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MINIMAL TOPOLOGIES AND L_p-SPACES

BY

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Abstract

This paper deals with minimal topologies on Riesz spaces. A minimal topology is a Hausdorff locally solid topology that is coarser than any other Hausdorff locally solid topology on the space. It is shown that every minimal topology satisfies the Lebesgue property, that an Archimedean Riesz space can admit a locally convex-solid topology that is minimal if and only if the space is discrete, that $C[0, 1]$ and $L_\infty([0, 1])$ do not admit a minimal topology, and that the topology of convergence in measure on $L_p([0, 1])$ $(0 \le p < \infty)$ is a minimal topology. A similar result is shown for certain Orlicz spaces.

1. Preliminaries

For notation and basic terminology concerning Riesz spaces not explained below, see [2] and [5]. We recall briefly the basic concepts needed for this paper. Let L be a Riesz space. A net $\{u_\alpha\}$ of L is *order convergent* to u, in symbols

$$u_\alpha \xrightarrow{(o)} u,$$

if there exists a net $\{v_\alpha\}$ with the same indexed set such that $|u_\alpha - u| \le v_\alpha$ for all α and $v_\alpha \downarrow \theta$. A subset S of L is said to be

(i) solid if $|u| \le |v|$ and $v \in S$ imply $u \in S$,
(ii) σ-order closed if S contains its sequential order limits,
(iii) order closed if S contains its order limits.

A *Riesz subspace* K of L is a vector subspace K of L such that $u \vee v \in K$ for every pair $u, v \in K$. The Riesz subspace K is said to be

(a) order dense if for every $\theta < u \in L$, there exists $\theta < v \in K$ with $\theta < v \le u$,
(b) super order dense if for every $\theta < u \in L$, there exists a sequence $\{u_n\} \subseteq K$ with $\theta \le u_n \uparrow u$ in L.

An *ideal* A of L is a solid vector subspace of L; every ideal is a Riesz subspace. A σ-order closed ideal is called a *σ-ideal*, and an order closed ideal is called a *band*.

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A *locally solid topology* τ on a Riesz space is a linear topology τ having a basis for zero consisting of solid sets. A locally solid topology τ on a Riesz space L is said to be

(1) Lebesgue if $u_\alpha \downarrow \theta$ in L implies $u_\alpha \xrightarrow{\tau} \theta$,
(2) Pre-Lebesgue if $\theta \leq u_n \downarrow$ implies that $\{u_n\}$ is a τ-Cauchy sequence,
(3) σ-Fatou if τ has a basis for zero consisting of solid and σ-order closed sets,
(4) Fatou if τ has a basis for zero consisting of solid and order closed sets.

A sequence $\{S_n\}$ of subsets of a vector space is said to be *normal* if $S_{n+1} + S_{n+1} \subseteq S_n$ holds for all n. If now L is a Riesz space and $\{S_n\}$ is a normal sequence of solid sets, then $N = \bigcap_{n=1}^{\infty} S_n$ is an ideal of L, called the null ideal of $\{S_n\}$. If, in addition, each S_n is σ-order closed (resp. order closed), then N is a σ-ideal (resp. a band) of L.

DEFINITION 1. Let τ be a locally solid topology on a Riesz space L, and let $\mathcal{N}$ denote the collection of all normal sequences of solid τ-neighborhoods of zero. The carrier C_τ of τ is defined by

$$C_\tau = \bigcup \left\{N^d\colon N = \bigcap_{n=1}^{\infty} V_n, \{V_n\} \in \mathcal{N}\right\}.$$

We note that the carrier C_τ can also be introduced in terms of Riesz pseudonorms. Thus, if for a Riesz pseudonorm ρ we put $N_\rho = \{u \in L\colon \rho(u) = 0\}$, then

$$C_\tau = \bigcup \{N_\rho^d\colon \rho \text{ is a } \tau\text{-continuous Riesz pseudonorm}\}.$$

The carrier C_τ is always a σ-ideal of L [1, Lemma 2.1, p. 4]. If two locally solid topologies on a Riesz space satisfy $\tau_1 \subseteq \tau_2$, then $C_{\tau_1} \subseteq C_{\tau_2}$. A locally solid topology will be called *entire* if its carrier is order dense. Note that every entire topology is necessarily Hausdorff. Every Hausdorff Fatou topology is entire [1, Lemma 2.1, p. 4].

A linear functional ϕ is called order continuous if $u_\alpha \downarrow \theta$ in L implies $\lim \phi(u_\alpha) = 0$. The band of the order continuous linear functionals on L is denoted by $L_n^\sim$. The null ideal of an order bounded linear functional ϕ is $N_\phi = \{u \in L\colon |\phi|(|u|) = 0\}$, and its carrier C_ϕ is defined by $C_\phi = N_\phi^d$.

We close this section by mentioning that for this paper *all topologies will be assumed to be Hausdorff.*

2. Minimal Topologies

If a Riesz space L admits a locally solid topology, then it also admits a maximal topology, that is, a locally solid topology that is finer than any other locally solid topology on L. The maximal topology is simply the topology generated by the collection of all Riesz pseudonorms on L.

It is, therefore, natural to ask whether a locally solid Riesz space admits a minimal topology, that is a locally solid topology that is coarser than any other locally solid topology on the Riesz space. Of course, if such a topology exists, it must be uniquely determined. In this section we shall discuss the properties of minimal topologies and present some interesting examples of Riesz spaces with minimal topologies. On the other hand, we shall see that not every locally solid Riesz space has a minimal topology.

DEFINITION 2. A Hausdorff locally solid topology ξ on a Riesz space L is called minimal if ξ is coarser than any other Hausdorff locally solid topology τ on L, i.e., $\xi \subseteq \tau$.

The basic properties of a minimal topology are included in the next theorem.

THEOREM 3. *Let L be a Riesz space with a minimal topology ξ. Then the following statements hold*:

(i) *Every disjoint sequence of L is ξ-convergent to zero.*
(ii) *ξ is a Lebesgue topology.*

Proof. Let $\mathscr{P}$ denote the set of all ξ-continuous Riesz pseudonorms on L. For each $\rho \in \mathscr{P}$ and $v \in L^+$ define $\rho_v(u) = \rho(|u| \wedge v)$ for $u \in L$; then ρ_v is also a Riesz pseudonorm.

(i) Now let $\{u_n\}$ be a positive disjoint sequence of L, and

$$S = \{v \in L^+ : \text{there exists } k \text{ with } v \wedge u_n = \theta \text{ for all } n \geq k\}.$$

Then the set of Riesz pseudonorms $\{\rho_v \colon \rho \in \mathscr{P} \text{ and } v \in S\}$ generates a locally solid topology τ on L which is Hausdorff (since $S^d = \{\theta\}$, see [5, Section 28, p. 160]). Clearly

$$u_n \xrightarrow{\tau} \theta.$$

Now by hypothesis ξ is coarser than τ, and thus

$$u_n \xrightarrow{\xi} \theta.$$

(ii) If $u_\alpha \downarrow \theta$ in L, fix an index β. Let $\varepsilon > 0$ and

$$T = \{v \in L^+ : v \wedge (u_\alpha - \varepsilon u_\beta)^+ = \theta \text{ for some } \alpha\}.$$

The set of Riesz pseudonorms $\{\rho_v \colon \rho \in \mathscr{P} \text{ and } v \in T\}$ defines a locally solid topology τ on L that must be Hausdorff. To see this, suppose $\rho_v(u) = 0$ for all $\rho \in \mathscr{P}$ and all $v \in T$. Since $(u_\alpha - \varepsilon u_\beta)^- \in T$ for each α, it follows that $|u| \wedge (u_\alpha - \varepsilon u_\beta)^- = \theta$ for each α. But $u_\alpha \downarrow \theta$, so $|u| \wedge u_\beta = \theta$. Thus $|u| \in T$, which implies $|u| = |u| \wedge |u| = \theta$ and hence $u = \theta$. Therefore τ is a Hausdorff topology.

It now follows by the definition of τ that

$$(u_\alpha - \varepsilon u_\beta)^+ \xrightarrow[\alpha]{\tau} \theta$$

and hence we must have

$$(u_\alpha - \varepsilon u_\beta)^+ \xrightarrow[\alpha]{\xi} \theta.$$

Since $\theta \leq u_\alpha \leq (u_\alpha - \varepsilon u_\beta)^+ + \varepsilon u_\beta$ and ε is arbitrary, it follows that $u_\alpha \xrightarrow{\xi} \theta$. ∎

Note. The above simple proof that replaces our original is due to the referee.

Remark. There is a close relationship between a Riesz space with a minimal topology and its universal completion. If ξ is a minimal topology on L, then Theorem 3 combined with [2, Theorem 24.3] shows that ξ has a Lebesgue extension to the universal completion L^u of L, and in fact L^u is the topological completion of (L, ξ).

Combining part (ii) of the last theorem with the fact that $C[0, 1]$ does not admit a Lebesgue topology [2, Example 8.2, p. 53], we obtain the following.

THEOREM 4. *$C[0, 1]$ has no minimal topology.*

Recall that a non-zero element u of a Riesz space is called a discrete element if $|v| \leq |u|$ implies $v = \lambda u$. A Riesz space L is called discrete if the band generated by the discrete elements is all of L. An atom is a non-zero element u of a Riesz space such that $\theta \leq v \leq |u|, \theta \leq w \leq |u|$, and $v \wedge w = \theta$ imply $v = \theta$ or $w = \theta$. In an Archimedean Riesz space an element is a discrete element if and only if it is an atom. Also the Archimedean discrete spaces are precisely the order dense Riesz subspaces of the Riesz spaces of the form $\mathbf{R}^X$ [2, Theorem 2.17, p. 17].

THEOREM 5. *An Archimedean Riesz space L admits a minimal topology that is also locally convex if and only if L is a discrete space.*

In this case, the minimal topology is the topology of pointwise convergence.

Proof. Assume that L is discrete. By [2, Theorem 2.17, p. 17], L is an order dense Riesz subspace of some $\mathbf{R}^X$. We claim that the locally convex-solid topology ξ of pointwise convergence on L is a minimal topology. To see this, let τ be a locally solid topology on L, and let $\{u_\alpha\} \subseteq L^+$ satisfy

$$u_\alpha \xrightarrow{\tau} \theta.$$

If $\lim u_\alpha(x) \neq 0$ for some x, then by passing to a subnet we can assume that there exists $\varepsilon > 0$ such that $u_\alpha(x) \geq \varepsilon > 0$ for all α. But then $\theta < \varepsilon\chi_{\{x\}} = v \in L$ satisfies $\theta < v \leq u_\alpha$ for all α, contradicting

$$u_\alpha \xrightarrow{\tau} \theta.$$

Therefore, ξ is minimal.

Now assume that L admits a minimal topology ξ that is in addition locally convex. By Theorem 3, ξ is a Lebesgue topology; therefore $L' \subseteq L_n^\sim$. Now let $\theta < u \in L$. Choose $\theta < \phi \in L'$ with $\phi(u) > 0$. Since $\phi \in L_n^\sim$, there exists $\theta < v \in C_\phi$ with $\theta < v \leq u$. If $[\theta, v]$ does not contain any discrete element, then there exists a disjoint sequence $\{v_n\}$ with $\theta < v_n \leq v$ for all n. For each n, choose λ_n with $\lambda_n \phi(v_n) = 1$. By Theorem 3,

$$\lambda_n v_n \xrightarrow{\xi} \theta;$$

consequently $\lim \phi(\lambda_n v_n) = 0$, a contradiction. The discreteness of L is now immediate. ∎

Repeating almost verbatim the arguments of the second part of the previous proof we can establish the following interesting result.

THEOREM 6. *Let L be an atomless Riesz space with a minimal topology ξ. Then $(L, \xi)' = \{\theta\}$.*

We now turn our attention to minimal topologies on L_p-spaces. Start with a finite measure space (X, Σ, μ). It is assumed that the Caratheodory completion process has been applied to the measure space (X, Σ, μ), so that Σ is the σ-algebra of all μ-measurable subsets of X. Let $\mathcal{M}$ denote the equivalence classes of all real valued μ-measurable functions on X; $\mathcal{M}$ is often denoted by $L_0(X, \Sigma, \mu)$. Then $\mathcal{M}$ is a Riesz space under the ordering $f \geq g$, whenever $f(x) \geq g(x)$ for μ-almost all x. Clearly, $\mathcal{M}$ is σ-Dedekind and σ-laterally complete (i.e., every positive disjoint sequence has a supremum). By the additivity of μ it follows that each disjoint subset of $\mathcal{M}$ is at most countable, and hence has a supremum, that is, $\mathcal{M}$ is laterally complete.

The Riesz pseudonorm

$$\rho(u) = \int_X \frac{|u|}{1 + |u|}\, d\mu$$

generates a metrizable locally solid topology ξ on $\mathcal{M}$. This topology is actually the *topology of convergence in measure*, since

$$u_n \xrightarrow{\xi} u$$

if and only if

$$\lim_n \mu(\{x \in X\colon |u_n(x) - u(x)| \geq \varepsilon\}) = 0 \quad \text{for all } \varepsilon > 0.$$

It is easy to see that ξ is a Lebesgue topology, and in fact the only Hausdorff locally solid topology that $\mathcal{M}$ can carry [2, Theorem 24.7, p. 185]. It follows that $\mathcal{M}$ has the countable sup property, and consequently $\mathcal{M}$ is Dedekind complete. Thus $\mathcal{M}$ is a universally complete Riesz space. Note also that if the measure μ is non-atomic, then $\mathcal{M}$ does not have discrete elements. Moreover, $(\mathcal{M}, \xi)$ is ξ-complete (prove it either directly or by using [2, Theorem 24.2, p. 182]).

A simple verification now shows that $\mathcal{M}$ is the universal completion of the Riesz spaces $L_p(X, \Sigma, \mu)$ for $0 < p \leq \infty$. In actuality each $L_p(X, \Sigma, \mu)$ is a super order dense ideal of $\mathcal{M}$. The restriction of ξ to any $L_p(X, \Sigma, \mu)$ is the topology of convergence in measure.

THEOREM 7. *Let (X, Σ, μ) be a finite measure space. Then the topology of convergence in measure is a minimal topology on each $L_p(X, \Sigma, \mu)$ for $0 < p < \infty$.*

Proof. For $0 < p < \infty$, the topology τ generated by the "L_p-norm" on $L_p(X, \Sigma, \mu)$ is a Lebesgue Fréchet topology. Thus by [2, Theorem 16.7, p. 112], τ is finer than any other locally solid topology on $L_p(X, \Sigma, \mu)$. In particular, every locally solid topology on $L_p(X, \Sigma, \mu)$ has the Lebesgue property.

Let ξ be the topology of convergence in measure on $L_p(X, \Sigma, \mu)$. Then every disjoint sequence of $L_p(X, \Sigma, \mu)$ is ξ-convergent to zero. Indeed, if $\{f_n\}$ is a disjoint sequence of $\mathcal{M}$, then $f = \sum_{n=1}^{\infty} n|f_n| \in \mathcal{M}$ and $|f_n| \leq n^{-1}f$ holds for all n, so by the Lebesgue dominated convergence theorem,

$$f_n \xrightarrow{\xi} \theta.$$

The proof now can be completed by invoking the authors' theorem: A Lebesgue topology ξ on a Riesz space L is coarser than any other σ-Fatou topology on L if and only if every disjoint sequence of L is ξ-convergent to zero; see [2, Theorem 24.3, p. 182]. ∎

A similar result holds for an arbitrary measure space (X, Σ, μ). That is, every abstract L_p-space has a minimal topology. In the general case ξ is generated by the family of Riesz pseudonorms $\{\rho_E : E \in S\}$, where

$$\rho_E(u) = \int_E \frac{|u|}{1 + |u|}\, d\mu \text{ for } u \in \mathcal{M} \quad \text{and} \quad S = \{E \in \Sigma : \mu(E) < \infty\}.$$

The reader can verify that the proof of Theorem 7 is valid. However, here the minimal topology on each $L_p(X, \Sigma, \mu)$ $(0 < p < \infty)$ is the topology of convergence in measure on the measurable subsets of X whose measure is finite.

The situation for $L_\infty(X, \Sigma, \mu)$ is quite different as the next theorem shows.

THEOREM 8. *Let (X, Σ, μ) be a non-atomic σ-finite measure space. Then $L_\infty(X, \Sigma, \mu)$ has no minimal topology.*

Proof. Let $L = L_\infty(X, \Sigma, \mu)$. Then the norm dual satisfies

$$L' = L_n^\sim \oplus (L_n^\sim)^d,$$

where $(L_n^\sim)^d$ is the band of all linear functionals disjoint from $L_n^\sim$. Let τ be the locally solid topology on L generated by the Riesz seminorms $\{\rho_\phi: \phi \in (L_n^\sim)^d\}$, where $\rho_\phi(u) = |\phi|(|u|)$ for $u \in L$. Since μ is non-atomic it follows that $(L_n^\sim)^d$ separates the points of L [3, p. 348], and thus τ is Hausdorff. Since $C_\phi = \{\theta\}$ for each $\phi \in (L_n^\sim)^d$ it follows easily that $C_\tau = \{\theta\}$.

Now suppose ξ is a minimal topology on L. Then $\xi \subseteq \tau$, so that $C_\xi \subseteq C_\tau$. However, ξ is a Lebesgue topology by Theorem 3, and so C_ξ is order dense in L [1, Lemma 2.1, p. 4] which contradicts $C_\tau = \{\theta\}$. Thus $L_\infty(X, \Sigma, \mu)$ has no minimal topology. ∎

Is a Lebesgue topology on a Riesz space L that is extendable to the universal completion L^u of L necessarily minimal? In general, the answer is negative according to the previous theorem. However, if every locally solid topology has an order dense carrier, then the answer is affirmative. The details follow. (Recall that a locally solid topology is called entire if its carrier is order dense.)

THEOREM 9. *Assume that ξ is a Lebesgue topology on a Riesz space L, and that ξ has a locally solid extension to the universal completion L^u of L. If every Hausdorff locally solid topology on L is entire, then ξ is minimal.*

Proof. Let τ be a locally solid topology on L. Choose a basis $\{V\}$ of zero for τ consisting of solid τ-neighborhoods. For each $V \in \{V\}$, let V^* denote the ξ-closure of V; clearly each V^* is solid and order closed. Thus $\{V^*\}$ defines a Fatou topology τ^* on L such that $\tau^* \subseteq \tau$.

We next show that τ^* is Hausdorff. By way of contradiction assume that $\theta < u \in V^*$ for all V^*. Since both C_ξ and C_τ are order dense, we can assume that $u \in C_\xi \cap C_\tau$. Pick a normal sequence $\{V_n\}$ of solid τ-neighborhoods of zero with $u \in N^d$, where $N = \bigcap_{n=1}^\infty V_n$. Similarly, pick another normal sequence $\{W_n\}$ of solid ξ-neighborhoods of zero with $u \in M^d$, where $M = \bigcap_{n=1}^\infty W_n$. Clearly $u \in V_n^*$ for all n. Now for each n, choose $u_n \in V_n$ with $\theta \le u_n \le u$ and $u - u_n \in W_n$. Then $\{u_n\} \subseteq M^d$ and

$$u_n \xrightarrow{\tau_1} u$$

in M^d, where τ_1 is the metrizable Lebesgue topology on M^d generated by $\{W_n \cap M^d\}$. Now by [2, Theorem 15.9, p. 107], there exists a subsequence $\{v_n\}$ of $\{u_n\}$ and a net $\{w_\alpha\}$ satisfying $\theta \le w_\alpha \uparrow u$ in M^d, and such that given α, there exists n_α with $w_\alpha \le v_n$ for all $n \ge n_\alpha$. It follows that $\{w_\alpha\} \subseteq N$. But since $\{w_\alpha\} \subseteq N^d$, we get $w_\alpha = \theta$ for all α. Hence, $u = \theta$, a contradiction.

Now by [2, Theorem 24.3, p. 182], we have $\xi \subseteq \tau^*$. Hence $\xi \subseteq \tau$, and therefore ξ is minimal. ∎

We next show how the previous theorem can be applied to Orlicz spaces. Consider a finite measure space (X, Σ, μ), and let L_Φ be an Orlicz space associated with a Young function Φ whose values are finite. Note that this excludes L_Φ being equal to $L_\infty(X, \Sigma, \mu)$. Then the Banach lattice L_Φ contains L_∞ as an ideal, and therefore $\mathscr{M}$ is the universal completion of L_Φ. On the other hand, every element f of L_∞ has an absolutely continuous norm, i.e., $|f| \geq f_n \downarrow \theta$ in L_∞ implies $\lim \|f_n\| = 0$, where $\|\cdot\|$ is, of course, the norm of L_Φ; see [4, Theorem 10.3, p. 87]. Next we show that each locally solid topology τ on L_Φ is entire. Since L_Φ is a Banach lattice, the norm topology is finer than τ. Hence, if τ^* is the restriction of τ to $L_\infty(X, \Sigma, \mu)$, it follows from the absolute continuity of the norm on L_∞ and the countable sup property, that τ^* is a Lebesgue topology. So, C_{τ^*} is order dense in L_∞, and hence order dense in L_Φ. It now follows easily that $C_{\tau^*} \subseteq C_\tau$, and hence C_τ is order dense in L_Φ, so that τ is entire. By applying Theorem 9 we now obtain the following result.

THEOREM 10. *Let (X, Σ, μ) be a finite measure space, and let Φ be a finite valued Young function. Then the topology of convergence in measure on the Orlicz space L_Φ is a minimal topology.*

We close the paper by presenting some classes of Riesz spaces having minimal topologies.

THEOREM 11. *Let L be an order dense Riesz subspace of an Archimedean Riesz space M. If L admits a minimal topology, then so does M.*

Proof. Let ξ be the minimal topology of L. Since L is order dense in M, the universal completion M^u of M is Riesz isomorphic to L^u. Combining Theorem 3 with [2, Theorem 24.3, p. 182], we see that ξ has a Lebesgue extension to M^u. We claim that the restriction of this extension to M is a minimal topology. Indeed, if τ is a locally solid topology on M, then τ restricted to L has an order dense carrier, and hence (as is easily seen) τ has an order dense carrier in M. The result now follows from Theorem 9. ∎

THEOREM 12. *If a σ-laterally complete Riesz space L admits a Hausdorff Lebesgue topology τ, then τ is minimal.*

Proof. This is Theorem 24.4 of [2], p. 184. ∎

A Riesz subspace M of a Riesz space L is said to be *full* in L if for every $u \in L$, there exists $v \in M$ with $|u| \leq v$. By Theorem 8 it can be seen that a minimal topology restricted to a super order dense ideal is no longer minimal. However, the following result holds regarding restrictions of minimal topologies. Its straightforward proof is left to the reader.

THEOREM 13. *Let ξ be a minimal topology on a Riesz space L. If A is either an order dense full Riesz subspace or a projection band of L, then ξ induces a minimal topology on A.*

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