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HANKEL OPERATORS WITH HILBERT SPACE RANGE

BY

JEFFREY R. BUTZ

1. Introduction

In what follows, L^2 will denote the class of Lebesgue measurable functions on the unit circle in the complex plane, and H^2 will denote the Hardy (closed) subspace of L^2 consisting of those functions whose Fourier coefficients vanish on the negative integers. Functions that differ only on zero sets will be considered equal.

Given a function $\phi \in L^\infty$, we define the corresponding *Toeplitz operator* T_ϕ: $H^2 \to H^2$ by $T_\phi x = P_+ \phi(e^{i\theta})x(e^{i\theta})$, where P_+: $L^2 \to H^2$ is the orthogonal projection. When ϕ is in H^∞, then the corresponding operator T_ϕ is said to be *analytic Toeplitz*, and, in such a case, the projection P_+ is unnecessary. Likewise, given a function ϕ, we define the corresponding *Hankel operator* H_ϕ: $H^2 \to H^2$ by $H_\phi x = P_+ \phi(e^{i\theta})x(e^{-i\theta})$. Clearly, two functions ϕ_1, ϕ_2 having the same projection onto H^2 determine the same Hankel operator. For each of the above operators T_ϕ, H_ϕ we refer to the function ϕ as the corresponding *symbol*.

Toeplitz operators and, to a somewhat lesser extent, Hankel operators have been the object of extensive study in the last twenty years, and a vast literature exists concerning them. In particular, it is well known and not difficult to show that the operators T_ϕ, with $\phi \in L^\infty$, are characterized by the property that $U_+^* T_\phi U_+ = T_\phi$, where U_+ is the unilateral shift, given on H^2 by multiplication by $e^{i\theta}$, and that $\|T_\phi\| = \|\phi\|_\infty$. Likewise, a Hankel operator H_ϕ is characterized by the equation $H_\phi U_+ = U_+^* H_\phi$. In the latter case, however, H_ϕ may exist as a bounded operator on H^2 even though ϕ is not in L^∞. Since, as noted above, the values of H_ϕ are independent of the part of ϕ in $L^2 \ominus H^2$, this is not surprising. An explicit criterion for the boundedness of H_ϕ in terms of the symbol function ϕ is given by the following famous theorem of Z. Nehari.

NEHARI'S THEOREM [3]. *A Hankel operator H_ϕ is bounded if, and only if, there exists a function $\psi \in L^2 \ominus H^2$ such that $\phi + \psi \in L^\infty$. In such a case, the function ψ can be chosen so that $\|\phi + \psi\|_\infty = \|H_\phi\|$.*

Generalizations and alternative proofs to this theorem have been obtained by several authors, cf. [1], [4]. In the following sections, we offer an extension of the Nehari theorem quite different from these earlier results. We do this by

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defining, in Section 2, a class of operators, called "generalized Hankel operators", mapping the Hardy space H^2 into an arbitrary (separable, complex) Hilbert space X. This class will include (choosing $X = H^2$) the analytic Toeplitz and the Hankel operators. In Section 3, we shall prove a boundedness criterion for the generalized Hankel operators that yields the Nehari theorem in the classical Hankel case.

2. Generalized Hankel operators: Definition

In order to obtain operators H_ϕ mapping the Hardy space H^2 to an arbitrary Hilbert space X and including among them the classical Hankel operators, we need certain results from the functional calculus developed by Sz.-Nagy and Foias which we now recall.

Let T be a contractive linear operator on a Hilbert space X. Such an operator is said to be *completely non-unitary* if it has no non-trivial reducing subspace N for which the restriction $T|N$ of T to N is unitary. The prototypical examples of such an operator are the unilateral shift and (hence also) its adjoint.

If $T: X \to X$ is a completely non-unitary contraction, then a result of Sz.-Nagy and Foias enables us to define a corresponding functional calculus for functions $u \in H^\infty$ This result is as follows:

THEOREM [7, p. 114]. *For a completely non-unitary contraction T on a Hilbert space X, the mapping $u \mapsto u(T)$ of H^∞ into $B(X)$, defined by*

$$(2.1) \qquad u(T) = \lim_{r \uparrow 1} \sum_{k=0}^{\infty} r^k c_k T^k \quad \textit{for} \quad u(e^{i\theta}) = \sum_{k=0}^{\infty} c_k e^{ik\theta} \in H^\infty,$$

is a norm-decreasing algebra homomorphism of H^∞ into $B(X)$.

Let us now choose a fixed vector $\phi \in X$ and a fixed completely non-unitary operator T on X. Using the above theorem, we can then define our generalized Hankel operator on H^∞ by setting $H_\phi u = u(T)\phi$. If we *assume* that ϕ has the property that there exists a constant C such that

$$(2.2) \qquad \|u(T)\phi\|_X \leq C\|u\|_{H^2} \quad \text{for all} \quad u \in H^\infty,$$

then, in such a case, we can extend, via continuity, the definition of H_ϕ to obtain an operator, still called H_ϕ, with domain all of H^2. More formally, we have the following:

DEFINITION. Let T: $X \to X$ be a completely non-unitary contraction on a Hilbert space X, and let $\phi \in X$ satisfy (2.2). Then we define the corresponding *generalized Hankel operator* H_ϕ to be the continuous extension to H^2 of the operator H_ϕ: $H^\infty \to X$ defined by $H_\phi u = u(T)\phi$, where $u(T)$ is given by (2.1). For $u \in H^2$, we write $H_\phi u = u(T)\phi$.

It is easy to see that we obtain the classical Hankel operators H_ϕ: $H^2 \to H^2$ by choosing $X = H^2$, $\phi = \phi(e^{i\theta})$, and $T = U_+^*$, the adjoint of the unilateral shift. Likewise, we obtain the analytic Toeplitz operators T_ϕ: $H^2 \to H^2$ by choosing $X = H^2$, $\phi = \phi(e^{i\theta})$, and $T = U_+$.

3. Generalized Hankel operators: Boundedness

For the generalized Hankel operator defined above, we wish to obtain a criterion for boundedness analogous to that given by Nehari's theorem. To do so, we shall use the structure theory for unitary dilations of Sz.-Nagy and Foias [7].

If T is a contraction on a Hilbert space X and W is a unitary operator on a Hilbert space $Y \supset X$ such that $T^n = P_X W^n | X$ for $n = 1, 2, \ldots$, where P_X: $Y \to X$ is the orthogonal projection onto X, then W is said to be a *unitary dilation* of T. It is proved in [7] that every contraction T has a unitary dilation W. Moreover, if W is assumed to be minimal in the sense that the smallest reducing subspace for W containing X is Y itself, then W is unique up to isomorphism. In particular, we note that the bilateral shift U on L^2 is the minimal unitary dilation of the unilateral shift U_+ on H^2.

In obtaining our condition for a generalized Hankel operator to be bounded, we shall use a result of Sz.-Nagy and Foias [6] which has also been reformulated, with an alternative proof, in [2]. This result, stated below, describes, for a contraction T, its commutant $\{T\}' = \{A \in B(X) \,|\, AT = TA\}$ in terms of the commutant of its minimal unitary dilation.

PROPOSITION 3.1 [2, Theorem 6]. *Suppose that T is a contraction on a Hilbert space X and that W is the unique minimal unitary dilation of T on the Hilbert space $Y \supset X$. Write $X = M \ominus N$, where M is the smallest invariant subspace for W containing X, and $WN \subset N$. Then the commutant of T consists of all operators of the form $P_X B | X$, where $B \in B(Y)$ is in the commutant of W and satisfies $BM \subset M$ and $BN \subset N$. Furthermore, if A is in the commutant of T, then A can be written as $A = P_X B | X$, where B has the properties described above and satisfies $\|B\| = \|A\|$.*

Remark. By a result of Sarason, cited in [2], the Hilbert space X always has a decomposition of the type $X = M \ominus N$, where M and N are as described in the above proposition.

If T_1 and T_2 are operators on Hilbert spaces X_1 and X_2, respectively, then we say than an operator A: $X_1 \to X_2$ is *intertwining* for T_1 and T_2 if $AT_1 = T_2 A$. Using the above proposition, we can state a lemma which relates intertwining operators A for contractions T_i: $X_i \to X_i$, $i = 1, 2$, to operators B which are intertwining for their minimal unitary dilations. Though surely well known, this result does not seem to appear anywhere in the literature, and we therefore indicate a proof.

LEMMA 3.2. *Suppose that T_i is a contraction acting on a Hilbert space X_i, for $i = 1, 2$. Let W_i be the unique minimal unitary dilation of T_i acting on the Hilbert space $Y_i \supset X_i$. Let $A\colon X_1 \to X_2$ be intertwining for T_1 and T_2, i.e., $AT_1 = T_2 A$. Then there exists an operator $B\colon Y_1 \to Y_2$ intertwining for W_1 and W_2, and satisfying*

$$BW_1 = W_2 B, \quad P_{X_2} B\,|\,X_1 = A, \quad \|B\| = \|A\|.$$

Proof. Define operators $\hat{T}$ and $\hat{A}$ on the Hilbert space $X_1 \oplus X_2$ by

$$\hat{T} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

Then,

$$\left\| \hat{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 = \|T_1 x_1\|^2 + \|T_2 x_2\|^2$$

$$\leq \|x_1\|^2 + \|x_2\|^2 = \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 \quad \text{for} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X_1 \oplus X_2,$$

so $\hat{T}$ is a contraction on $X_1 \oplus X_2$. Also, $AT_1 = T_2 A$ implies that

$$\hat{A}\hat{T} = \begin{pmatrix} 0 & 0 \\ AT_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ T_2 A & 0 \end{pmatrix} = \hat{T}\hat{A}.$$

Now the minimal unitary dilation of $\hat{T}$ is the operator

$$\hat{W} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}$$

acting on $Y_1 \oplus Y_2$ since

$$P_{X_1 \oplus X_2} \hat{W}^n\,|\,X_1 \oplus X_2 = \begin{pmatrix} P_{X_1} W_1^n | X_1 & 0 \\ 0 & P_{X_2} W_2^n | X_2 \end{pmatrix} = \begin{pmatrix} T_1^n & 0 \\ 0 & T_2^n \end{pmatrix} = \hat{T}^n.$$

Therefore, by Proposition 3.1 applied to the operator $\hat{T}$ on the Hilbert space $X_1 \oplus X_2$ with minimal unitary dilation $\hat{W}$ on $Y_1 \oplus Y_2$, there exists an operator

$$\hat{B} = \begin{pmatrix} Z_1 & Z_2 \\ B & Z_3 \end{pmatrix}$$

such that

$$\begin{pmatrix} Z_1 W_1 & Z_2 W_2 \\ BW_1 & Z_3 W_2 \end{pmatrix} = \hat{B}\hat{W} = \hat{W}\hat{B} = \begin{pmatrix} W_1 Z_1 & W_1 Z_2 \\ W_2 B & W_2 Z_3 \end{pmatrix},$$

$$P_{X_1 \oplus X_2} \hat{B}\,|\,X_1 \oplus X_2 = \hat{A} \quad \text{and} \quad \|\hat{B}\| = \|\hat{A}\|.$$

It follows immediately that the operator B has the desired properties, and thus the proof is complete.

With the help of the above lemma, we can prove necessary and sufficient conditions for a generalized Hankel operator to be bounded. Recall that if H_ϕ is a classical Hankel matrix with $\phi \in H^2$, then Nehari's theorem states that H_ϕ is bounded if and only if there exists a function $\psi \in L^\infty$ such that $P_{H^2}\psi = \phi$, in which case we can choose ψ so that the operator J_ψ: $L^2 \to L^2$ defined by $J_\psi u = \psi(e^{i\theta})u(e^{-i\theta})(= u(U^*)\psi)$ satisfies

$$P_{H^2}J_\psi | H^2 = H_\phi, \quad J_\psi U = U^*J_\psi \quad \text{and} \quad \|J_\psi\| = \|H_\phi\|,$$

where U: $L^2 \to L^2$ is the bilateral shift. The following theorem shows that similar conditions hold for the generalized Hankel operators.

THEOREM 3.3. *Let T be a completely non-unitary contraction on a Hilbert space X. Let W: $Y \to Y$ be the minimal unitary dilation of T. Let $\phi \in X$ define a generalized bounded Hankel operator H_ϕ: $H^2 \to X$ by $H_\phi u = u(T)\phi$. Then there exists an element $\psi \in Y$ such that $P_X\psi = \phi$ and the operator J_ψ: $L^2 \to Y$ defined by $J_\psi u = u(W)\psi$ satisfies*

$$P_X J_\psi | H^2 = H_\phi, \quad J_\psi U = WJ_\psi \quad \text{and} \quad \|J_\psi\| = \|H_\phi\|.$$

Proof. From the definition of H_ϕ, we have that $H_\phi U_+ = TH_\phi$, where U_+ is the unilateral shift on H^2. Since the bilateral shift U is the minimal unitary dilation of U_+, by Lemma 3.1 there exists an operator J: $L^2 \to Y$ such that $P_X J | H^2$, $JU = WJ$, $\|J\| = \|H_\phi\|$. Let $\psi = J1$. Then $J(e^{in\theta}) = JU^m 1 = W^n J1 = W^n\psi$ for all integers n. (For $n < 0$, we have used a well-known theorem of Putnam [5, Theorem 1.6.2] which states that if N_1, N_2 are bounded normal operators and if B is a bounded operator such that $BN_1 = N_2B$, then $BN_1^* = N_2^*B$.) The trigonometric polynomials are dense in L^2. Therefore, we have obtained an element $\psi \in Y$ such that $Ju = u(W)\psi$ for $u \in L^2$. Hence, J is the operator J_ψ of the theorem, and we have $P_X\psi = P_X J_\psi 1 = H_\phi 1 = \phi$. This completes the proof.

Though we have not made it explicit in the statement of the theorem, it is clear that, as a converse, an operator H: $H^2 \to X$ defined via the formula $H = P_X J_\psi | H^2$ for such an operator J_ψ as above is a generalized bounded Hankel operator with symbol $\phi = P_X\psi$.

Remarks 1. We note that the classical criteria for boundedness discussed in Section 1 follow as an easy consequence of the above theorem. In the analytic Toeplitz case (where $X = H^2$, $T = U_+$), the theorem gives that the operator $J(= J_\phi)$ must satisfy $JU = UJ$, and it is well known that such an operator must be multiplication by an L^∞ function. In the classical Hankel case (where $X = H^2$, $T = U_+^*$), we have $J_\psi U = U^*J_\psi$. But then for the operator R: $L^2 \to L^2$ given by $Ru(e^{i\theta}) = u(e^{-i\theta})$, we have $RJ_\psi U = RU^*J_\psi = URJ_\psi$ and, therefore, RJ_ψ must be multiplication by an L^∞ function. It is easy to see that this function must be $\psi(e^{-i\theta})$.

2. A result of Sz.-Nagy and Foias [7, Proposition II.2.2] states that if $T: X \to X$ is a contraction with minimal isometric dilation $W_+: Y_+ \to Y_+$ and if $G: Z \to Z$ is an isometry, then the solutions $K: Z \to X$ of the equation $TL = KG$ are precisely those operators of the form $K = P_X L$, where L is a bounded operator on Y_+ satisfying $W_+ L = LG$.

Since the unilateral shift $U_+: H^2 \to H^2$ is an isometry, we can apply this result in the context of generalized Hankel operators $H_\phi: H^2 \to X$. We then conclude that for the minimal isometric dilation $W_+: Y_+ \to Y_+$ of T there exists an operator $J_+: H^2 \to Y_+$ such that $P_X J_+ = H_\phi$, $J_+ u = u(W_+)\psi_+$, for $\psi_+ = J_+ 1 \in Y_+$, and $J_+ U_+ = W_+ J_+$. (This, of course, cannot be used to obtain an analogue of Nehari's theorem, as is obvious from the case where T is an isometry, so that $T = W_+$, $X = Y_+$, and $\psi_+ = \phi$.)

3. We could alter slightly the definition of generalized Hankel operators so as to include all of the Toeplitz operators. To do this we would allow the symbol ϕ to be chosen out of the domain space Y for the minimal unitary dilation W of T, subject to the condition that $\|P_X u(W)\phi\|_X \leq C\|u\|_{H^2}$ for some constant C and all $u \in H^\infty$. With routine changes, the conclusion of Theorem 3.3 continues to hold.

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University of Oklahoma
Norman, Oklahoma