in add M_1, factors through

$$(0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \xrightarrow{[(0,\mathrm{id}),(f_0,f_0)]} (r_\Lambda Q, Q, i).$$

The indecomposable $T_2(\Lambda)$-modules in add M_1 are described in the proof of Lemma 4 and are of the types

$$(0, Q/r^j_\Lambda Q, 0) \quad \text{and} \quad (Q/r^{i+1}_\Lambda Q, Q/r^i_\Lambda Q, p)$$

where Q is indecomposable projective Λ-module and p is the natural epimorphism if $r^i_\Lambda Q \neq 0$ and the identity if $r^i_\Lambda Q = 0$. Assume first that

$$N_1 = (0, Q'/r^j_\Lambda Q', i)$$

for an indecomposable projective Λ-module Q' and

$$\phi\colon N_1 = (0, Q'/r_\Lambda^j Q', 0) \to (r_\Lambda Q, Q, 0).$$

Then $\phi = (0, \beta)$ for a Λ-homomorphism $\beta\colon Q'/r_\Lambda^j Q' \to Q$. Now look at the diagram

$$\begin{tikzcd}
 & & 0 \arrow[dl, "0"'] \arrow[d] \\
0 \arrow[urr, "0"] \arrow[r, "0"] \arrow[d, "0"'] & r_\Lambda Q \arrow[d, "i"] & Q \arrow[dl, "\mathrm{id}"] \\
Q'/r_\Lambda^j Q' \arrow[r, "\beta"'] \arrow[urr, "\beta"] & Q &
\end{tikzcd}$$

which commutes, i.e., $\phi = (0, \beta) = (0, \mathrm{id}) \circ (0, \beta)$. So ϕ factors through

$$(0, Q, 0) \xrightarrow{(0,\mathrm{id})} (\mathbf{r}_\Lambda Q, Q, i)$$

and therefore through

$$(0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \xrightarrow{[(0,\mathrm{id}),(f_0,f_0)]} (r_\Lambda Q, Q, i).$$

It now remains to prove that every morphism

$$\phi\colon N_1 = (Q'/r_\Lambda^{j+1} Q', Q'/r_\Lambda^j Q', p') \to (r_\Lambda Q, Q, i),$$

where Q' is an indecomposable projective Λ-module and $\phi = (\alpha, \beta)$ factors through

$$(0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \xrightarrow{[(0,\mathrm{id}),(f_0,f_0)]} (r_\Lambda Q, Q, i).$$

Consider therefore the commuting diagram

$$\begin{tikzcd}
Q'/r_\Lambda^{j+1} Q' \arrow[r, "\alpha"] \arrow[d, "p'"] & r_\Lambda Q \arrow[d, "i"] \\
Q'/r_\Lambda^j Q' \arrow[r, "\beta"'] & Q.
\end{tikzcd}$$

p' is epi so $\mathrm{Im}\,\beta = \mathrm{Im}\,\beta \circ p' = \mathrm{Im}\, i \circ \alpha \subseteq r_\Lambda Q$. Therefore there exists an homomorphism $g\colon Q'/r_\Lambda^j Q' \to Q_0$ such that the diagram

$$\begin{tikzcd}
 & & Q_0 \arrow[dl, "f_0"'] \arrow[d, "\mathrm{id}"] \\
Q'/r_\Lambda^{j+1} Q' \arrow[r, "\alpha"] \arrow[d, "p'"] & \mathbf{r}_\Lambda Q \arrow[d, "i"] & Q_0 \arrow[dl, "f_0"] \\
Q'/r_\Lambda^j Q' \arrow[r, "\beta"'] \arrow[urr, "g"] & Q &
\end{tikzcd}$$

commutes. Now let h: $Q'/r_\Lambda^{j+1}Q' \to Q_0$ be given by $h = g \circ p'$. Then

$$\psi = (h, g): (Q'/r_\Lambda^{j+1}Q', Q'/r_\Lambda^j Q', p') \to (Q_0, Q_0, \mathrm{id}).$$

Further $i \circ \alpha = \beta \circ p' = f_0 \circ g \circ p' = f_0 \circ h$ and $i = \mathrm{id}\, r_\Lambda Q$, so we must have $\alpha = f_0 \circ h$. So $(f_0, f_0) \circ (h, g) = (\alpha, \beta) = \phi$ and therefore ϕ factors through

$$(Q_0, Q_0, \mathrm{id}) \xrightarrow{(f_0, f_0)} (r_\Lambda Q, Q, i),$$

i.e., through

$$(0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \xrightarrow{[(0,\mathrm{id}),(f_0,f_0)]} (r_\Lambda Q, Q, i).$$

Observe that everything goes well if p' is the identity as well as if p' is a proper epimorphism. It follows then that $(M_1, -)$ preserves the exactness of

$$(0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \to (r_\Lambda Q, Q, i) \to 0.$$

We have now got the following commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0 \arrow[r] \arrow[d, "i_0"] & 0 \amalg Q_0 \arrow[r, "f_0"] \arrow[d] & r_\Lambda Q \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & Q_0 \arrow[r, "{[-f_0,\mathrm{id}]}"] & Q \amalg Q_0 \arrow[r, "{[\mathrm{id},f_0]}"] & Q \arrow[r] & 0
\end{tikzcd}$$

where i_0 is the natural inclusion. Now assume that $K_j = \operatorname{Ker} f_j$ and that i_j: $K_j \to Q_j$ is the natural inclusion and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{j+1} \arrow[r] \arrow[d, "i_{j+1}"] & 0 \amalg Q_{j+1} \arrow[r, "f_{j+1}"] \arrow[d] & K_j \arrow[r] \arrow[d, "i_j"] & 0 \\
0 \arrow[r] & Q_{j+1} \arrow[r, "{[-f_{j+1},\mathrm{id}]}"] & Q_j \amalg Q_{j+1} \arrow[r, "{[\mathrm{id},f_{j+1}]}"] & Q_j \arrow[r] & 0.
\end{tikzcd}$$

It follows that i_{j+1} is the natural inclusion $K_{j+1} = \operatorname{Ker} f_{j+1} \subseteq Q_{j+1}$. If we now are able to prove that $(M_1, -)$ preserves the exactness of

$$0 \to (K_{j+1}, Q_{j+1}, i_{j+1}) \to (0, Q_j, 0) \amalg (Q_{j+1}, Q_{j+1}, \mathrm{id}) \to (K_j, Q_j, i_j) \to 0$$

we will have proved the lemma. This last proof goes exactly as the one with $(r_\Lambda Q, Q, i)$ at the right end of the sequence, so we leave the details for the reader.

An easy consequence of this lemma is the following.

COROLLARY 6. pd $(M_1, r_{T_2(\Lambda)}(Q, Q, \mathrm{id})) =$ pd $(M_0, r_\Lambda Q) + 1$ *when* Λ, $T_2(\Lambda)$, M_0, M_1 *and* Q *are as in Lemma* 5.

LEMMA 7. *Let* Λ, $T_2(\Lambda)$, M_0, *and* M_1 *be as in Lemma* 5. *Then*

$$\mathrm{pd}\ (M_1, r_{T_2(\Lambda)}(0, Q, 0)) = \mathrm{pd}\ (M_0, r_\Lambda Q)$$

for every indecomposable projective Λ-*module* Q.

The proof of this lemma is trivial and it is left to the reader.

LEMMA 8. *Let Λ, $T_2(\Lambda)$, M_0 and M_1 be as in Lemma 5 and assume that the pair (Λ, M_0) has the following properties.*

(1) *For every indecomposable Λ-module A in* add M_0 *there exists a $k \in \mathbf{N}$ such that* Soc $A = r_\Lambda^k A$.

(2) *For every indecomposable nonsimple projective Λ-module Q there exists a minimal resolution*

$$0 \to Q_m \xrightarrow{f_m} Q_{m-1} \to \cdots \to Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} \mathbf{r}_\Lambda Q \to 0$$

of $r_\Lambda Q$ in add M_0 *such that:*

(a) *Every Q_i is projective.*
(b) *For each fixed i, the Loewy length of the summands in Q_i are all the same.*
(c) $\mathscr{L}(Q_0) = \mathscr{L} r_\Lambda(Q)$, $\mathscr{L}(Q_i) = \mathscr{L}(Q_{i-1}) - 1$, $1 \le i \le m$.
(d)

$$\begin{aligned} 0 \to Q_m/r_\Lambda^{j-m}Q_m &\xrightarrow{f_{m,j}} Q_{m-1}/r_\Lambda^{j-m+1}Q_{m-1} \to \cdots \\ &\to Q_1/r_\Lambda^{j-1}Q_1 \xrightarrow{f_{1,j}} Q_0/r_\Lambda^{j}Q_0 \xrightarrow{f_{0,j}} r_\Lambda Q/r_\Lambda^{j+1}Q \to 0 \end{aligned}$$

is a minimal resolution of $r_\Lambda Q/r_\Lambda^{j+1}Q$ in add M_0. *Then the pair $(T_2(M), M_1)$ has the same properties.*

Proof. From Lemma 4 it follows that the pair $(T_2(\Lambda), M_1)$ has property (1). The indecomposable projective $T_2(\Lambda)$-modules of the form $(0, Q, 0)$ where Q is an indecomposable projective Λ-module have trivially the property (2) when Q as an Λ-module has this property. So it remains only to show that property (2) is valid for every indecomposable projective $T_2(\Lambda)$-module of the form (Q, Q, id) when Q is an indecomposable projective Λ-module. Now let (Q, Q, id) be any such one. Then there exists by assumption a minimal resolution

$$0 \to Q_m \xrightarrow{f_m} Q_{m-1} \to \cdots \to Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} r_\Lambda Q \to 0$$

of $r_\Lambda Q$ in add M_0, such that (2) is satisfied. It then follows from Lemma 5 that

$$\begin{aligned} 0 \to (0, Q_m, 0) \to (0, Q_{m-1}, 0) \amalg (Q_m, Q_m, \mathrm{id}) \to \cdots \to (0, Q_0, 0) \amalg (Q_1, Q_1, \mathrm{id}) \\ \to (0, Q, 0) \amalg (Q_0, Q_0, \mathrm{id}) \to r_{T_2(\Lambda)}(Q, Q, \mathrm{id}) = (r_\Lambda Q, Q, i) \to 0 \end{aligned}$$

is a minimal resolution of $r_{T_2(\Lambda)}(Q, Q, \mathrm{id})$ in add M_1. Then one easily sees that property 2(a), 2(b), and 2(c) are satisfied. Now by assumption

$$\begin{aligned} 0 \to Q_m/r_\Lambda^{j-m}Q_m &\xrightarrow{f_{m,j}} Q_{m-1}/r_\Lambda^{j-m+1}Q_{m-1} \to \cdots \\ &\to Q_1/r_\Lambda^{j-1}Q_1 \xrightarrow{f_{1,j}} Q_0/r_\Lambda^{j}Q_0 \xrightarrow{f_{0,j}} r_\Lambda Q/r_\Lambda^{j+1}Q \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 \to Q_m/r_\Lambda^{j-m-1}Q_m &\xrightarrow{f_{m,j-1}} Q_{m-1}/r_\Lambda^{j-m}Q_{m-1} \to \cdots \\ &\to Q_1/r_\Lambda^{j-2}Q_1 \xrightarrow{f_{1,j-1}} Q_0/r_\Lambda^{j-1}Q_0 \xrightarrow{f_{0,j-1}} r_\Lambda Q/r_\Lambda^{j}Q \to 0 \end{aligned}$$

are minimal resolutions of $r_\Lambda Q/r_\Lambda^{j+1}Q$ and $r_\Lambda Q/r_\Lambda^j Q$ in add M_0 respectively. From this we want to show that

$$
\begin{aligned}
&0 \to (0,\, Q_m/r_\Lambda^{j-m-1}Q_m,\, 0)\\
&\xrightarrow{\left[\substack{(0, f_{m,j-1})\\ (0,\mathrm{id})}\right]} (0,\, Q_{m-1}/r_\Lambda^{j-m}Q_{m-1},\, 0) \amalg (Q_m/r_\Lambda^{j-m}Q_m,\, Q_m/r_\Lambda^{j-m-1}Q_m,\, p_m) \to \cdots\\
&\to (0,\, Q/r_\Lambda^j Q,\, 0) \amalg (Q_0/r_\Lambda^j Q_0,\, Q_0/r_\Lambda^{j-1}Q_0,\, p_0)\\
&\xrightarrow{[(0,\mathrm{id}),(f_{0,j}, f_{0,j-1})]} (r_\Lambda Q/r_\Lambda^{j+1}Q,\, Q/r_\Lambda^j Q,\, \Psi) = r_{T_2(\Lambda)}(Q, Q, \mathrm{id})/r_{T_2(\Lambda)}^{j+1}(Q, Q, \mathrm{id}) \to 0
\end{aligned}
$$

with $\Psi\colon r_\Lambda Q/r_\Lambda^{j+1}Q \to Q/r_\Lambda^j Q$, $p_k\colon Q_k/r_\Lambda^{j-k}Q \to Q_k/r_\Lambda^{j-k-1}Q$ and all other homomorphisms natural, is a minimal resolution of

$$(r_\Lambda Q/r_\Lambda^{j+1}Q,\, Q/r_\Lambda^j Q,\, \Psi) = r_{T_2(\Lambda)}(Q, Q, \mathrm{id})/r_{T_2(\Lambda)}^{j+1}(Q, Q, \mathrm{id})$$

in add M_1. It follows trivially that this sequence is exact so it remains to prove that $(M_1, -)$ preserves this exactness. We will start by proving that

$$(0,\, Q/r_\Lambda^j Q,\, 0) \amalg (Q_0/r_\Lambda^j Q_0,\, Q_0/r_\Lambda^{j-1}Q_0,\, p_0) \to (r_\Lambda Q/r_\Lambda^{j+1}Q,\, Q/r_\Lambda^j Q,\, \Psi) \to 0$$

remains exact after applying $(M_1, -)$. It is, as in Lemma 5, enough to prove that every homomorphism $\phi\colon N_1 \to (r_\Lambda Q/r_\Lambda^{j+1}Q,\, Q/r_\Lambda^j Q,\, \Psi)$ factors through

$$(0,\, Q/r_\Lambda^j Q,\, 0) \amalg (Q_0/r_\Lambda^j Q_0,\, Q_0/r_\Lambda^{j-1}Q_0,\, p_0) \to (r_\Lambda Q/r_\Lambda^{j+1}Q,\, Q/r_\Lambda^j Q,\, \Psi)$$

for every indecomposable $T_2(\Lambda)$-module N_1 in add M_1. If N_1 is of type

$$(0,\, Q'/r_\Lambda^i Q',\, 0)$$

for an indecomposable projective Λ-module Q' the proof goes exactly as in Lemma 5. So assume that

$$N_1 = (Q'/r_\Lambda^{i+1}Q',\, Q'/r_\Lambda^i Q',\, p')$$

for an indecomposable projective Λ-module Q' and p' the natural epimorphism (identity if $r_\Lambda^i Q = 0$). Let

$$\phi = (\alpha, \beta)\colon (Q'/r_\Lambda^{i+1}Q',\, Q'/r_\Lambda^i Q',\, p') \to (r_\Lambda Q/r_\Lambda^{j+1}Q,\, Q/r_\Lambda^j Q,\, \Psi).$$

Consider then the following commutative diagram:

$$
\begin{tikzcd}
 & & Q_0/r_\Lambda^j Q_0 \arrow[dl, "f_{0,j}"'] \arrow[d, "p_0"] \\
Q'/r_\Lambda^{i+1}Q' \arrow[r, "\alpha"] \arrow[d, "p'"] & r_\Lambda Q/r_\Lambda^{j+1}Q \arrow[d, "\Psi"] & Q_0/r_\Lambda^{j-1}Q_0 \arrow[dl, "f_{0,j-1}"] \\
Q'/r_\Lambda^i Q' \arrow[r, "\beta"] & Q/r_\Lambda^j Q &
\end{tikzcd}
$$

Then there exists a $g\colon Q'/r_\Lambda^{i+1}Q' \to Q_0/r_\Lambda^j Q_0$ such that $\alpha = f_{0,j} \circ g$. But now

$$g(r_\Lambda^i Q'/r_\Lambda^{i+1}Q') \subseteq g(\mathrm{Soc}\ Q'/r_\Lambda^{i+1}Q') \subseteq \mathrm{Soc}\ Q_0/r_\Lambda^j Q_0 = r_\Lambda^{j-1}Q_0/r_\Lambda^j Q_0.$$

So there exists an $h\colon Q'/r_\Lambda^i Q' \to Q_0/r_\Lambda^{j-1}Q_0$ such that $p_0 \circ g = h \circ p'$, i.e., we have got a homomorphism

$$\phi' = (g, h)\colon (Q'/r_\Lambda^{i+1}Q', Q'/r_\Lambda^i Q', p') \to (Q_0/r_\Lambda^j Q_0, Q_0/r_\Lambda^{j-1}Q_0, p_0)$$

such that $f_{0,j} \circ g = \alpha$. But now

$$\beta \circ p' = \Psi \circ \alpha = \Psi \circ f_{0,j} \circ g = f_{0,j-1} \circ p_0 \circ g = f_{0,j-1} \circ h \circ p'$$

and p' is epic so $\beta = f_{0,j-1} \circ h$. Totally we have

$$(f_{0,j}, f_{0,j-1}) \circ (g, h) = (\alpha, \beta) = \phi.$$

We then have that ϕ factors through

$$(Q_0/r_\Lambda^j Q_0, Q_0/r_\Lambda^{j-1}Q_0, p_0) \to (r_\Lambda Q/r_\Lambda^{j+1}Q, Q/r_\Lambda^j Q, \Psi)$$

and therefore through

$$(0, Q/r_\Lambda^j Q, 0) \amalg (Q_0/r^j Q_0, Q_0/r_\Lambda^{j-1}Q_0, p_0) \to (r_\Lambda Q/r_\Lambda^{j+1}Q, Q/r_\Lambda^j Q, \Psi).$$

So $(M_1, -)$ preserves the exactness of

$$(0, Q/r_\Lambda^j Q, 0) \amalg (Q_0/r_\Lambda^j Q_0, Q_0/r_\Lambda^{j-1}Q_0, p_0) \to (r_\Lambda Q/r_\Lambda^{j+1}Q, Q/r_\Lambda^j Q, \Psi) \to 0.$$

Now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, "\Psi_1"] & Q_0/r_\Lambda^j Q_0 \amalg 0 \arrow[r, "f_{0,j}"] \arrow[d, "\begin{bmatrix} p_0 & 0 \\ 0 & 0 \end{bmatrix}"] & r_\Lambda Q/r_\Lambda^{j+1}Q \arrow[r] \arrow[d, "\Psi"] & 0 \\
0 \arrow[r] & K_1' \arrow[r] & Q_0/r_\Lambda^{j-1}Q_0 \amalg Q/r_\Lambda^j Q \arrow[r, "{[f_{0,j-1}, \mathrm{id}]}"] & Q/r_\Lambda^j Q \arrow[r] & 0
\end{tikzcd}$$

in mod Λ. One easily sees that $K_1' \simeq Q_0/r_\Lambda^{j+1}Q_0$ because the last sequence splits and that $\operatorname{Ker}\Psi_1 = \operatorname{Soc} K_1$. Therefore we get a natural homomorphism $(Q_1/r_\Lambda^{j-1}Q_1, Q_1/r_\Lambda^{j-2}Q_2, p_1) \to (K_1, K_1', \Psi_1)$ and if we look at the natural homomorphism

$$(0, Q_0/r_\Lambda^{j-1}Q_0, 0) \amalg (Q_1/r_\Lambda^{j-1}Q_1, Q_1/r_\Lambda^{j-2}Q_1, p_1) \to (K_1, K_1', \Psi_1)$$

we see that the kernel of this has the same property as (K_1, K_1', Ψ_1). Now an analogous argument as before gives that $(M_1, -)$ preserves the exactness.

We observe that the Artin rings Λ that satisfy condition (2) in Lemma 8 have the property that $\operatorname{pd}(M_0, r_\Lambda Q/r_\Lambda^j Q) \leq \operatorname{pd}(M_0, r_\Lambda Q)$ for every indecomposable projective Λ-module Q. Therefore these rings satisfy the condition in Lemma 3.

Now follows a lemma which says that the class of rings that satisfy condition (1) and (2) in Lemma 8 contains the class of lower triangular matrix rings over a field k.

LEMMA 9. *Let*

$$\Lambda_m = \begin{pmatrix} k & 0 & & 0 \\ k & k & & 0 \\ \vdots & \vdots & & \vdots \\ k & & \cdots & k \end{pmatrix}$$

be the full lower triangular matrix ring over a field k with $\mathcal{L}(\Lambda_m) = m \geq 2$. Then Λ_m satisfies condition (1) *and* (2) *in Lemma* 8.

Proof. This follows from the fact that $r_{\Lambda_m} Q$ is an indecomposable projective Λ_m-module for every indecomposable projective Λ_m-module Q.

LEMMA 10. *The class $\mathcal{T}$ of all rings $T_2^n(\Lambda_m)$, $m \geq 2$, $n \geq 0$ where Λ_m is as in Lemma* 9 *satisfies conditions* (1) *and* (2) *in Lemma* 8.

Proof. Follows directly from Lemma 8 and 9.

LEMMA 11. *Let Λ_m be as in Lemma* 9 *and let*

$$M_{n,m} = T_2^n(\Lambda_m) \amalg T_2^n(\Lambda_m)/r_{T_2^n(\Lambda_m)}^{m+n-1} \amalg \cdots \amalg T_2^n(\Lambda_m)/r_{T_2^n(\Lambda_m)}$$

for $n \geq 0$, $m \geq 2$. Then gl. dim End $(M_{n+1,m})^{\text{op}} =$ gl. dim End $(M_{n,m})^{\text{op}} + 1$.

Proof. By the observation after Lemma 8 and Lemma 3 we have that

$$\begin{aligned}
&\text{gl. dim End } (M_{n+1,m})^{\text{op}} \\
&= \max \{\text{pd } (M_{n+1,m}, r_{T_2^{n+1}(\Lambda_m)} Q \mid \\
&\qquad\qquad Q \text{ indecomposable projective } T_2^{n+1}(\Lambda_m)\text{-module}\} + 2 \\
&= \max \{\text{pd } (M_{n+1,m}, r_{T_2^{n+1}(\Lambda_m)} (Q, Q, \text{id})), \text{pd } (M_{n+1,m}, r_{T_2^{n+1}(\Lambda_m)} (0, Q, 0) \mid \\
&\qquad\qquad Q \text{ indecomposable projective } T_2^n(\Lambda_m)\text{-module}\} + 2 \\
&= \max \{\text{pd } (M_{n,m}, r_{T_2^n(\Lambda_m)} Q) + 1, \text{pd } (M_{n,m}, r_{T_2^n(\Lambda_m)} Q) \mid \\
&\qquad\qquad Q \text{ indecomposable projective } T_2^n(\Lambda_m)\text{-module}\} + 2 \\
&= \max \{\text{pd } (M_{n,m}, r_{T_2^n(\Lambda_m)} Q) \mid \\
&\qquad\qquad Q \text{ indecomposable projective } T_2^n(\Lambda_m)\text{-module}\} + 3 \\
&= \text{gl. dim End } (M_{n,m})^{\text{op}} + 1.
\end{aligned}$$

We are now able to prove the main result in this paper.

THEOREM 12. *For every pair of integers n and m where $n \geq m \geq 2$ there exists an Artin algebra Λ with $\mathcal{L}(\Lambda) = n$ and* gl. dim End $(M)^{\text{op}} = m$ *where*

$$M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda.$$

Proof. As before, let Λ_i be the full lower triangular matrix ring over a field k with $\mathscr{L}(\Lambda_i) = i \geq 2$. Now select $i = 2 + n - m \geq 2$ and consider $T_2^{m-2}(\Lambda_{2+n-m})$. Then $\mathscr{L}(T_2^{m-2}(\Lambda_{2+n-m})) = m - 2 + 2 + n - m = n$ and

$$\begin{aligned}&\text{gl. dim End } (M_{m-2,2+n-m})^{\text{op}} \\ &= m - 2 + \text{gl. dim End } (M_{0,2+n-m})^{\text{op}} = m - 2 + 2 = m\end{aligned}$$

where $M_{i,j}$ are as in Lemma 11.

COROLLARY. *The inequality in the theorem referred to in the introduction is optimal.*

Proof. Select $m = n \geq 2$ in Theorem 12.

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