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A SPLITTING CRITERION FOR PAIRS OF LINEAR TRANSFORMATIONS

BY

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Introduction

A system, or more exactly a $\mathbf{C}^2$-system, is a pair of complex vector spaces V and W together with a system operation which is a $\mathbf{C}$-bilinear map $(e, v) \mapsto ev$ of $\mathbf{C}^2 \times V$ into W. For a fixed basis of $\mathbf{C}^2$, a system determines and is determined by a pair of linear transformations from V to W. See [3]. A homomorphism of a system (S, T) into a system (X, Y) is a pair (ϕ, ψ) of linear transformations $\phi: S \to X$ and $\psi: T \to Y$ such that $e\phi s = \psi es$ for all $e \in \mathbf{C}^2$ and $s \in S$.

The category of systems is equivalent to the category of modules over the subring of $M_3(\mathbf{C})$ consisting of matrices of the form

$$\begin{bmatrix} \beta & 0 & \alpha_1 \\ 0 & \beta & \alpha_2 \\ 0 & 0 & \gamma \end{bmatrix},$$

and contains subcategories equivalent to the category of modules over $\mathbf{C}[\zeta]$, the ring of complex polynomials in one variable. Systems in these subcategories are called nonsingular systems. See [1]. Many concepts and theorems in the theory of modules over $\mathbf{C}[\zeta]$ carry over to the category of systems.

In this paper we prove:

(1) A system of bounded height (defined below) is a direct sum of finite-dimensional indecomposable systems. The nonsingular analogue of this is Kulikov's well-known theorem on bounded modules. See [5, Theorem 6].

(2) Systems of bounded height are pure injective.

(3) A torsion system, (X, Y) has the property that every mixed system (U, Z) with (X, Y) as torsion part splits if and only if (X, Y) is a direct sum of a divisible system and a bounded system.

An analogous result for abelian groups is Baer's characterization of torsion cotorsion groups [4, Theorem 100.1].

In the light of the above results and others in the literature it is interesting that an easy but important result in the theory of modules over $\mathbf{C}[\zeta]$ fails to hold for systems, namely: The intersection of pure subsystems in a torsion-free system is not necessarily pure.

This will be shown by means of a simple example.

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1. Preliminaries

This section is for the convenience of the reader and may be skipped by those familiar with our references.

DEFINITION 1.1. (a) A system is a pair of vector spaces (V, W) together with a system operation which is a **C**-bilinear map $(e, v) \mapsto ev$ of $\mathbf{C}^2 \times V$ into W. (V, W) is said to be finite-dimensional if $\dim V + \dim W < \infty$.

(b) A system (V, W) is nonsingular if there exists $e \in \mathbf{C}^2$ such that the map $v \mapsto ev$ is an isomorphism of V onto W.

A system (V, W) is ordinary if $V = W$ and there is an $e \in \mathbf{C}^2$ that acts like the identity on V. (Every nonsingular system is isomorphic to an ordinary system [1, p. 281].)

DEFINITION 1.2. (a) A system (V, W) is said to be torsion-free in case all the linear transformations $v \mapsto ev$, $0 \neq e \in \mathbf{C}^2$, are injective.

(b) Let (a, b) be a basis of $\mathbf{C}^2$. $\theta \in \tilde{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is said to be an eigenvalue of a system, (V, W), if $b_\theta v = 0$ for some $0 \neq v \in V$. ($b_\theta = b - \theta a$ if $\theta \neq \infty$; if $\theta = \infty$, $b_\theta = a$).

For any system (V, W) there exists a smallest subsystem $t(V, W)$, of (V, W) such that $(V, W)/t(V, W)$ is torsion-free [1, p. 324]. (V, W) is said to be torsion if $t(V, W) = (V, W)$.

(c) Let X, Y be subsets of V, W respectively. There exists a smallest subsystem, (V^1, W^1), of (V, W) with $X \subset V^1$, $Y \subset W^1$ such that $(V, W)/(V^1, W^1)$ is torsion-free. (V^1, W^1) is called the torsion-closure of (X, Y) and is denoted by $tc_{(V,W)}(X, Y)$. A subsystem (X, Y), of (V, W) is said to be torsion-closed if (X, Y) is the torsion-closure of (X, Y) i.e., if $(V, W)/(X, Y)$ is torsion-free.

(d) A system, (V, W), is said to be of rank 1 if $(V, W) = tc_{(V,W)}(\phi, w)$ for all $0 \neq w \in W$ [2, p. 433 and Lemma 2.2].

(e) A system, (V, W), is said to be a divisible system if $eV = W$ for all $0 \neq e \in \mathbf{C}^2$.

Observe that the definition of eigenvalue depends on the choice of basis of $\mathbf{C}^2$. However, the property of having no eigenvalues is not so dependent because a system is torsion-free if and only if it has no eigenvalues. In any case, a change of basis of $\mathbf{C}^2$ involves a Moebius transformation of the parameters giving the eigenvalues [1, p. 282]. As a result we conclude that the number of eigenvalues of a system is an invariant of the system.

DEFINITION 1.3. Let (V, W) be a system, $v_i \in V$, $w_i \in W$.

(a) A chain $((v_1, v_2, \ldots, v_{m-1}), (w_1, w_2, \ldots, w_m))$ is said to be of type III^m if $av_1 = w_1$, $av_i = w_i = bv_{i-1}$, $i = 2, \ldots, m-1$, $bv_{m-1} = w_m$. If $m = 1$, the chain is (ϕ, w_1).

(b) A chain $((v_1, v_2, \ldots, v_m), (w_1, w_2, \ldots, w_m))$ is said to be of type II^m_θ if $b_\theta v_1 = 0$, $av_i = w_i = b_\theta v_{i+1}$, $i = 1, \ldots, m-1$, $av_m = w_m$.

Let V' and W' be the respective spans of the v_i's and w_j's. The subsystem, (V', W'), of (V, W) is called the subsystem spanned by $((v_i), (w_j))$. In case the v_i's and w_j's form bases of V' and W' respectively (V', W') is itself called a subsystem of type III^m or II_θ^m depending on the type of chain which spans it.

Remark 1.4. (a) In [1, p. 282] the types are defined in a way that makes it obvious that being of type III^m is independent of the choice of a basis of $\mathbf{C}^2$. However, a change of basis of $\mathbf{C}^2$ changes a system of type II_θ^m to one of type II_η^m (same m) with η related to θ by a Moebius transformation; see the remark following 1.2. The equivalence of our definition of the types to that in [1] is the content of [1, Proposition 2.6].

(b) Systems of type III^m are torsion-free and of rank one [2, Lemma 2.2].

(c) A subsystem of a system of type III^m is isomorphic to

$$(V_1, W_1) \oplus \cdots \oplus (V_n, W_n)$$

for some positive integer, n, where (V_i, W_i) is of type III^{m_i}, $m_i \leq m$ for all $i = 1, 2, \ldots, n$. The decomposition follows from [1, Theorem 4.3] and (b) above and the inequality holds because $\sum_{i=1}^{n} m_i \leq m$.

For a fixed positive integer m and a basis (a, b) of $\mathbf{C}^2$, the chains of type III^m in a system (U, Z) form a vector space, denoted in [1] by $CIII^m(a, b; U, Z)$.

It has a subspace, $\hat{C}III^m(a, b; U, Z)$, consisting of all chains of type III^m in (U, Z) which are sums of two type III^m chains,

$$((x_1^1, \ldots, x_{m-1}^1), (y_1^1, \ldots, y_m^1)) \quad \text{and} \quad ((x_1^2, \ldots, x_{m-1}^2), (y_1^2, \ldots, y_m^2)),$$

such that $y_1^1 = bx_0^1$ for some $x_0^1 \in U$ and $y_m^2 = ax_m^2$ for some $x_m^2 \in U$. The quotient space $CIII^m(a, b; U, Z)/\hat{C}III^m(a, b; U, Z)$ is denoted by $QIII^m(a, b; U, Z)$.

Given a chain $((x_1, \ldots, x_m), (y_1, \ldots, y_n))$ *in* (U, Z) *the subsystem of* (U, Z) *spanned by the chain is the smallest subsystem* (X, Y) *satisfying* $x_1, \ldots, x_m \in X$ *and* $y_1, \ldots, y_n \in Y$.

2. Bounded systems

LEMMA 2.1. *Let* (U, Z) *be a torsion-free system and* (V, W) *a torsion-closed subsystem of* (U, Z). *If* (X, Y) *is a rank* 1 *subsystem of* (U, Z) *not contained in* (V, W) *then* $(V, W) \cap (X, Y) = 0$. *In particular, distinct torsion closed rank* 1 *subsystems of* (U, Z) *intersect trivially.*

Proof. Suppose $(V, W) \cap (X, Y) \neq 0$. By torsion-freeness this implies that $W \cap Y \neq 0$. Let $0 \neq y \in W \cap Y$. Since (X, Y) has rank 1,

$$(X, Y) = tc_{(X,Y)}(\phi, y).$$

But $tc_{(X,Y)}(\phi, y) \subseteq tc_{(U,Z)}(\phi, y) = tc_{(V,W)}(\phi, y) \subseteq (V, W)$. The last equality comes from the fact that (V, W) is torsion-closed and [2, 2.1(e)]. So $(X, Y) \subseteq (V, W)$, a contradiction.

DEFINITION 2.2. (a) [3, p. 736] A subsystem (S, T) of (U, Z) is said to be pure in (U, Z) provided for every intermediate subsystem (X, Y), $(S, T) \subset (X, Y) \subset (U, Z)$ such that $(X, Y)/(S, T)$ is finite-dimensional, (S, T) is a direct summand of (X, Y).

(b) A system (S, T) is said to be pure injective if it is a direct summand of any system containing it as a pure subsystem.

We shall now derive a corollary to 2.1.

COROLLARY 2.3. *In a torsion-free system of rank at most two the intersection of pure subsystems is again pure.*

Proof. Torsion-free systems of rank 1 are purely simple, i.e., have no proper pure subsystems [2, p. 433].

Now pure subsystems of a torsion-free system are torsion-closed by [2, 2.1(g)]. So if (U, Z) has rank 2, nontrivial pure subsystems have rank 1 by [2, 2.4]. Therefore in this case the corollary follows from Lemma 2.1.

Remark 2.4. (a) *Unlike the situation for modules over* $\mathbf{C}[\zeta]$ *in an arbitrary torsion-free system the intersection of pure subsystems is not necessarily pure.* Since a subsystem of a finite-dimensional system is pure if and only if it is a direct summand [1, Theorem 5.5], this is shown by the following example of a finite-dimensional system with two direct summands whose intersection is not a direct summand: Let (a, b) be a basis of $\mathbf{C}^2$ and

$$(V, W) = (X, Y) \oplus (S, T) \quad \text{where } (X, Y) = (X_1, Y_1) \oplus (X_2, Y_2)$$

with $((x_1), (y_1, y_2'))$, $((x_2), (y_2, y_3'))$ spanning (X_1, Y_1) and (X_2, Y_2) respectively, where $ax_1 = y_1$, $bx_1 = y_2'$, $ax_2 = y_2$, $bx_2 = y_3'$ with x_i's and y_j's and y_j''s bases of X and Y respectively; (S, T) is spanned by $((s_1, s_2), (t_1, t_2, t_3))$ where $as_1 = t_1$, $bs_1 = t_2 = as_2$, $bs_2 = t_3$ with the s_i's and t_j's bases of S and T respectively.

(V, W) is also equal to $(X^1, Y^1) \oplus (S, T)$ where (X^1, Y^1) is spanned by

$$((x_1 + x_2 + s_1), (y_1 + y_2 + t_1, y_2' + y_3' + t_2)) \oplus ((x_1 - x_2 + s_2), (y_1 - y_2 + t_2, y_2' - y_3' + t_3))$$

with a and b acting as above.

$$(X, Y) \cap (X^1, Y^1) = (X \cap X^1, Y \cap Y^1) = (0, \mathbf{C}(y_2' + y_3' - y_1 + y_2)).$$

By the uniqueness up to isomorphism of decomposition of a finite-dimensional system into a direct sum of indecomposables [1, p. 309], the subsystem $(0, \mathbf{C}(y_2' + y_3' - y_1 + y_2))$ cannot be a direct summand in (V, W).

(b) It is easy to show that for (V, W) a nonsingular torsion-free system the following property characterizes the nonsingular torsion-free systems of rank not exceeding two:

(1) *Any two distinct nontrivial pure subsystems of* (V, W) *have zero intersection.*

However any singular system $(V, W) = (0, \mathbf{C}y_1) \oplus (V^1, W^1)$ where $y_1 \neq 0$ and (V^1, W^1) is purely simple of rank ≥ 2, satisfies Property (1) even though rank $(V, W) \geq 3$.

DEFINITION 2.5. Let (V, W) be a torsion-free system. An element $w \in W$ is said to give a chain of type III^m in (V, W) if there exists $v_1, \ldots, v_{m-1}$, in V, w_1, $w_2, \ldots, w_m$ in W with $w_1 = w$ such that $((v_1, v_2, \ldots, v_{m-1}), (w_1, w_2, \ldots, w_m))$ is a chain of type III^m.

This definition depends on the choice of basis (a, b) of $\mathbf{C}^2$. However, if (V, W) is of type III^m then for any choice of basis of $\mathbf{C}^2$ there always exists a nonzero element in W that gives a chain of type III^m. This follows Remark 1.4(a) and our definition of type III^m. The following is immediate:

Let $(V, W) = \prod_J (V_j, W_j)$, J an arbitrary indexing set. Then $(w_j)_{j \in J}$, $w_j \in W_j$ gives a chain of type III^m if and only if each w_j does the same in (V_j, W_j) for all $j \in J$.

LEMMA 2.6. *Let (V, W) be a torsion-free system and (X, Y) a subsystem spanned by a type III^m chain $((x_1, x_2, \ldots, x_{m-1}), (y_1, y_2, \ldots, y_m))$. Then (X, Y) is of type III^m if and only if at least one of the x_i's or y_i's is not zero.*

Proof. Suppose at least one of the x_i's or y_i's is not zero. Let (S, T) be a system of type III^m spanned by a chain

$$(s_j)_{j=1}^{m-1}, (t_j)_{j=1}^{m}$$

in $CIII^m(a, b; S, T)$. Define linear maps $\phi: S \to X$, $\psi: T \to Y$ by the requirements $\phi(s_j) = x_j$, $\psi(t_j) = y_j$. Then $(\phi, \psi): (S, T) \to (X, Y)$ is an epimorphism of systems. By assumption, $(\phi, \psi) \neq (0, 0)$. Hence by [2, Lemma 3.1], (ϕ, ψ) is a monomorphism. Hence $(X, Y) \cong (S, T)$.

Conversely if all of the x_i's and y_j's are zero then (X, Y) would be the zero system.

The remark following 1.2 and Remark 1.4(a) make the following definition independent of the basis of $\mathbf{C}^2$.

DEFINITION 2.7. (a) A torsion system (X, Y) is said to be bounded if and only if it satisfies the following conditions:

(i) (X, Y) has finitely many eigenvalues.

(ii) There exists a positive integer M such that (X, Y) has no subsystem of any type II_θ^m with $m > M$.

(b) A torsion-free system (V, W) is said to be of bounded height if and only if there exists a positive integer M such that (V, W) has no subsystem of type III^m with $m > M$. In this case we say that (V, W) is of bounded height not exceeding $M - 1$.

(c) Let (X, Y) be the torsion part of a system (U, Z). (U, Z) is said to be of bounded height if and only if (X, Y) is bounded and $(U, Z)/(X, Y)$ is of bounded height.

LEMMA 2.8. *A torsion-free system, (V, W), of bounded height not exceeding $M - 1$ is a direct sum of finite-dimensional indecomposable subsystems of the types III^m, $m \leq M$.*

Proof. Every indecomposable finite-dimensional subsystem of (V, W) is of type III^m by [1, Theorem 4.3] and by 2.7(b), $m \leq M$.

Choose chains $(\Gamma_m^j)_{j \in J_m}$ in $CIII^m(a, b; V, W)$ representing a basis of $QIII^m(a, b; V, W)$. Let (V_m^j, W_m^j) denote the subsystem of (V, W) spanned by the chain Γ_m^j. By [1, Theorem 6.7], (V_m^j, W_m^j) is of type III^m and

(2) $(V_0, W_0) = \sum_{m=1}^{M} \sum_{j \in J_m} (V_m^j, W_m^j)$ is a maximal pure direct sum of finite-dimensional indecomposable subsystems.

Claim. $(V_0, W_0) = (V, W)$.

We shall assume the contrary and derive a contradiction to (2). $(V_0, W_0) \neq (V, W)$ implies that $W_0 \neq W$ because if $W_0 = W$, then $(V, W)/(V_0, W_0)$ is isomorphic to $(V/V_0, 0)$. The latter must be torsion-free because (V_0, W_0) is pure in (V, W) [2, Lemma 2.1(g)]. This happens if and only if $V = V_0$ leading us to $(V_0, W_0) = (V, W)$. So let $w \in W \backslash W_0$, and $(X_1, Y_1) = (0, Cw)$. The subsystem (X_1, Y_1) is of type III^1 and $(X_1, Y_1) \cap (V_0, W_0) = (0, 0)$. Assume that for an integer $1 \leq m \leq M$ we have found (X_m, Y_m) where $(X_m, Y_m) \subset (V, W)$ is of type III^m and $(V_0, W_0) \cap (X_m, Y_m) = (0, 0)$.

Let Δ^m denote a chain of type III^m spanning (X_m, Y_m). By the choice of $(\Gamma_m^j)_{j \in J_m}$, $(\Gamma_m^j)_{j \in J_m} \cup \Delta^m$ cannot be independent modulo $\hat{C}III^m(a, b; V, W)$. Therefore, there exists a finite subset K of J_m such that

$$\Delta = \Delta^m - \sum_{j \in K} \alpha_j \Gamma_m^j \in \hat{C}III^m(a, b; V, W), \quad \alpha_j \in \mathbf{C}.$$

i.e.,
$$\Delta = ((x_j)_{j=1}^{m-1}, (y_j)_{j=1}^{m}) + ((x_j^1)_{j=1}^{m-1}, (y_j^1)_{j=1}^{m})$$

where the chains are extendible to chains

$$((x_j)_{j=0}^{m-1}, (y_j)_{j=0}^{m}) \quad \text{and} \quad ((x_j^1)_{j=1}^{m}, (y_j^1)_{j=1}^{m+1})$$

of $CIII^{m+1}(a, b; V, W)$. Let (X_{m+1}, Y_{m+1}), (X_{m+1}^1, Y_{m+1}^1) denote the subsystems of (V, W) spanned by the latter. By 2.6 and the fact that Δ is not the zero chain (since that would imply that $(X_m, Y_m) \subset (V_0, W_0)$), at least one of (X_{m+1}, Y_{m+1}), (X_{m+1}^1, Y_{m+1}^1) is of type III^{m+1}. We have

$$(X_m, Y_m) \subset (X_{m+1}, Y_{m+1}) + (X_{m+1}^1, Y_{m+1}^1) + (V_0, W_0),$$

so (V_0, W_0) does not contain at least one of (X_{m+1}, Y_{m+1}) and (X_{m+1}^1, Y_{m+1}^1). Say (V_0, W_0) does not contain (X_{m+1}, Y_{m+1}). By Lemma 2.1, $(V_0, W_0) \cap (X_{m+1}, Y_{m+1}) = 0$.

By induction we find that (V, W) contains a subsystem of type III^{M+1}, contradicting (2). Therefore $(V_0, W_0) = (V, W)$ as required.

THEOREM 2.9. *A system (U, Z) of bounded height is a direct sum of finite-dimensional indecomposable subsystems.*

Proof. Let $t(U, Z)$ denote the torsion part of (U, Z). By hypothesis it has only finitely many eigenvalues, so by [1, p. 338], it corresponds to a module over $\mathbf{C}[\zeta]$. Our definition of bounded system implies that the corresponding module is bounded in the sense of modules over $\mathbf{C}[\zeta]$. See [5, p. 36 and p. 16] for the definition. Such modules are direct sum of modules of the form $\mathbf{C}[\zeta]/(\zeta - \theta)^n\mathbf{C}[\zeta]$, n a positive integer. These modules correspond to systems of type II_θ^n and such systems are indecomposable [1, Proposition 2.2]. We have

$$E\colon 0 \to t(U, Z) \to (U, Z) \to (U, Z)/t(U, Z) \to 0.$$

By Lemma 2.8, $(U, Z)/t(U, Z)$ is a direct sum of systems of type III^m. Ext $(\bigoplus_{i \in I} III^{m_i}, t(U, Z))$ is isomorphic to $\prod_I$ Ext $(III^{m_i}, t(U, Z))$ which is 0 as is readily seen by [1, Proposition 9.12] and the definition of purity. Therefore

$$(U, Z) \cong t(U, Z) \oplus (U, Z)/t(U, Z),$$

and by the first part of the proof, we are done.

Remark. The assumption on $t(U, Z)$ in Theorem 2.9 can be relaxed by using Kulikov's theorem on primary $\mathbf{C}[\zeta]$-modules. We considered only the systems of bounded height in the sense of Definition 2.7 because these are the systems which play a role in Theorem 3.5.

3. Mixed systems

We need some facts on pure injective systems that can be proved directly or deduced from results in [6]. The author in [6] speaks of purity with respect to a family of objects in an abelian category. In her terminology, purity as we have defined it is $\mathscr{I}$-purity where $\mathscr{I}$ is the family of finite-dimensional systems.

PROPOSITION 3.1. (a) *A direct product of pure injective systems is pure injective.*

(b) *A direct summand of a pure injective system is pure injective.*

PROPOSITION 3.2. *Let m be any fixed integer.*

(a) *Let J be an infinite indexing set and (V, W) a system of type $\prod_J III^m$ ($\prod_J II_\theta^m$, for a fixed θ). Then (V, W) is a system of type $\bigoplus_{J_0} III^m$ ($\bigoplus_{J_0} II_\theta^m$) where* card $(J_0) = 2^{\text{card}\,(J)}$.

(b) *Let J be any indexing set. Then systems of type $\bigoplus_J III^m$ ($\bigoplus_J II_\theta^m$) are pure injective.*

Proof. (a) Let (X, Y) be a given system of type III^m and y a nonzero element of Y. *The system* $tc_{(X,Y)}(\phi, y)$ is, by 1.4(b) and 1.2(d) equal to

(X, Y) hence is of type III^m. Therefore *by Lemma* 2.6, y gives a chain of type III^l, $l \leq m$. By 1.4(c) *and the remark following* 2.5, a similar statement holds for subsystems of (X, Y). Since (V, W) is of type $\prod_J III^m$, we conclude from the last statement and the remark following 2.5 that (V, W) is of bounded height not exceeding $m - 1$. Therefore by Lemma 2.8 it is a direct sum of subsystems of type III^{n_k} with $n_k \leq m$. By the remark following 2.5 any non-zero element, w, in W that gives a chain of type III^m is contained in a sum of range spaces of components of (V, W) in the direct sum decomposition with $n_k = m$. Let $W_1 = \prod_{j \in J} \mathbf{C}w_j$ be the vector subspace of W, where $0 \neq w_j$ gives a chain of type III^m in (V_j, W_j) (such w_j's exist for each $j \in J$ by the remark following the definition in 2.5). Note also that if w gives a chain of type III^m in any system so does $\alpha \cdot w$ for any $\alpha \in \mathbf{C}$). W_1 is isomorphic to Hom $(\bigoplus_J \mathbf{C}, \mathbf{C}) = (\bigoplus_J \mathbf{C})^*$ hence W_1 has dimension $2^{\text{Card}\,(J)}$ (see N. Jacobson, *Lectures in algebra*, vol. 2, page 247, Theorem 2). Any element in a basis of W_1 gives a chain of type III^m in (V, W) by the next to last sentence in brackets and 2.5. Hence there are at least $2^{\text{Card}\,(J)}$ linearly independent elements, w, in W giving chains of type III^m. Hence $n_k = m$ for $2^{\text{Card}\,(J)}$ components in the direct sum decomposition, again by the remark following 2.5. We now prove that $n_k = m$ for all the components by showing that (V, W) cannot have a direct summand of type III^l with $l < m$.

Suppose $\Gamma = ((v_1, \ldots, v_{l-1}), (w_1, \ldots, w_l)) \in CIII^l(a, b;\ V, W)$ spans such a summand. Let $(V, W) = \prod_J (V^j, W^j)$ where (V^j, W^j) is of type III^m for all $j \in J$. The projection $\pi_j\colon V \to V^j$ is defined by $\pi_j((v^h)_{h \in J}) = v^j$; $\rho_j\colon W \to W^j$ is defined similarly and

$$(\pi_j, \rho_j)\colon (V, W) \to (V^j, W^j)$$

is a system epimorphism and $(\pi_j, \rho_j)\Gamma = \Gamma^j$ is a chain in $CIII^l(a, b;\ V^j, W^j)$. Since (V^j, W^j) is indecomposable, $\Gamma^j \in \hat{C}III^l(a, b;\ V^j, W^j)$ by [1, Theorem 6.6]. It is easy to see that this implies that $\Gamma \in \hat{C}III^l(a, b;\ V, W)$ which is a contradiction again by [1, Theorem 6.6]. By [1, p. 338, a system of type II_θ^m is isomorphic to an ordinary system which in turn corresponds to a $\mathbf{C}[\zeta]$ module. In that case the conclusion that (V, W) is of type $\bigoplus_{2^{\text{Card}\,(J)}} II_\theta^m$ follows from [5, Theorem 17.2], cardinality considerations and the analogue of the remark following Definition 2.5 for order of an element in $\prod_J \mathbf{C}[\xi]/(\xi - \theta)^m\mathbf{C}[\xi]$ at the irreducible polynomial $\xi - \theta$.

(b) If card $(J) < \infty$, (b) follows from [1, Theorem 5.5]. So we may assume J is an infinite set. A system (V, W) of type $\bigoplus_J III^m$ is isomorphic to a direct summand of a system of type $\bigoplus_{J_0} III^m$ where card $(J_0) = 2^{\text{card}\,(J)}$. The latter is isomorphic to a system of type $\prod_J III^m$ by 3.2(a), which is pure injective by [1, Theorem 5.5] and Proposition 3.1(a). So (V, W) is pure injective, by 3.1(b). Replace III^m by II_θ^m throughout to get the proof for (V, W) of type $\bigoplus_J II_\theta^m$.

THEOREM 3.3. *A system of bounded height is pure injective.*

Proof. Let (U, Z) be a system of bounded height. By the proof of 2.9,

$$(U, Z) \cong t(U, Z) \oplus (U, Z)/t(U, Z).$$

It is enough, by Proposition 3.1, to show that each component is pure injective. $t(U, Z) = \sum_{\theta \in \mathbf{C}} t(U, Z)_\theta$, where $t(U, Z)_\theta$ is the smallest subsystem of $t(U, Z)$ such that $t(U, Z)/t(U, Z)_\theta$ does not have θ as an eigenvalue [1, Proposition 9.19]. Since $t(U, Z)$ is bounded $t(U, Z)_\theta = 0$ except for finitely many θ. Since $t(U, Z)_\theta$ is bounded, it is a system of type $\sum_{k=1}^{l} (\bigoplus_{J_k} (II_\theta^{n_k}))$. Each $\bigoplus_{J_k} (II_\theta^{n_k})$ is pure injective by 3.2(b). Hence by Proposition 3.1(a), $t(U, Z)_\theta$ is pure injective. $(U, Z)/t(U, Z)$ can be shown to be pure injective in a similar fashion using 2.8, 3.2(b) and 3.1(a). This completes the proof of Theorem 3.3.

DEFINITION 3.4. (a) A mixed system, (V, W), is a system with the property that both $t(V, W)$ and $(V, W)/t(V, W)$ are nonzero.

(b) A mixed system is said to split if the torsion part is a direct summand.

(c) A splitting criterion is a condition on a torsion module, T, such that every sequence $0 \to T \to G \to G/T \to 0$ with G/T torsion-free splits.

In any torsion theory in an abelian category one may ask for a splitting criterion for mixed objects in the category. Our last result gives such a criterion for the category of systems.

THEOREM 3.5. *A torsion system (X, Y) has the property that every mixed system (U, Z) with (X, Y) as torsion part splits if and only if (X, Y) is a direct sum of a divisible system and a bounded system.*

Proof. We have $E\colon 0 \to (X, Y) \to (U, Z) \to (U, Z)/(X, Y) \to 0$. By [1, Proposition 9.12], (X, Y) is pure in (U, Z). Suppose

$$(X, Y) = (X^1, Y^1) \dotplus (X^2, Y^2),$$

where (X^1, Y^1) is divisible and (X^2, Y^2) is bounded. By [1, Theorem 9.15], (X^1, Y^1) is pure injective and by Theorem 3.3, (X^2, Y^2) is pure injective. Therefore by 3.1(a), (X, Y) is pure injective, hence E splits.

For the converse, we have $(X, Y) = (X^1, Y^1) \dotplus (X^2, Y^2)$ where (X^1, Y^1) is divisible and (X^2, Y^2) is reduced, i.e., has no nonzero divisible subsystem [1, Corollary 9.16]. Since

$$\operatorname{Ext}((V, W), (X^1, Y^1) \dotplus (X^2, Y^2))$$

is isomorphic to

$$\operatorname{Ext}((V, W), (X^1, Y^1)) \oplus \operatorname{Ext}((V, W), (X^2, Y^2))$$

and $\operatorname{Ext}((V, W), (X^1, Y^1)) = 0$ if (V, W) is torsion-free, by [1, 9.12 and 9.15], it is enough in the proof of the converse to assume that (X, Y) is reduced and unbounded and prove that there is a torsion-free system (V, W) such that $\operatorname{Ext}((V, W), (X, Y)) \neq 0$. We want to reduce to the case that (X, Y) is nonsingular. $(X, Y) = \sum_{\theta \in \mathbf{C}} (X, Y)_\theta$. If there exists a θ in $\tilde{\mathbf{C}}$ that is not an eigenvalue, i.e., $(X, Y)_\theta = 0$, then (X, Y) is nonsingular by [1, p. 338]. So we may assume that every $\theta \in \tilde{\mathbf{C}}$ is an eigenvalue. In that case $\sum_{\theta \in \mathbf{C}} (X, Y)_\theta$ is an unbounded and nonsingular direct summand of (X, Y). It is therefore sufficient to show that

$$\operatorname{Ext}\left((V, W), \sum_{\theta \in \mathbf{C}} (X, Y)_\theta\right) \neq 0$$

for some torsion-free system (V, W). Since a nonsingular system is isomorphic to an ordinary system, it suffices to treat the case of an ordinary system (X, X). In [4, Theorem 100.1] it is shown that if a reduced torsion group G is not bounded, then there is a nonsplitting mixed group, H, with G as torsion part. The same result goes through for modules over $\mathbf{C}[\zeta]$ by replacing all the primes that occur in the proof by appropriate irreducible polynomials in $\mathbf{C}[\zeta]$. So by the correspondence between modules over $\mathbf{C}[\zeta]$ and nonsingular systems we get a nonsplitting exact sequence

$$0 \to (X, X) \to (U, U) \to (V, V) \to 0$$

with (V, V) torsion-free.

Remark. It would be interesting to know what systems (V, W) have the property that $\operatorname{Ext}((V, W), (X, Y)) = 0$ for all torsion systems (X, Y). One can prove the following partial result: *Let* (V, W) *be a system with* $\operatorname{Ext}((V, W), (X, Y)) = 0$ *for all torsion systems* (X, Y) *then* (V, W) *is a singular system with no nonsingular subsystem.*

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