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COHOMOLOGY OF FLAG VARIETIES IN CHARACTERISTIC p

BY

WALTER LAWRENCE GRIFFITH, JR.[1]

Section 1

(1.1) Let V be a three-dimensional vector space over an algebraically closed field K. Let X be the projective plane of lines in V and let Y be the dual projective plane of planes in V. The flag variety F is the subvariety of $X \times Y$ consisting of pairs (l, s) where the line l is contained in the plane s. F is a homogeneous space under the action of $SL(V)$.

Let π_X (respectively π_Y) be the projection $X \times Y \to X$ (respectively $X \times Y \to Y$). Let $\mathscr{M}(i, j) = \pi_X^* \mathcal{O}_X(i) \otimes \pi_Y^* \mathcal{O}_Y(j)$, for any pair of integers (i, j) which is a line bundle on $X \times Y$. Let $\mathscr{L}(i, j)$ denote its restriction to F. A line bundle is called singular if any of the following conditions hold: $i = -1$, $j = -1$, or $i + j = -2$. If a line bundle is non-singular, its index is defined to be the number of negative integers in the set $\{i + 1, j + 1, i + j + 2\}$.

There is a general theorem of Bott [5] giving the structure of the cohomology of flag varieties if char $(K) = 0$. In the case of F it is:

THEOREM 1.1. *Let* char $(K) = 0$. *The cohomology vector space* $H^q(\mathscr{L}(i, j)) \neq (0)$ *iff* $\mathscr{L}$ *is non-singular and* q *is the index of* $\mathscr{L}$.

The purpose of this paper is to determine the analogous theorem when char $(K) = p > 0$.

The following theorem is a special case of a theorem of Kempf [12].

THEOREM 1.2. *Let* char (K) *be arbitrary. Assume* q *is either* 0 *or* 3. *The following are equivalent:*

(i) $H^r(\mathscr{L}(i, j)) \neq (0)$ *if and only if* $r = q$.

(ii) $\mathscr{L}(i, j)$ *is non-singular of index* q.

The next theorem is the major result of this paper. It shows that Bott's theorem is false in positive characteristic.

THEOREM 1.3. *Assume* char $(K) = p$. *Let* a *and* b *be positive integers with* $a < p$.

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(i) *Assume* $i+1$ *and* $j+1$ *have opposite signs. If*

$$ap^b < |i+1|,\ |j+1| < (a+1)p^b$$

then $H^1(\mathscr{L}(i,j)) \neq (0) \neq H^2(\mathscr{L}(i,j))$.

(ii) *If the hypothesis in part* (i) *does not hold, the conclusion of Theorem* 1.1 *is correct even in the non-zero characteristic case.*

Since $H^q(\mathscr{L}) = (0)$ for any $\mathscr{L}$ if $q > \dim F = 3$, theorem 1.3 completely solves the problem of when $H^q(\mathscr{L})$ vanishes.

(1.2) In Section 2 projective duality and Serre duality will be used to help reduce the proof of Theorem 1.3 to a special case. In Section 3, the fact that F is a Cartier divisor in $X \times Y$ will be used to relate the cohomology of $\mathscr{L}(i,j)$ to the cohomology of $\mathscr{M}(i,j)$. As an incidental bonus, a proof of Theorem 1.2 different from Kempf's is obtained. In Section 4, the cohomology of the "boundary", of the region in Theorem 1.3 (i), will be computed by representation theoretic techniques. In Section 5, the proof of Theorem 1.3 is completed and some corollaries are given.

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Section 2

Let $\mathscr{L}(i,j)$ be a line bundle on F. Since all assertions to be proven involve the cohomology of $\mathscr{L}$, duality theorems may be used to restrict the range of values of (i,j) that must be considered. The canonical sheaf $\mathscr{C}$ on F has degrees $(-2, -2)$[2], hence by Serre duality $H^q(\mathscr{L})$ and $H^{3-q}(\mathscr{L}^{-1} \otimes \mathscr{C})$ are dual vector spaces over K. The degrees of $\mathscr{L}^{-1} \otimes \mathscr{C}$ are $(-i-2, -j-2)$. If $j = -1$, then $H^q(\mathscr{L}) = (0)$ for all q (independently of i as in [12] or [13]. So it may be assumed that $j \geq 0$, replacing $\mathscr{L}$ by $\mathscr{L}^{-1} \otimes \mathscr{C}$ is necessary.

In the case when $i < 0$ but $i + j > -2$ a further reduction may be made by using projective duality. Let X_0, X_1, X_2 be projective coordinates on X; let Y_0, Y_1, Y_2 be dual projective cordinates. By [11], F is the subvariety of $X \times Y$ defined by the single equation $X_0 Y_0 + X_1 Y_1 + X_2 Y_2 = 0$, and hence is a Cartier divisor in $X \times Y$, since $X \times Y$ is irreducible.

Exchanging a flag (l, s) with its dual (s^*, l^*) yields an automorphism of $\mathbf{P}^2 \times \mathbf{P}^2$ leaving F invariant and exchanging X_i and Y_i. Since the sections of line bundles are rational functions in the X_i's and Y_i's, a map of sheaves $\mathscr{L} \to \mathscr{L}^*$ is induced. It is easy to see that if $\mathscr{L} = \mathscr{L}(i,j)$, then $\mathscr{L}^* = \mathscr{L}(j,i)$ and further that the induced map $H^q(\mathscr{L}) \to H^q(\mathscr{L}^*)$ is an isomorphism for all q in the category of K-vector spaces. Combining with Serre duality yields the fact that $H^q(\mathscr{L})$ is dual to $H^{3-q}((\mathscr{L}^*)^{-1} \otimes \mathscr{C})$. The degrees of $(\mathscr{L}^*)^{-1} \otimes \mathscr{C}$ are $(-j-2, -i-2)$. If $i = -1$, $H^q(\mathscr{L}) = (0)$ as above, so assume $i \leq -2$. Then it follows easily that the involution taking $\mathscr{L}$ to $(\mathscr{L}^*)^{-1} \otimes \mathscr{C}$ takes line bundles of degree (i,j) with $i \leq -2, i+j \geq -2, j \geq 0$ into line bundles of degree (i,j) with

$i+j \leq -2$, $j \geq 0$. So for the purposes of proving Theorem 1.3 it suffices to assume $i+j \leq -2$, $j \geq 0$.

Section 3

(3.1) Since F is a divisor in $X \times Y$ of degrees (1, 1) as seen in the previous section, there is a short exact sequence of sheaves

$$(1) \qquad 0 \to \mathscr{M}(i-1, j-1) \to \mathscr{M}(i, j) \to \mathscr{L}(i, j) \to 0.$$

First assume that $i, j \geq 0$. (1) can be used to prove Theorem 1.2. Since

$$\mathscr{M}(i, j) = \pi_X^* \mathcal{O}_X(i) \otimes \pi_Y^* \mathcal{O}_Y(j),$$

$H^0(\mathscr{M}(i, j)) \cong H^0(\mathcal{O}_X(i)) \otimes H^0(\mathcal{O}_Y(j))$ and $H^q(\mathscr{M}) = (0)$ for $q > 0$ by the Kunneth formula and Serre's computation of the cohomology of line bundles on projective spaces [10, III, 2.1, 12]. Note that a similar discussion also holds for $\mathscr{M}(i-1, j-1)$, since $i-1, j-1 \geq -1$ and $H^q(\mathcal{O}_X(-1)) = H^q(\mathcal{O}_Y(-1)) = (0)$ for all q. Taking the long exact sequence in cohomology corresponding to (1) yields

$$(2) \qquad 0 \to H^0(\mathscr{M}(i-1, j-1)) \to H^0(\mathscr{M}(i, j)) \to H^0(\mathscr{L}(i, j)) \to 0$$

and $H^q(\mathscr{L}(i, j)) = (0)$ for $q > 0$. This proves Theorem 1.2 for the case of index $(\mathscr{L}) = 0$. If index $(\mathscr{L}) = 3$ an analogous proof holds, or one may simply use Serre duality to reduce to the case above. Note that Theorem 1.3 is also proven in the case where i and j are of the same sign.

Assume now that $i < 0$, $j \geq 0$. By the Kunneth formula and Serre's result cited above, $H^2(\mathscr{M}(i, j)) = H^2(\mathcal{O}_X(i)) \otimes H^0(\mathcal{O}_Y(j))$ and $H^q(\mathscr{M}) = (0)$ if $q \neq 2$. Further, $H^2(\mathcal{O}_X(i))$ is the K-vector space generated by monomials $X_0^a X_1^b X_2^c$, where $a + b + c = i$ and $a, b, c < 0$. $H^0(\mathcal{O}_Y(j))$ is the K-vector space of elements homogeneous of degree j in the symmetric algebra generated by Y_0, Y_1, Y_2. The long exact sequence in cohomology now reads

$$(3) \quad 0 \to H^1(\mathscr{L}(i, j)) \to H^2(\mathscr{M}(i-1, j-1)) \to H^2(\mathscr{M}(i, j)) \to H^2(\mathscr{L}(i, j)) \to 0$$

and

$$H^0(\mathscr{L}(i, j)) = H^3(\mathscr{L}(i, j)) = (0).$$

THEOREM 3.1. *$H^1(\mathscr{L}(-(a+1)p^b, ap^b)) \neq (0)$, where $a, b \in Z^+$ and $a < p$.*

To prove Theorem 3.1 a lemma is needed. Let $c \in H^2(\mathscr{M}(i-1, j-1))$. From above, c can be written in the form

$$(4) \qquad c = c_1 X_0^{-1} + c_2 X_0^{-2} + \cdots + c_{-i+1} X_0^{i-1}$$

where each c_k is an expression involving X_1^{-1}, X_2^{-1}, Y_0, Y_1, Y_2. Let

$$E = X_0 Y_0 + X_1 Y_1 + X_2 Y_2 = X_0 Y_0 + r.$$

The map $H^2(\mathscr{M}(i-1, j-1)) \to H^2(\mathscr{M}(i, j))$ is induced by multiplication by E.

LEMMA 3.2. (i) *$E \cdot a = 0$ implies Y_0^i divides $r^i c_i$.*

(ii) *If d is an expression in X_1^{-1}, X_2^{-1}, Y_0, Y_1, Y_2 such that Y_0^i divides $r^i d$, then there exists a unique $c \in H^2(\mathscr{M}(i-1, j-1))$ as above such that $c_i = d$.*

Proof. Write

$$E \cdot c = X_0^{-1}(rc_1 + Y_0 c_2) + X_0^{-2}(rc_2 + Y_0 c_3) + \cdots + X_0^{i-1}(rc_{i+1}).$$

Then $E \cdot c = 0$ if and only if $-rc_1 = Y_0 c_2, \ldots, rc_{i+1} = 0$. Note that Y_0 is not a zero divisor. By successive substitutions $(-1)^k Y_0^k c_{k+1} = r^k c_1$ for $1 \leq k \leq i$. The case of $k = i-2$ immediately yields (i). Given c_1, c_{k+1} can be uniquely determined by the preceding equation, hence (ii) also follows.

Proof of Theorem 3.1. By the exact sequence (3), it suffices to show that

$$H^2(\mathscr{M}(-(a+1)p^b - 1, ap^b - 1)) \to H^2(\mathscr{M}(-(a+1)p^b, ap^b))$$

has non-zero kernel. By Lemma 3.2 (ii) this will be true if a suitable non-zero c_1 can be found. Let $c_1 = X_1^{-p^b} X_2^{-ap^b} Y_0^{ap^b - 1}$.Then

$$r^{ap^b} c_1 = (X_1^{p^b} Y_1^{p^b} + X_2^{p^b} Y_2^{p^b})^a X_1^{-p^b} X_2^{-ap^b} Y_0^{ap^b - 1}.$$

Since $a < p$,

$$(5) \qquad (X_1^{p^b} Y_1^{p^b} + X_2^{p^b} Y_2^{p^b})^a = \sum_{a_1 + a_2 = a} \frac{a!}{a_1!\, a_2!} X_1^{a_1 p^b} X_2^{a_2 p^b} Y_1^{a_1 p^b} Y_2^{a_2 p^b}.$$

Recall from Serre's computation [10] that if $s > 0$, $t < 0$, $X_1^s X_1^t = 0$ as a cohomology class if $|t| < s$ (and similarly for X_2). Each term of the right-hand side of (5) hence annihilates $X_1^{-p^b} X_2^{-ap^b} Y_0^{ap^b - 1}$, since if $a_1 = 0$, $X_2^{a_2 p^b} \cdot X_2^{-ap^b} = X_2^{ap^b} \cdot X_2^{-ap^b} = 0$, whereas if $a_1 > 0$, $X_1^{a_1 p^b} \cdot X_1^{-p^b} = 0$.

Hence $r^{ap^b} c_1 = 0$ and the hypothesis of Lemma 3.2 (ii) is satisfied. This yields a non-zero element of $H^1(\mathscr{M}(-(a+1)p^b, ap^b)$, proving Theorem 3.1.

LEMMA 3.3. *If $\mathscr{L}$ is of index* 2, $H^2(\mathscr{L}) \neq (0)$.

Proof. Let $h^i(\mathscr{L}) = \dim H^i(\mathscr{L})$. Since $h^2(\mathscr{M}(i-1, j-1))$ and $h^2(\mathscr{M}(i, j))$ do not depend on the characteristic of K by the computation preceding (3), $\chi(\mathscr{L}) = h^2(\mathscr{L}) - h^1(\mathscr{L})$ does not depend on char (K) either by (3). By Theorem 1.1 and the hypothesis on $\mathscr{L}$, $\chi(\mathscr{L}) > 0$. So $h^2(\mathscr{L}) > 0$, proving Lemma 3.3.

(3.2) In this section certain Schubert subvarieties of F will be used to study the cohomology. The various assumptions made above continue to be in force.

Define a Cartier divisor S in F by the global equation $X_2 = 0$. (Hence $X_0 Y_0 = X_1 Y_1$ on S.) Define a divisor T in S by the following local equations: on the open set $U_1 = \{X_0 \neq 0\}$ in S, the equation of T is $Y_1 X_0^{-1} = 0$; on the open set $U_2 = \{X_1 \neq 0\}$ the equation of T is $Y_0 X_1^{-1} = 0$. Since $Y_1 X_0^{-1} = -Y_0 X_1^{-1}$ on $U_1 \cap U_2$ this gives rise to a Cartier divisor. S and T are examples of Kempf varieties (see [12]).

Note that S is the variety of flags (l, s) such that l lies in the plane $x_2 = 0$ and T is the subvariety such that s is that plane. Hence $T \cong \mathbf{P}^1$.

(3.3) The cohomology of line bundles on S and T will now be calculated. It follows from the last paragraph of (3.2) that $\mathscr{L}(i, j)|_T \cong \mathcal{O}_{\mathbf{P}^1}(i)$; the isomorphism multiplies a local section of $\mathcal{O}(i)$ (expressed in terms of X_0 and X_1) by Y_2^j to obtain the corresponding section of $\mathscr{L}(i, j)|_T$. Hence $H^q(T, \mathscr{L}(i, j)) = (0)$ if $q \neq 1$ and $H^1(T, \mathscr{L}(i, j))$ is the K-vector space generated by monomials of the form $X_0^{a_0} X_1^{a_1} Y_2^j$, where $a_0 + a_1 = i$ and $a_0, a_1 < 0$.

Next consider S. The exact sequence of the divisor T in S is

$$0 \to \mathscr{L}(i+1, j-1)|_S \to \mathscr{L}(i, j)|_S \to \mathscr{L}(i, j)|_T \to 0. \tag{6}$$

LEMMA 3.4. *$H^1(S, \mathscr{L}(i, j)|_S)$ is naturally isomorphic as a K-vector space to the vector space generated by the monomials*

$$X_0^{i+r} X_1^{-r} Y_1^s Y_2^{j-s} \quad \textit{for} \quad 0 \leq s \leq j, 1 \leq r \leq -i-s-1.$$

Also $H^q(\mathscr{L}(i, j)|_S) = (0)$ if $q \neq 1$.

Proof. By increasing induction on j. If $j = -1$, $H^q(\mathscr{L}(i, j)|_S) = (0)$ for all q [2], [12], [13]. If $j = 0$, then $H^1(\mathscr{L}(i, j)|_S)$ is isomorphic to $H^1(\mathscr{L}(i, j)|_T)$, by the long exact sequence derived from (6) and the case of $j = -1$ just mentioned. Since the map $\mathscr{L}|_S \to \mathscr{L}|_T$ is restriction, the monomials $X_0^{a_0} X_1^{a_1}$ which generate $H^1(\mathscr{L}|_T)$ have preimages $X_1^{a_0} X_1^{a_1}$ in $H^1(\mathscr{L}|_S)$. Taking $r = -a_1$ proves Lemma 3.4 if $j = 0$, since $H^q(\mathscr{L}|_T) = (0)$ implies $H^q(\mathscr{L}|_S) = (0)$ for all $q \neq 1$.

Assume the lemma holds for $j - 1$. From (6),

$$0 \to H^1(\mathscr{L}(i+1, j-1)|_S) \to H^1(\mathscr{L}(i, j)|_S) \to H^1(\mathscr{L}(i, j)|_T) \to 0. \tag{7}$$

The map $\mathscr{L}(i+1, j-1)|_S \to \mathscr{L}(i, j)|_S$ is given by multiplication by $Y_1 X_0^{-1}$ $(= -Y_0 X_1^{-1})$, so the image of $H^1(\mathscr{L}(i+1, j-1)|_S \to H^1(\mathscr{L}(i, j)|_S)$ is generated by

$$X_0^{i+r} X_1^{-r} Y_1^{s+1} Y_2^{j-s-1}, \quad 0 \leq s \leq j-1, \quad 1 \leq r \leq i-s-2.$$

Replacing $s + 1$ by s gives

$$X_0^{i+r} X_1^{-r} Y_1^s Y_2^{j-s}, \quad 1 \leq s \leq j, \quad 1 \leq r \leq -i-s-1.$$

The preimage of $H^1(\mathscr{L}(i, j)|_T)$ is generated by $X_0^{a_0} X_1^{a_1} Y_2^j$; this just gives the $s = 0$ case. Since by induction $H^q(\mathscr{L}(i+1, j-1)|_S) = (0)$ if $q \neq 1$ and $H^q(\mathscr{L}(i, j)|_T) = (0)$ if $q \neq 1$, then the cohomology sequence derived from (6) implies $H^q(\mathscr{L}(i, j)|_S) = (0)$, $q \neq 1$. This proves Lemma 3.4.

(3.4) LEMMA 3.5. *Let $\mathscr{L}'$ be the line bundle of degrees* $(1, 0)$ (*resp.* $(0, 1)$). *Assume that the degrees i and j of both $\mathscr{L}$ and $\mathscr{L} \otimes \mathscr{L}'$ satisfy $i + j \leq 2, j \geq 0$. Let u be X_0 (resp. Y_0). Then the map*

$$C_u \colon H^1(\mathscr{L}) \to H^1(\mathscr{L} \otimes \mathscr{L}')$$

(*induced by the cup product $H^1(\mathscr{L}) \otimes H^0(\mathscr{L}') \to H^1(\mathscr{L} \otimes \mathscr{L}')$*) *is injective.*

Proof. Assume first that $\mathscr{L}'$ has degrees (1, 0). The variety defined by the vanishing of u is S. Since the map C_u acts on $H^1(\mathscr{L})$ (as it has been represented here) by multiplication by u, it is clearly the map in the exact sequence

$$H^0(\mathscr{L} \otimes \mathscr{L}'|_S) \to H^1(\mathscr{L}) \xrightarrow{C_u} H^1(\mathscr{L} \otimes \mathscr{L}') \tag{8}$$

obtained from the exact sequence of the divisor $u = 0$:

$$0 \to \mathscr{L} \to \mathscr{L} \otimes \mathscr{L}' \to \mathscr{L} \otimes \mathscr{L}'|_S \to 0$$

Hence it suffices to show $H^0(\mathscr{L} \otimes \mathscr{L}'|_S) = (0)$. This follows immediately from Lemma 3.4.

Assume now that $\mathscr{L}'$ has degrees (0, 1). Let the subvariety of F defined by $Y_0 = 0$ be denoted by S'. As above it suffices to show that $H^0(\mathscr{L} \times \mathscr{L}'|_{S'}) = (0)$. As noted in Section 2, there is a map $F \to F$ obtained by interchanging X_i and Y_{i}. This map clearly takes S' to S and vice versa. So the involution $\mathscr{L} \to (\mathscr{L}^*)^{-1} \otimes \mathscr{C}$ of Section 2 takes $\mathscr{L} \otimes \mathscr{L}'|_{S'}$ to $\mathscr{L}_1|_S$, where $\mathscr{L}_1$ has degrees (i, j) such that $i + j \geq -2, j \geq 0$. To show $H^0(\mathscr{L} \otimes \mathscr{L}'|_S) = (0)$ it suffices then to show $H^2(\mathscr{L}_1|_S) = (0)$, by Serre duality.

By taking the long exact sequence in cohomology coming from (6) the following exact sequence is obtained:

$$H^2(\mathscr{L}(i+1, j-1)|_S) \to H^2(\mathscr{L}(i, j)|_S) \to H^2(\mathscr{L}(i, j)|_T). \tag{9}$$

$H^2(\mathscr{L}(i, j)|_T) = (0)$ by the first paragraph of (3.3), so

$$H^2(\mathscr{L}(i+1, j-1)|_S) \to H^2(\mathscr{L}(i, j)|_S)$$

is surjective. By induction $H^2(\mathscr{L}(i+r, j-r)|_S) \to H^2(\mathscr{L}(i, j)|_S)$ is surjective for any $r \geq 0$. In particular

$$H^2(\mathscr{L}(i+j+1, -1)|_S) \to H^2(\mathscr{L}(i, j)|_S)$$

is surjective. $H^2(\mathscr{L}(i+j+1, -1)|_S) = (0)$ by [12, Lemma 1], so $H^2(\mathscr{L}_1|_S) = (0)$, proving Lemma 3.5.

Section 4

(4.1) The following material is adapted from [8] to the case of char $(K) = p$. Assume that $\mathscr{L}$ is a line bundle of degrees (i, j) such that $i + j \leq -2, j \geq -1$. In this section it will be shown that $H^1(\mathscr{L}) \neq (0)$ if $j = ap^b - 1$ for $a, b \in \mathbf{Z}^+$ with $a < p$.

Let S: $F \to X$ be the map induced by the projection $X \times Y \to X$. Let i_1 be a non-negative integer of the form $ap^b - 1$ for $1 \leq a < p$, $b > 0$. Let $\mathscr{L}_1$ be the line bundle on F of degrees (1, 0). $S_*(\mathscr{L})$ is a $SL(3)$-homogeneous vector bundle on $\mathbf{P}^2$. If $\mathscr{C}_1$ denotes the canonical sheaf of F over $\mathbf{P}^2$ (i.e., the sheaf of differentials of F_3 as a $\mathbf{P}^2$-scheme via S), then by [8] there is an equivariant duality $S_*(\mathscr{L}_1) \otimes S_*(\mathscr{L}_1^{-1} \otimes \mathscr{C}^{-1}) \to \mathcal{O}_{\mathbf{P}^2}$, since S is a $\mathbf{P}^1$—fibering (the fiber of S over a plane t representing a point of $\mathbf{P}^2$ is the set of all lines contained in t, which is isomorphic to $\mathbf{P}^1$). This clearly extends to a map

$$\text{Sym}^{i_1}(S_*(\mathscr{L}_1)) \otimes \text{Sym}^{i_1}(S_*(\mathscr{L}_1^{-1} \otimes \mathscr{C}_1^{-1})) \to \mathcal{O}_{\mathbf{P}^2}. \tag{10}$$

(10) is in fact an equivariant duality. Because the sheaves are SL-homogeneous, it suffices to verify (10) after passing to the stalks at any particular point y in $\mathbf{P}^2$. Choose y to have homogeneous coordinates $(Y_0, Y_1, Y_2) = (0, 0, 1)$. The equation of F then reduces to $X_2 = 0$ in the fiber of S over y. Hence the stalk of $\text{Sym}^{i_1}(S_*(\mathscr{L}_1))$ is $\text{Sym}^{i_1} V$, where V is the vector space generated by X_0, X_1 over K. By [8, Section 6] $\mathscr{C}_1$ is the line bundle of degrees $(-2, 1)$. Then the first degree of $\mathscr{L}_1^{-1} \otimes \mathscr{C}_1^{-1}$ is also 1, so its stalk is $\text{Sym}^{i_1} \hat{V}$, where $\hat{V}$ is the dual of V. It is only necessary then to verify that $\text{Sym}^{i_1} V$ is irreducible as an $SL(2)$-space (the subgroup of $SL(3)$ preserving the stalk, modulo the subgroup which acts trivially on the stalk, is isomorphic to $SL(2)$).

LEMMA 4.1. *The representation of $SL(2, K)$ on* $\text{Sym}^{i_1} V$ *is irreducible, if $i_1 = ap^b - 1$ and K is algebraically closed.*

Proof. Let π be the representation of $SL(2, K)$ on V. The irreducible module

$$\left(\bigotimes_{k=1}^{p-1} \pi \otimes \pi^{Fr} \otimes \cdots \pi^{Fr^{b-1}}\right) \otimes \left(\bigotimes_{k=1}^{a-1} \pi^{Fr^b}\right)$$

which is irreducible by [16], is easily verified to be the representation of $SL(2)$ on $\text{Sym}^{i_1} V$. This proves Lemma 4.1.

(4.2) Hence (10) is an equivariant duality. Since

$$\text{Sym}^{i_1}(S_*(\mathscr{L}_1)) \cong S_*(\mathscr{L}_1^{i_1}) \quad \text{and} \quad \text{Sym}^{i_1}(S_*(\mathscr{L}_1^{-1} \otimes \mathscr{C}_1^{-1})) \cong S_*(\mathscr{L}_1^{-i_1} \otimes \mathscr{C}_1^{-i_1}),$$

then (10) translates as

$$S_*(\mathscr{L}(i_1, 0)) \otimes S_*(\mathscr{L}(i_1, -i_1)) \to \mathcal{O}_{\mathbf{P}^2}$$

being a perfect pairing. Hence

$$[\mathcal{O}(j_1) \otimes S_*(\mathscr{L}(i_1, 0))] \otimes [\mathcal{O}(-j_1) \otimes S_*(\mathscr{L}(i_1, -i_1))] \to \mathcal{O}_{\mathbf{P}^2}$$

is a perfect pairing. But

$$\mathcal{O}(j_1) \otimes S_*(\mathscr{L}(i, 0)) \cong S_*(\mathscr{L}(i_1, j_1))$$

and

$$\mathcal{O}(-j_1) \otimes S_*(\mathscr{L}(i_1, -i_1)) \cong S_*(\mathscr{L}(i_1, -i_1 - j_1)),$$

so

$$S_*(\mathscr{L}(i_1, j_1)) \otimes S_*(\mathscr{L}(i_1, -i_1 - j_1)) \to \mathcal{O}_{\mathbf{P}^2}$$

is a perfect pairing. Let $\mathscr{L} = \mathscr{L}(i_1, j_1)$. Then the pairing is

$$S_*(\mathscr{L}) \otimes S_*(\mathscr{L}^{-1} \otimes \mathscr{C}_1^{-i_1}) \to \mathcal{O}_{\mathbf{P}^2}$$

By Serre duality $S_*(\mathscr{L}^{-1} \otimes \mathscr{C}_1^{-i_1})$ is dual to $R^1S_*(\mathscr{L} \otimes \mathscr{C}_1^{i_1+1})$ so

$$S_*(\mathscr{L}) \cong R^1S_*(\mathscr{L} \otimes \mathscr{C}_1^{i_1+1}) \tag{11}$$

Since $\mathscr{L} \cong S^*(\mathscr{O}(j_1)) \otimes \mathscr{L}(i_1, 0)$,

$$R^qS_*(\mathscr{L}) \cong \mathscr{O}(j_1) \otimes R^qS_*(\mathscr{L}(i_1, 0)).$$

But $R^qS_*(\mathscr{L}(i_1, 0)) = (0)$ if $q > 0$ since $i_1 \geq 0$. So $R^qS_*(\mathscr{L}) = (0)$ if $q > 0$. Similarly

$$\begin{aligned} R^qS_*(\mathscr{L} \otimes \mathscr{C}^{i_1+1}) &\cong R^qS_*(\mathscr{L}(-i, -2, i_1 + j_1 + 1) \\ &\cong \mathscr{O}(i_1 + j_1 + 1) \otimes R^qS_*(\mathscr{L}(-i_1 - 2, 0)), \end{aligned}$$

which is 0 if $q \neq 1$.

Then the Leray Spectral Sequence implies

$$H^q(\mathscr{L}) \cong H^q(S_*(\mathscr{L})), \qquad H^{q+1}(\mathscr{L} \otimes \mathscr{C}^{i_1+1}) \cong H^q(R^1S_*(\mathscr{L} \otimes \mathscr{C}^{i_1+1}))$$

so

$$H^1(\mathscr{L} \otimes \mathscr{C}_1^{i_1+1}) \cong H^0(\mathscr{L}), \qquad H^q(\mathscr{L} \otimes \mathscr{C}_1^{i_1+1}) \cong (0), \quad q \neq 1 \tag{12}$$

Applying the involution $\mathscr{L} \to (\mathscr{L}^*)^{-1} \otimes \mathscr{C}$ of (2.1) (recall $\mathscr{C} \cong \mathscr{L}(-2, -2)$) one obtains from (12),

$$\begin{aligned} &H^2(((\mathscr{L} \otimes \mathscr{C}_1^{i_1+1})^*)^{-1} \otimes \mathscr{C}) \cong H^0(\mathscr{L}), \\ &H^q(((\mathscr{L} \otimes \mathscr{C}_1^{i_1+1})^*)^{-1} \otimes \mathscr{C}) \cong (0), \quad q \neq 2, \end{aligned} \tag{13}$$

$((\mathscr{L} \otimes \mathscr{C}_1^{i_1+1})^*)^{-1} \otimes \mathscr{C} \cong \mathscr{L}(-i_1 - j_1 - 3,\ i_1)$. Suppose $\mathscr{L}'(i, j)$ is any line bundle such that $i + j \leq -2, j \geq -1$, and j is of the form $ap^b - 1$. By choosing $i_1 = j$ and $j_1 = -i - j - 3$, we find that

$$\mathscr{L}' = ((\mathscr{L}(j, -i - j - 3) \otimes \mathscr{C}_1^{i_1+1})^*)^{-1} \otimes \mathscr{C}.$$

Note that j and $-i - j - 3 \geq -1$.

Note that (13) and the last paragraph relate the higher cohomology of some line bundle to the global sections of another. In particular a condition for the vanishing of $H^1(\mathscr{M})$ for certain $\mathscr{M}$ in implied.

Section 5

(5.1) This subsection is the proof of Theorem 1.3. By Section 2 it suffices to consider $\mathscr{L}(i, j)$ with $i + j \leq -2, j \geq 0$. Suppose there exists a line bundle $\mathscr{L}'$ with degrees i, $(a + 1)p^b - 1$ such that

$$ap^b < j \leq (a + 1)ap^b - 1,$$

but $i + (a + 1)ap^b - 1 \leq -2$. $H^1(\mathscr{L}') = 0$ by (13) above, since $\mathscr{L}'$ is of the form required. Repeated application of Lemma 3.5 shows that there is an injection $H^1(\mathscr{L}) \to H^1(\mathscr{L}')$, hence $H^1(\mathscr{L}) = (0)$.

The inequalities $ap^b < j \leq (a+1)p^b - 1$ and $i + (a+1)p^b - 1 \leq -2$ can be satisfied for some a, b as in Theorem 1.3 if and only if $ap^b < |i+1|, |j+1| < (a+1)p^b$ cannot be satisfied for some a, b as in Theorem 1.3. Hence to prove Theorem 1.3 it now suffices to show $ap^b < |i+1|, |j+1| < (a+1)p^b$ implies $H^1(\mathscr{L}) \neq (0)$. Since $i \geq -(a+1)p^b$, $j > ap^b$, there exists an injection

$$H^1(\mathscr{L}(-(a+1)p^b, ap^b)) \to H^1(\mathscr{L}(i, j))$$

by Lemma 3.5. Since $H^1(\mathscr{L}(-(a+1)p^b, ap^b) \neq (0)$ by Theorem 3.1, Theorem 1.3 follows immediately.

(5.2) In this subsection two corollaries of Theorem 1.3 are proven. In the case of Corollary 5.2 below, the reader is assumed to know pertinent definitions and results from [2], [3], and [4].

COROLLARY 5.1. *Assume* char $(K) = p$. *There always exist both singular and non-singular line bundles* $\mathscr{L}$ *for which* $H^q(\mathscr{L}) \neq (0)$, $q = 1, 2$. (*This result was announced in* [4] *prior to the presentation of the author's thesis.*)

Proof. Let $\mathscr{L}$ be $\mathscr{L}(-p-2, p)$; respectively let $\mathscr{L}$ be $\mathscr{L}(-2p^2, p^2)$. Apply Theorem 1.3.

COROLLARY 5.2. *Let* $F_n = SL(n)/B$.

(1) *For any* $n \geq 3$, *there exist line bundles* $\mathscr{L}$ *on* F_n *such that* $H^q(\mathscr{L}) \neq (0)$ *for at least two values of* q.

(2) *For any* $n \geq 3$, *there exist line bundles* $\mathscr{L}$ *on* F_n, *with neither* $\mathscr{L}$ *nor* $\mathscr{L}'$ (*the Serre dual of* $\mathscr{L}$) *in the dominant chamber or its adjacent walls, such that* $H^q(\mathscr{L}) \neq (0)$ *for exactly one value of* q.

Proof. By induction on n. For $n = 3$, the corollary is logically weaker than Theorem 1.3.

Suppose the corollary holds for $n - 1$. There is a morphism $S: F_n \to \mathbf{P}^{n-1}$, where $\mathbf{P}^{n-1}$ is considered as the projective space of $(n-1)$—dimensional subspaces in $\mathbf{A}^n$. S takes a flag $(V_1, \ldots V_{n-1})$ to V_{n-1}. The fiber of S over the point represented by V'_{n-1} is $\{(V_1, \ldots V_{n-2}, V'_{n-1})\}$ and hence is isomorphic to F_{n-1}.

The dominant Weyl chamber (including the adjacent walls) consists of line bundles with all degrees ≥ -1; the dual chamber (including the walls) consists of characters with all degrees ≤ -1. By [2], any line bundle satisfying (1) must lie outside these chambers.

By the Leray Spectral Sequence $H^{q_1}(\mathbf{P}^{n-1}, R^{q_2}S_*(\mathscr{L}))$ converges as an E^2 sequence to $H^{q_1+q_2}(\mathscr{L})$, for any line bundle $\mathscr{L}$ on F_n. $\mathscr{L}$ restricted to the fiber of S over a point p in $\mathbf{P}^{n-1}$ is the line bundle with degrees equal to the first $n-2$ degrees of $\mathscr{L}$. Choose $\mathscr{L}$ such that $\mathscr{L}$ restricted to a fiber (considered as F_{n-1} as above) satisfies (1) (respectively (2)) in the corollary for $n - 1$. Then $R^{q_2}S_*(\mathscr{L}) \neq (0)$ for at least two values of q_2 (respectively exactly one value of q_2). Replace $\mathscr{L}$ by $\mathscr{L} \otimes S^*(\mathcal{O}(m))$, where m is so large that $R^qS_*(\mathscr{L} \otimes S^*(\mathcal{O}(m))$

is very ample for all q. Then the Leray sequence degenerates and

$$H^q(\mathscr{L} \otimes S^*(\mathcal{O}(m))) \cong H^0(\mathbf{P}^{n-1}, \quad R^qS_*(\mathscr{L}) \otimes \mathcal{O}(m))$$

Since $R^qS_*(\mathscr{L}) \neq (0)$ for at least two values of q (respectively exactly one value of q), $H^q(\mathscr{L} \otimes S^*(\mathcal{O}(m))) \neq (0)$ for at least two values of q (respectively exactly one value of q).

Since the degrees of $\mathscr{L}$ are neither all ≥ -1 or ≤ -1, the same is true of $\mathscr{L} \otimes S^*(\mathcal{O}(m))$ (whose first $n-2$ degrees are those of $\mathscr{L}$ and the $(n-1)$-st degree is m). Hence (1) holds for n (respectively (2) holds).

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University of Missouri—St. Louis
St. Louis, Missouri