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BANACH-MODULE VALUED DERIVATIONS ON C^*-ALGEBRAS

BY

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1. Introduction

Let $\mathscr{A}$ be a unital Banach algebra. A Banach space X is a Banach $\mathscr{A}$-module if X is a unital $\mathscr{A}$-bimodule whose actions $(a, x) \to ax$, $(a, x) \to xa$ are bilinear maps of $\mathscr{A} \times X$ into X which are continuous relative to the natural norm topologies. B. E. Johnson has defined and studies a cohomology theory for Banach algebras with coefficients in a Banach module, and this theory (or ones very close to it has been developed extensively by several authors [1], [3], [4], [5], [10], etc.

One of the central problems of this theory is the determination of when the first cohomology group H^1 is trivial. One considers a unital Banach algebra $\mathscr{A}$ and derivations of $\mathscr{A}$ into a Banach $\mathscr{A}$-module X, i.e., linear mappings $\delta\colon \mathscr{A} \to X$ which satisfy $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in \mathscr{A}$. Each element $x \in X$ induces a derivation δ of $\mathscr{A}$ into X via the formula $\delta(a) = ad(x)(a) = xa - ax$; such derivations are termed *inner*. Following the cohomological notations, we let $Z^1(\mathscr{A}, X)$ be the space of all derivations of $\mathscr{A}$ into X, $B^1(\mathscr{A}, X)$ the space of all inner derivations of $\mathscr{A}$ into X. We define $H^1 = H^1(\mathscr{A}, X)$ as the quotient group $Z^1(\mathscr{A}, X)/B^1(\mathscr{A}, X)$; H^1 is trivial if $Z^1(\mathscr{A}, X) = B^1(\mathscr{A}, X)$, i.e., if every derivation is inner.

Now assume $\mathscr{A}$ is a unital C^*-algebra. In this paper, we will be concerned with proving that certain classes of Banach $\mathscr{A}$-module-valued derivations are inner. To state our results precisely, some definitions and notation are needed.

Let $\delta\colon \mathscr{A} \to X(\mathscr{A} \to X^*)$ be a derivation from the unital C^*-algebra $\mathscr{A}$ to a unital Banach (dual) $\mathscr{A}$-module $X(X^*)$. The *hull* of δ, denoted by hull δ, is by definition the set

$$\text{co}\,\{\delta(u)u^*\colon u \in U(\mathscr{A})\},$$

where $U(\mathscr{A})$ is the unitary group of $\mathscr{A}$ and co $\mathscr{S}$ denotes the norm $(\sigma(X^*, X))$-closed convex hull of a subset $\mathscr{S}$ of $X(X^*)$. Motivated by the concept of strong amenability for C^*-algebras (see [2], Definition, p. 70], we say that δ is *strongly inner* if there exists an $x \in$ hull δ such that $\delta = \text{ad}\, x$.

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Next set

$$Z_s^1(\mathscr{A}, X) = \{\delta \in Z^1(\mathscr{A}, X)\colon \text{hull } \delta \text{ is norm-separable}\},$$

$$Z_{WC}^1(\mathscr{A}, X) = \{\delta \in Z^1(\mathscr{A}, X)\colon \text{hull } \delta \text{ is } \sigma(X, X^*)\text{-compact}\},$$

$$B_s^1(\mathscr{A}, X) = \{\delta \in Z^1(\mathscr{A}, X)\colon \delta \text{ is strongly inner}\},$$

$$Z_F^1(\mathscr{A}, X) = \{\delta \in Z^1(\mathscr{A}, X)\colon \delta \text{ has finite rank}\}.$$

Recently, J. Rosenberg [10] has shown that one may have $B_s^1(\mathscr{A}, X) \subsetneqq B^1(\mathscr{A}, X)$. Our goal is to show that each of the sets $Z_s^1(\mathscr{A}, X^*)$, $Z_{WC}^1(\mathscr{A}, X)$, and the uniform closure of $Z_F^1(\mathscr{A}, X)$ consist entirely of strongly inner derivations.

We note once and for all that every element of $Z^1(\mathscr{A}, X)$ is uniformly continuous by [9, Theorem 2]. Unless otherwise specified, $\mathscr{A}$ will always denote a unital C^*-algebra, X a unital Banach $\mathscr{A}$-module, and X^* the dual Banach $\mathscr{A}$-module obtained from the module actions induced by X on X^*.

2. Strongly inner derivations

To show that a derivation is strongly inner, we exploit the main idea of Johnson and Ringrose's proof [8, Theorem 3.6] of Sakai's theorem on derivations of W^*-algebras. Let $\delta \in Z^1(\mathscr{A}, X^*)$. For each $u \in U(\mathscr{A})$, let

$$T_u x = \delta(u)u^* + uxu^*, \quad x \in X^*.$$

T_u is an affine, $\sigma(X^*, X)$-continuous map of X^* into X^*. Let $u, v \in U(\mathscr{A})$. Since δ is a derivation,

$$\begin{aligned} T_u T_v x &= u(\delta(v)v^* + vxv^*)u^* + \delta(u)u^* \\ &= (uv)x(uv)^* + u\delta(v)(uv)^* + \delta(u)u^* \\ &= (uv)x(uv)^* + \delta(uv)(uv)^* - \delta(u)v(uv)^* + \delta(u)u^* \\ &= (uv)x(uv)^* + \delta(uv)(uv)^* \\ &= T_{uv}x, \quad x \in X^*. \end{aligned}$$

Thus $G = \{T_u\colon u \in U(\mathscr{A})\}$ is a group, and G leaves

$$\text{hull } \delta = \text{co } \{T_u(0)\colon u \in U(\mathscr{A})\}$$

invariant. If we can show that G has a fixed point $x \in \text{hull } \delta$, then $\delta(u) = xu - ux$, $u \in U(\mathscr{A})$. Since $\mathscr{A}$ is spanned by its unitary group, it follows that δ is strongly inner. This is done in the proof of the following theorem:

2.1. THEOREM. *$Z_s^1(\mathscr{A}, X^*)$ is uniformly closed and each element is strongly inner.*

Proof. Let $\delta \in Z_s^1(\mathscr{A}, X^*)$. By the foregoing discussion, it suffices to show that G has a fixed point $x \in \text{hull } \delta$. This is achieved by applying an appropriate version of the Ryll-Nardjewski fixed point theorem [11], which we now describe.

Let Y be a Banach space, $Q \subseteq Y$, $\mathscr{S}$ a family of maps of Q into Q. $\mathscr{S}$ is *noncontracting* if for each pair of distinct points x and y in Q, $0 \notin \{Tx - Ty: T \in \mathscr{S}\}^-$, where $^-$ denotes norm closure. The next proposition can be proven by a simple adaptation of the arguments of [7]:

PROPOSITION. *Let Y be a Banach space, and let Q be a norm-separable, norm-bounded, $\sigma(Y^*, Y)$-compact, convex subset of Y^*. Suppose $\mathscr{S}$ is a non-contracting semigroup of $\sigma(Y^*, Y)$-continuous, affine maps of Q into Q. Then there exists $x \in Q$ such that $Tx = x$, for all $T \in \mathscr{S}$.*

We want to apply this proposition with $\mathscr{S} = G$, $Q = \text{hull } \delta$. By hypothesis, hull δ is norm-separable. One easily checks that $\|x\| \leq \|\delta\| < \infty$, for all $x \in \text{hull } \delta$, and so hull δ is norm-bounded and $\sigma(X^*, X)$-compact.

We claim that G is non-contracting. Since $T_u x - T_u y = u(x - y)u^*$, we must show that

$$(2.1) \qquad 0 \notin \gamma = \{uxu^*: u \in U(\mathscr{A})\}^- \quad \text{if } x \in X^*, x \neq 0.$$

Suppose $0 \in \gamma$. Let M denote the maximum of the norms of the mappings $(a, x) \to ax$, $(a, x) \to xa$, $a \in \mathscr{A}$, $x \in X^*$. Since X^* is unital, it follows that $M^{-2}\|x\| \leq \|uxu^*\|$, for all $u \in U(\mathscr{A})$, and so (2.1) follows. By the above proposition, we conclude that G has a fixed point $x \in \text{hull } \delta$, whence δ is strongly inner.

To show that $Z_s^1(\mathscr{A}, X^*)$ is uniformly closed, let $\delta \in Z_s^1(\mathscr{A}, X^*)^-$, and choose a sequence $\{\delta_k\} \subseteq Z_s^1(\mathscr{A}, X^*)$ such that $\|\delta - \delta_k\| \to 0$. We claim that

$$(2.2) \qquad \text{hull } \delta \subseteq \left(\bigcup_{k=1}^{\infty} \text{hull } \delta_k \right)^- \qquad (^- = \text{norm closure}).$$

This evidently implies $\delta \in Z_s^1(\mathscr{A}, X^*)$.

To prove (2.2), fix $x \in \text{hull } \delta$, $\varepsilon > 0$. Choose k so that $\|\delta - \delta_k\| < \varepsilon/2M$, where M is as defined previously.

Now, let Σ denote the family of all nonvoid finite subsets of Ball $X = \{x \in X: \|x\| \leq 1\}$, directed by inclusion. Let $\sigma \in \Sigma$. By definition of hull δ, there exists a convex combination

$$x'_\sigma = \sum_{j=1}^{n} \lambda_j \delta(u_j)u_j^*, \quad \{u_1, \ldots, u_n\} \subseteq U(\mathscr{A}),$$

such that

$$|x(y) - x'_\sigma(y)| < \varepsilon/2, \quad \text{for all } y \in \sigma.$$

Let $x_\sigma = \sum_{j=1}^{n} \lambda_j \delta_k(u_j)u_j^*$. Then

$$\begin{aligned}\|x'_\sigma - x_\sigma\| &= \left\| \sum_{j=1}^{n} \lambda_j(\delta(u_j) - \delta_k(u_j))u_j^* \right\| \\ &\leq \sum_{j=1}^{n} \lambda_j \|(\delta(u_j) - \delta_k(u_j))u_j^*\| \\ &\leq M\|\delta - \delta_k\| \sum_{j=1}^{n} \lambda_j < \varepsilon/2.\end{aligned}$$

Thus,

$$|x(y) - x_\sigma(y)| < \varepsilon, \quad \text{for all } y \in \sigma. \tag{2.3}$$

It follows from (2.3) that if $\sigma, \tau \in \Sigma$ with $\sigma \supseteq \tau$, then

$$|x_\sigma(y) - x_\tau(y)| < 2\varepsilon, \quad \text{for all } y \in \tau. \tag{2.4}$$

Since hull δ_k is norm-bounded, it is $\sigma(X^*, X)$-compact, and so the net $\{x_\sigma: \sigma \in \Sigma\}$ in hull δ_k accumulates at a point $x_0 \in$ hull δ_k in the $\sigma(X^*, X)$ topology.

Let $y \in$ Ball X, and set $\sigma = \{y\}$. By (2.3), $|x(y) - x_\sigma(y)| < \varepsilon$. Choose $\tau \supseteq \sigma$ such that $|x_\tau(y) - x_0(y)| < \varepsilon$. By (2.4), $|x_\sigma(y) - x_\tau(y)| < 2\varepsilon$. Thus, $|x(y) - x_0(y)| < 4\varepsilon$. Since $y \in$ Ball X is arbitrary, this shows that $\|x - x_0\| < 4\varepsilon$, proving (2.2). Q.E.D.

2.2. COROLLARY. *Suppose X^* is separable. Then every element of $Z^1(\mathscr{A}, X^*)$ is strongly inner.*

2.3. LEMMA. *Let $\delta \in Z_F^1(\mathscr{A}, X)$. Then the submodule $\mathscr{A}\delta(\mathscr{A})\mathscr{A}$ of X generated by the range of δ is finite-dimensional; in particular, the hull of δ is contained in a finite-dimensional subspace of X.*

Proof. Let $\{\langle a_1\rangle, \ldots, \langle a_n\rangle\}$ denote a basis of cosets for $\mathscr{A}/\ker \delta$. A simple computation shows that

$$\begin{aligned}\mathscr{S} = \mathscr{A}\delta(\mathscr{A})\mathscr{A} \subseteq \text{linear span of } \Bigg[&\{a_i\delta(a_j)a_k\}_{i,j,k,=1}^{n} \\ &\cup \left(\bigcup_{i=1}^{n} (\ker\delta)\delta(a_i)(\ker\delta)\right) \\ &\cup \left(\bigcup_{i,j=1}^{n} a_i\delta(a_j)(\ker\delta)\right) \\ &\cup \left(\bigcup_{i,j=1}^{n} (\ker\delta)\delta(a_i)a_j\right)\Bigg].\end{aligned} \tag{2.5}$$

Let $x, y \in \ker \delta$, $a \in \mathscr{A}$. Since

$$\delta(xay) = x\delta(ay) + \delta(x)ay = xa\delta(y) + x\delta(a)y = x\delta(a)y,$$

$$\delta(ax) = a\delta(x) + \delta(a)x = \delta(a)x, \quad \text{and} \quad \delta(xa) = x\delta(a) + \delta(x)a = x\delta(a),$$

it follows that for $i = 1, \ldots, n$,

$$\delta((\ker \delta)\delta(a_i)(\ker \delta)) = (\ker \delta)\delta(a_i)(\ker \delta),$$

$$\delta(a_i) \ker \delta = \delta(a_i \ker \delta), \quad \text{and} \quad (\ker \delta)\delta(a_i) = \delta((\ker \delta)a_i).$$

Since δ has finite rank, we conclude that $(\ker \delta)\delta(a_i)(\ker \delta)$, $\delta(a_i) \ker \delta$, and $(\ker \delta)\delta(a_i)$, $i = 1, \ldots, n$ are all finite-dimensional. We therefore conclude by (2.5) that $\mathscr{S}$ is contained in a finite-dimensional subspace of X. Q.E.D.

2.4 THEOREM. *$Z^1_{WC}(\mathscr{A}, X)$ is uniformly closed and every element is strongly inner.*

Proof. Let $\delta \in Z^1_{WC}(\mathscr{A}, X)$. If for each $u \in U(\mathscr{A})$, we define T_u as before, it follows by the reasoning of Theorem 2.1 that $\{T_u \colon u \in U(\mathscr{A})\}$ is a norm-contracting group of affine maps for which hull δ is invariant. Since each T_u is a bounded linear perturbation of a constant map, T_u is $\sigma(X, X^*)$-continuous. Since hull δ is weakly compact by hypothesis, the usual form of the Ryll-Nardjewski fixed point theorem hence implies as before that δ is strongly inner.

Let $\delta \in Z^1_{WC}(\mathscr{A}, X)^-$. We must show that hull δ is $\sigma(X, X^*)$-compact. Identify hull δ with its canonical embedding in X^{**}; with this done, the weak compactness of hull δ will follow by showing that the $\sigma(X^{**}, X^*)$-closure of hull δ is contained in X (recall that hull δ is norm-bounded, and so its $\sigma(X^{**}, X^*)$-closure is $\sigma(X^{**}, X^*)$-compact).

Choose a sequence $\{\delta_k\} \subseteq Z^1_{WC}(\mathscr{A}, X)$ with $\|\delta - \delta_k\| \to 0$. Let x be a $\sigma(X^{**}, X^*)$-accumulation point of hull δ, and let

$$\left\{\sum_i \lambda_{i,\alpha}\, \delta(u_{i,\alpha})u^*_{i,\alpha}\right\}_\alpha$$

be a net of convex combinations of elements from hull δ approaching x in the $\sigma(X^{**}, X^*)$ topology. Set $y^{(k)}_{i,\alpha} = \delta_k(u_{i,\alpha})u^*_{i,\alpha}$. Then if M is the norm of the map $(x, a) \to xa$, we have

$$\|y^{(k)}_{i,\alpha} - \delta(u_{i,\alpha})u^*_{i,\alpha}\| \leq M\|\delta - \delta_k\|. \tag{1}$$

By $\sigma(X, X^*)$-compactness of hull δ_k, we may assume that $\{\sum_i \lambda_{i,\alpha} y^{(k)}_{i,\alpha}\}_\alpha$ converges $\sigma(X, X^*)$ to $x_k \in X$. By (1) and $\sigma(X^*, X^{**})$-semicontinuity of the norm in X^*, it follows that $\|x - x_k\| \leq M\|\delta - \delta_k\|$. Since $\|\delta - \delta_k\| \to 0$, we conclude that $x \in X$. Q.E.D.

The following result generalizes a theorem of Kamowitz [6].

2.5. COROLLARY. *$Z^1_F(\mathscr{A}, X)^-$ consists entirely of strongly inner derivations.*

Proof. By Lemma 2.3, $Z_F^1(\mathscr{A}, X) \subseteq Z_{WC}^1(\mathscr{A}, X)$. Now apply Theorem 2.4. Q.E.D.

Remark. The norm closure in Corollary 2.5 cannot be replaced by the point-norm closure. Let M_2 be the set of 2×2 matrices, and let A denote the restricted C^*-direct sum of the constant sequence $\{M_2\}$, i.e.,

$$A = \{(A_n): A_n \in M_2, \|A_n\| \to 0\},$$

equipped with pointwise operations and the sup norm. The multiplier algebra of A is $M_2 \oplus M_2 \oplus \ldots$, so if we choose a projection $E_n \in M_2$ for each n and set $E = \bigoplus_n E_n$, then $\delta = \text{ad } E$ induces a derivation of $\mathscr{A} = \mathbf{C} + A$, i.e. an element of $Z^1(\mathscr{A}, \mathscr{A})$, and if in addition one requires that for each $n \geq 2$, $\|E_n - \lambda\| \geq |\lambda|$, for all $\lambda \in \mathbf{C}$, then δ is outer, i.e. $\delta \notin B^1(\mathscr{A}, \mathscr{A})$.

Set $\mathscr{A}_n = \{a \in \mathscr{A}: a_k = 0, \text{ for all } k \geq n + 1\}$. Then $\mathscr{A}_n$ is a finite dimensional direct summand of $\mathscr{A}$, so if π_n is the projection of $\mathscr{A}$ onto $\mathscr{A}_n$ and $\delta_n = \pi_n \circ \delta \circ \pi_n$, then $\{\delta_n\} \subseteq Z_F^1(\mathscr{A}, \mathscr{A})$ and $\delta_n \to \delta$ in the point-norm topology.

3. Two questions

Corollary 2.5 shows that a uniform limit of finite-rank derivations is inner. Since all such derivations are compact as linear mappings of A into X, this naturally raises the following questions:

(1) Is every compact derivation of A into X the uniform limit of finite-rank derivations?

(2) Is every compact derivation of A into X inner?

In a forthcoming paper [12], Charles Akemann and the author answer question (1), and hence question (2), in the affirmative when $X = A$. They also determine the structure of weakly compact derivations in this case, and give some corallaries of these results, among which are conditions both necessary and sufficient for a C^*-algebra to admit a nonzero compact or nonzero weakly compact derivation. Beyond these results, the status of questions (1) and (2) is unknown.

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