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ON THE FOURIER SERIES OF CERTAIN SMOOTH FUNCTIONS[1]

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1. Introduction and statement of results

By $w(t) = w(f, t)$ we shall denote the L^1-modulus of continuity of a period function belonging to $L^1(-\pi, \pi)$, namely

$$w(t) = \sup_{|h|<t} \int_{\pi}^{\pi} |f(x+h) - f(x)| \, dx. \tag{1.1}$$

A classical result of Marcinkiewicz shows that if

$$\int_0^1 w(t) \frac{dt}{t} < \infty,$$

then the Fourier Series of f converges a.e. The aim of this paper is to show a connection between the smoothness of a function and the growth of the partial sums of its Fourier Series.

THEOREM 1. *Suppose that $w(f, t) < c/|\log t|$; then*

$$S_n(f) = o[\log \log n(\log \log \log n)^{1+\varepsilon}] \quad \text{a.e.} \quad \varepsilon > 0.$$

More generally:

THEOREM 2. *Let $w(t)$ be the L^1-modulus of continuity of f. Let $\phi(t)$ be a continuous increasing function of the variable t such that*

$$\int_0^1 w(t)\phi(t) \frac{dt}{t} < \infty, \quad \phi(0) = 0.$$

Then

$$S_n(f) = o\left(\phi\left[\frac{1}{n}\right]\right)^{-1} \text{ a.e.}$$

REMARK. If $w(t)$ satisfies the Dini condition, there $S_n(f)(x)$ converges a.e. On the other hand, the closer $w(t)$ gets to satisfying the Dini condition the slower the growth of $S_n(f)$ is.

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2. Proof of the results

We shall prove Theorem 2 only since Theorem 1 is a particular case.

(2.1) LEMMA. *Let $w(t)$ and $\phi(t)$ be as in the statement of Theorem 2. Then for each $\lambda > 0$ it is possible to decompose f as $\bar{f} + \varphi$ so that the following hold.*

(i) $|\bar{f}| < c_1 \lambda$ a.e.

(ii) *$\bar{f} = f$ on a closed set F. Its complement G is covered by a denumerable union interval $\bigcup_1^\infty I_k \supset G$ such that each point $[-\pi, \pi]$ belongs to at most N intervals.*

$$\text{(iii)} \qquad \sum_1^\infty |I_k| \le \frac{C_2}{\lambda}\left(\|f\|_1 + \int_0^1 w(t)\phi(t)\frac{dt}{t}\right).$$

$$\text{(iv)} \qquad \sum_1^\infty \int_{I_k} |\varphi|\, dt \int_{|I_k|}^1 \phi(t)\frac{dt}{t} < C_3\left(\|f\|_1 + \int_0^1 w(t)\phi(t)\frac{dt}{t}\right).$$

This lemma is a specialization to $[-\pi, \pi]$ of Lemmas (2.2) and (2.3) in [1] and its proof follows the same lines. The constants C_1, C_2, C_3 and N do not depend on λ or f. Select $\lambda > 0$ and consider only the partial sums $S_n(\varphi)$ ($S_n(\bar{f})$ converges a.e. by Carleson's Theorem [2]). Let us denote by $2I_k$ the dialation of I_k two times about its center. Let $G_\lambda^* = \bigcup_1^\infty 2I_k$; Lemma (2.1) gives the estimate

$$\text{(2.2.1)} \qquad |G_\lambda^*| < 2\frac{C_2}{\lambda}\left(\|f\|_1 + \int_0^1 w(t)\phi(t)\frac{dt}{t}\right).$$

Let $S_*(f) = \sup_n |\phi(1/n)S_n(f)|$ and denote by $M(f)(x)$ the Hardy-Littlewood maximal operator. Then

$$\text{(2.2.2)} \qquad S_*(\varphi) \le CM(\varphi) + \sup_n \phi\left(\frac{1}{n}\right)\int_{|x-y|>1/n} \frac{1}{|x-y|}\, |\varphi(y)|\, dy$$

if $x \in [-\pi, \pi] - G_\lambda^*$.

Also

$$\text{(2.2.3)} \qquad \begin{aligned} &\phi\left(\frac{1}{n}\right)\int \frac{1}{|x-y|}\, |\varphi(y)|\, dy \\ &\le \sum_{k=1}^\infty \phi\left(\frac{1}{n}\right)\int_{\{|x-y|>1/n\}\cap I_k} \frac{1}{|x-y|}\, |\varphi(y)|\, dy \\ &\le \sum_{k=1}^\infty \int_{I_k} \frac{\phi(|x-y|)}{|x-y|}\, |\varphi(y)|\, dy \\ &= \Delta(x). \end{aligned}$$

Consequently

$$\text{(2.2.4)} \qquad S_*(\varphi) \le C(M(\varphi)(x) + \Delta(x))$$

whenever $x \in [-\pi, \pi] - G_\lambda^*$.

It should be pointed out that $M(\varphi)(x) < c\lambda$ on $[-\pi, \pi] - G_\lambda^*$. This follows from the proofs of Lemmas (2.2) and (2.3) in [1]. Integrating $S_*(\varphi)$ over $[-\pi, \pi] - G_\lambda^*$ and using (iv) of Lemma 2.1 we get

$$S_n(\varphi) = O\left[\phi\left(\frac{1}{n}\right)\right]^{-1} \quad \text{a.e. in} \quad [-\pi, \pi] - G_\lambda^*. \tag{2.2.5}$$

In order to get "o" we choose λ large so that $M(\varphi)$ is small except for a small set and use the estimate

$$\begin{aligned} &\overline{\lim} \left|\phi\left(\frac{1}{n}\right)S_n(\varphi)\right| \\ &\quad \leq CM(\varphi) + \lim \left|\sum_{k=1}^{k_0} \phi\left(\frac{1}{n}\right) \int_{I_k} D_n(x-y)\varphi(y)\,dy\right| \\ &\quad + \sum_{k_0}^{\infty} \int_{I_k} \frac{\phi(x-y)}{|x-y|} |\varphi(y)|\,dy, \\ &\quad x \in [-\pi, \pi] - \bigcup_1^\infty 2I_k. \end{aligned} \tag{2.2.6}$$

In the above expression $D_n(y)$ stands for the Dirichlet kernel. For $x \in [-\pi, \pi] - G_\lambda^*$,

$$\sum_{k=1}^{k_0} \phi\left(\frac{1}{n}\right) \int_{I_k} D_n(x-y)\varphi(y)\,dy$$

tends to zero because of the smallness of $\phi(1/n)$ and of Riemann-Lebesgue's Theorem applied to each one of the k_0 terms of the form $\int_{I_k} D_n(x-y)\varphi(y)\,dy$.

Finally, by selecting k_0 large enough,

$$\sum_{k_0}^{\infty} \int_{I_k} \frac{\phi(|x-y|)}{|x-y|} |\varphi(y)|\,dy$$

can be made arbitrarily small on $[-\pi, \pi] - G_\lambda^*$ except for a subset of small measure. This finishes the proof.

References

1. C. P. Calderón, *Smooth functions and convergences of Singular integrals*, Illinois J. Math., vol. 23 (1979), pp. 497–509.
2. L. Carleson, *On convergence and growth of partial sums of Fourier Series*, Acta Math., vol. 116 (1966), pp. 135–157.

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