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SOME TOPOLOGICAL ANTI-PROPERTIES

BY

I. L. REILLY AND M. K. VAMANAMURTHY

1. Introduction

This paper continues the study of the method introduced by Bankston [1] for generating new topological properties from old. Given a class $\mathscr{K}$ of topological spaces ($\mathscr{K}$ is closed under homeomorphism), the class of anti-$\mathscr{K}$ spaces is defined so that $X \in$ anti-$\mathscr{K}$ if and only if the only subspaces of X which are in $\mathscr{K}$ are those having cardinalities which require them to be in $\mathscr{K}$. The anti-connected spaces are the totally disconnected spaces, the anti-perfect spaces are the scattered spaces, and the anti-compact spaces are those whose only compact subspaces are finite. This latter class of spaces has been studied quite extensively in [3], [4], [5], [6].

If $\mathscr{K}$ is a topological class the spectrum of $\mathscr{K}$, denoted spec ($\mathscr{K}$), is the class of cardinal numbers κ such that any topology on a set of power κ lies in $\mathscr{K}$. Anti-$\mathscr{K}$ is defined to be the class of spaces X such that whenever $Y \subset X$ then $Y \in \mathscr{K}$ if and only if $|Y| \in$ spec ($\mathscr{K}$). This paper follows the set-theoretic and notational conventions of Bankston [1]. The symbol ▌ denotes the end of a proof, and N denotes the set of positive integers.

We show that the anti-$(\cdot)$ operation does not discriminate well between classes of spaces defined by different separation properties. In fact, the anti-$(\cdot)$ operation distinguishes only the T_0 spaces from spaces with any higher separation property. It maps the class of T_0 spaces onto the class of indiscrete spaces, and the class of T_i spaces ($i \geq 1$) onto the class of spaces with totally ordered topologies. Here T_3 means regular and T_1, T_4 means normal and T_1 and so on.

We also consider the relationships between the anti-spaces of some classes of spaces defined by compactness type properties. Anti-compactness and anti-sequential compactness were considered in [6], and anti-Lindelof and anti-κ-compact spaces were studied in [1].

Throughout this paper we make extensive use of the following basic result due to Bankston [1, Proposition 1.2].

PROPOSITION 1. *If $\mathscr{K}$ and $\mathscr{M}$ are classes of spaces, $\mathscr{K} \subset \mathscr{M}$ and* spec ($\mathscr{K}$) = spec ($\mathscr{M}$) *then* anti-$\mathscr{K} \supset$ anti-$\mathscr{M}$.

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2. Separation properties

Since any set containing at least two distinct points can have a non-Hausdorff topology defined on it, we have that Spec ({Hausdorff spaces}) = $\{0, 1\} = 2$. Thus X is anti-Hausdorff if and only if no pair of distinct points in X have disjoint neighborhoods. For example, the set of real numbers $\mathscr{R}$ with the left hand topology $\mathscr{L}$ which has as a base the family of sets $\{(-\infty, a)\colon a \in \mathscr{R}\}$ is anti-Hausdorff. We observe that $(\mathscr{R}, \mathscr{L})$ is T_0. We show that this is a best possible example in the sense that T_1 anti-Hausdorff spaces do not exist.

Remark. Let $\mathscr{K}_i$ be the class of topological spaces having the separation property T_i, $i = 0, 1, 2, 3, 3\frac{1}{2}, 4, 5, \alpha, \beta, m, t$, where T_α = discrete, T_β = indiscrete, T_m = metrizable, T_t = totally ordered. Then

$$\mathscr{K}_0 \supset \mathscr{K}_1 \supset \mathscr{K}_2 \supset \mathscr{K}_3 \supset \mathscr{K}_{3\frac{1}{2}} \supset \mathscr{K}_4 \supset \mathscr{K}_5 \supset \mathscr{K}_m \supset \mathscr{K}_\alpha,$$

while spec $(\mathscr{K}) = \{0, 1\}$ for all these classes. Hence Proposition 1 implies that the opposite inclusions hold for the Anti $(\mathscr{K})$ classes.

THEOREM 1. (a) Anti $(\mathscr{K}_0) = \mathscr{K}_\beta$.
(b) Anti $(\mathscr{K}_i) = \mathscr{K}_t$ *for* $i \in \{1, 2, 3, 3\frac{1}{2}, 4, 5, m, \alpha\}$.
(c) Anti $(\mathscr{K}_\beta) = \mathscr{K}_0$.
(d) Anti $(\mathscr{K}_t) = \mathscr{K}_1$.

Proof. (a) Let $X \in$ Anti $(\mathscr{K}_0)$. If X is not indiscrete, let A and B be two distinct nonempty open sets such that $A \not\subset B$. Let $a \in A - B$ and $b \in B$. Then the subspace $\{a, b\}$ is T_0, contradiction.

Conversely, let X be indiscrete. If $X \notin$ Anti $(\mathscr{K}_0)$, then there exists a subspace Y which is T_0 and $|Y| \geq 2$. But Y must be indiscrete, hence not T_0, contradiction.

(b) Let $X \in$ Anti $(\mathscr{K}_\alpha)$. If the topology on X is not totally ordered let A and B be nonempty open sets such that $A \not\subset B$ and $B \not\subset A$. Let $a \in A - B$, and $b \in B - A$. Then the subspace $\{a, b\}$ is discrete, contradiction.

Next, let the topology on X be totally ordered. If $X \notin$ Anti $(\mathscr{K}_1)$, then there exists a subspace Y such that Y is T_1 and $|Y| \geq 2$. Let $\{a, b\} \subset Y$, $a \neq b$. Then $\{a, b\}$ is T_1, hence discrete, and there exist sets A, B open in X such that $A \cap \{a, b\} = \{a\}$ and $B \cap \{a, b\} = \{b\}$. Thus $A \not\subset B$, $B \not\subset A$, contradiction. Hence (b) follows, in view of the above remark.

The proof of (c) is similar to that of (a). Finally for (d), let $X \in$ Anti $(\mathscr{K}_t)$. If $X \notin \mathscr{K}_1$, let $\{a, b\} \subset X$ such that for each set V, open in X, $a \in V \Rightarrow b \in V$. Hence the subspace $\{a, b\}$ has totally ordered topology, contradiction, since spec $(\mathscr{K}_t) = \{0, 1\}$.

Conversely, let $X \in \mathscr{K}_1$. If $X \notin$ Anti $(\mathscr{K}_t)$, then there exists a subspace Y such that $Y \in \mathscr{K}_t$, $|Y| \geq 2$. Let $\{a, b\} \subset Y$, $a \neq b$. Since $X \in \mathscr{K}_1$, there exist sets A, B, open in X, such that $a \in A - B$, $b \in B - A$. Then $A \cap Y$, $B \cap Y$ are open sets in Y such that $A \cap Y \not\subset B \cap Y$ and $B \cap Y \not\subset A \cap Y$, contradiction, since $Y \in \mathscr{K}_t$. ∎

COROLLARY 1. (a) Anti (Anti $(\mathscr{K}_i)) = \mathscr{K}_i$ *for* $i \in \{0, \beta, t\}$.
(b) Anti (Anti $(\mathscr{K}_i)) = \mathscr{K}_1$ *for* $i \in \{1, 2, 3, 3\frac{1}{2}, 4, 5, m, \alpha\}$.

Proof. Follows immediately from Theorem 1.

A natural question to consider is the behaviour of the properties of regularity, complete regularity and normality with respect to the anti-$(\cdot)$ operation. The first two yield to a treatment similar to Theorem 1 and Corollary 1. Let $\mathscr{R}_0$, $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{K}_{cr}$ and $\mathscr{K}_\alpha$ denote the classes of R_0, R_1, regular, completely regular and discrete spaces respectively. R_0 and R_1 are the regularity properties discussed by Davis [2]. Then we have $\mathscr{R}_0 \supset \mathscr{R}_1 \supset \mathscr{R}_2 \supset \mathscr{K}_{cr} \supset \mathscr{K}_\alpha$, while Spec $\mathscr{K} = \{0, 1\}$ for $\mathscr{K}$ any of these classes. Hence, by Proposition 1, reverse inclusions hold for the Anti-$(\mathscr{K})$ classes.

THEOREM 2. Anti $(\mathscr{R}_i) =$ Anti $(\mathscr{K}_{cr}) = \mathscr{K}_0 \cap \mathscr{K}_t$ *for* $i = 0, 1, 2$.

Proof. First let $X \in \mathscr{K}_0 \cap \mathscr{K}_t$. If $X \notin$ Anti $(\mathscr{R}_0)$ then there is a subspace $E = \{a, b\}$ which is in $\mathscr{R}_0 \cap \mathscr{K}_0$ and hence discrete, contradiction.

Next let $X \in$ Anti $(\mathscr{K}_{cr})$. If $X \notin \mathscr{K}_0 \cap \mathscr{K}_t$, then there are two points a and b such that every neighbourhood of either point also contains the other. Thus $\{a, b\}$ is indiscrete and hence completely regular, contradiction. ∎

Remark. It is interesting to note that while $\mathscr{R}_i \cap \mathscr{K}_0 = \mathscr{K}_{i+1}$ for $i = 0, 1, 2$ (see [2]), the above result shows that Anti $(\mathscr{R}_i) = \mathscr{K}_0 \cap$ Anti $(\mathscr{K}_{i+1})$, for $i = 0, 1, 2$.

When it comes to normal spaces the situation seems to be more complicated. Any topology on a set of at most two points is normal. Furthermore, if $|X| \geq 3$, let $X = \{a, b, c\} \cup E$ where the union is disjoint, and define a topology $\mathscr{T}$ on X by $\mathscr{T} = \{\emptyset, X, \{a\} \cup E, \{a, b\} \cup E, \{a, c\} \cup E\}$. Then $(X, \mathscr{T})$ is not normal, since $\{b\}$ and $\{c\}$ are disjoint closed sets which cannot be separated. Thus Spec ({normal spaces}) $= \{0, 1, 2\}$. We have not been able to characterize the class of anti-normal spaces.

3. Compactness properties

Here we consider the anti-spaces of classes $\mathscr{K}$ of topological spaces defined by some compactness property $\mathscr{P}$. If $\mathscr{P}$ is any of the properties, finiteness, compactness, countable compactness, sequential compactness or pseudocompactness then Spec $(\mathscr{K}) = \omega$. If $\mathscr{P}$ is σ-compactness or Lindelofness then Spec $(\mathscr{K}) = \Omega$.

THEOREM 3. *The following inclusions are all proper.*

(a) *Anti-pseudocompact* $\subset$ *Anti-countably compact* $\subset$ *Anti-compact* $\subset$ *Anti-finite.*
(b) *Anti-compact* $\subset$ *Anti-sequentially compact.*
(c) *Anti-countably compact* $\subset$ *Anti-sequentially compact.*
(d) *Anti-Lindelof* $\subset$ *Anti-σ-compact.*
(e) *Anti-compact* $\subset$ *Anti-σ-compact.*

Proof. The inclusions (a), (c) and (d) are proved by observing the equality of the spectra of the classes of spaces involved and then using Proposition 1. Inclusion (b) was shown to be proper in [6]. To show (e) let X be anti-compact and A be a σ-compact subset of X. Then $A = \bigcup \{K_n: n \in N\}$, where each K_n is compact in X and therefore finite. Hence A is countable, so that X is anti-σ-compact.

Examples (i), (ii) and (iii) show that the inclusions in (a) are proper, (iv) shows that (c) is proper, (i) shows that (d) is proper, and (v) shows that (e) is proper.

(i) Let X be an uncountable set with the cocountable topology. Then no infinite subset E of X is countably compact. For, let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of distinct points in E and let $V = X - \{x_n: n \in N\}$. Let $V_n = V \cup \{x_1, x_2, \ldots, x_n\}$ for each $n \in N$. Then $\{V, V_1, V_2, \ldots, V_n, \ldots\}$ is a countable open cover of E which has no finite subcover. Thus X is anti-countably compact. But, since there are no disjoint open sets in X, any real-valued continuous function on X is constant, and hence X is pseudocompact, and, in particular, not anti-pseudocompact. Furthermore, X is Lindelof and uncountable, so it is not anti-Lindelof.

(ii) Let X be the subspace of βN described by Walker [8, page 189]. Then X is countably compact and $|X| \leq c$. In fact, since βN is anti-sequentially compact [6] it follows that $|X| = c$. But infinite compact subsets of βN are of cardinality 2^c, Walker [8, Theorem 2, page 71]. Hence X is anti-compact and not anti-countably compact.

(iii) Any topological space is anti-finite, so a non pseudo-finite space is anti-finite but not anti-compact.

(iv) The Stone-Cech compactification of the integers, βN, is countably compact and hence is not anti-countably compact. Suppose A is a sequentially compact infinite subset of βN. Then there is a sequence $\{x_n: n \in N\}$ of distinct points in A, which has a convergent subsequence $\{x_{n_k}: k \in N\}$ converging to a point x in A. Then $\{x\} \cup \{x_{n_k}: k \in N\}$ is an infinite compact subset of βN of cardinality less than 2^c. Hence βN is anti-sequentially compact.

(v) Any compact space which is countably infinite is anti-σ-compact, indeed anti-Lindelof, but not anti-compact. ∎

Bankston [1, 1.3 (iii)] has shown that anti-compactness and anti-

Lindelofness are implicationally unrelated. Examples (i) and (v) above provide an alternative proof of that result.

Local characterizations of anti-compact and anti-sequentially compact spaces were given in [6]. The proof for anti-countably compact spaces is an obvious modification of the proof of Theorem 3 of [6].

THEOREM 4. *X is anti-compact (anti-countably compact) if and only if for each point p in X and each infinite subset A of X there is an open set G containing p such that $A - G$ is not compact (countably compact).*

The icn property discussed in [3], [6] is a local characterization of anti-sequentially compact spaces [6, Theorem 5].

THEOREM 5. *X is anti-sequentially compact if and only if for each point p in X and for each infinite subset A of X there is an open set G containing p such that $A - G$ is infinite.*

We have not been able to produce a local characterization of anti-pseudocompactness. The obvious modification of Theorem 4 is true in one direction but false in the other. Let X be an uncountable set with the cocountable topology, Example (i) above. Then X is not anti-pseudocompact since X is pseudocompact. If A is an infinite subset of X and $p \in X$, take $S = \{x_n : n \in N\}$ to be a sequence of distinct points from $A - \{p\}$. Then $G = X - S$ is open and contains p. Furthermore, $A - G = S$, so that $A - G$ has the discrete topology as a subspace of X and so is not pseudo-compact.

For anti-Lindelofness we have the following result.

THEOREM 6. *X is anti-Lindelof if and only if for each point p in X and for each uncountable subset A of X there is an open set G containing p such that $A - G$ is not Lindelof.*

Proof. Let X be anti-Lindelof. Then $A \cup \{p\}$ is uncountable and hence not Lindelof. So there is an open cover $\mathscr{C}$ of A with no countable subcover. Now $p \in G$ for some $G \in \mathscr{C}$. Then $A - G$ is not Lindelof, for otherwise $\mathscr{C}$ has a countable subcover.

Conversely, let A be an uncountable subset of X. Let $p \in X$ and V be an open set containing p such that $A - V$ is not Lindelof. Let $\mathscr{C}$ be an open cover of $A - V$ which has no countable subcover. Then $\mathscr{C} \cup \{V\}$ is an open cover of A with no countable subcover. Hence A is not Lindelof, so that X is anti-Lindelof. ∎

We observe that the obvious modification of Theorem 6 does not provide a local characterization of anti-σ-compactness. Again we appeal to Example (i), X an uncountable set with the cocountable topology. X is anti-σ-compact since it is anti-countably compact. But for each point p in X and for each open set G containing p we have $X - G$ is countable, and hence is σ-compact.

A sequence characterization of anti-sequentially compact spaces was given in [6, Theorem 4].

THEOREM 7. *X is anti-sequentially compact if and only if no sequence of distinct points in X has a convergent subsequence.*

We now show that a double application of the anti-$(\cdot)$ operation to the class of compact spaces yields the class of the hereditarily compact spaces of Stone [7]. First we need a lemma.

LEMMA 1. *Every non-compact space has a countably infinite subset which can be written as a sequence whose initial segments are relatively open. Furthermore, if X is R_0, it has an infinite discrete subspace.*

Proof. There is an open cover $\mathscr{C}$ of X which has no finite subcover. Let $x_1 \in X$, so there is a $U_1 \in \mathscr{C}$ with $x_1 \in U_1$. Let $x_2 \in X - U_1$, and $U_2 \in \mathscr{C}$ with $x_2 \in U_2$. Let $x_3 \in X - (U_1 \cup U_2)$, and $U_3 \in \mathscr{C}$ with $x_3 \in U_3$, and so on. Then $\{x_1, x_2, \ldots, x_n, \ldots\}$ is the required subset.

If X is R_0, then cl $\{x_1\} \subset U_1$, and $G_2 = X -$ cl $\{x_1\}$ is open and contains x_2. Hence $V_2 = U_2 \cap G_2$ is open and $x_2 \in V_2, x_1 \notin V_2$. By induction, for each n we obtain an open set V_n such that $x_n \in V_n$ and $x_i \notin V_n$ for $i = 1, 2, \ldots, n-1$. Hence $\{x_1, x_2, \ldots, x_n, \ldots\}$ is a countably infinite discrete subspace of X. ▮

Example. Let N be the set of natural numbers with a topology consisting of all sets of the form $\{1, 2, \ldots, n\}$, $n \in N$, together with ϕ and N. Then N is anti-compact, anti-R_0, and the only discrete subsets are singletons.

THEOREM 8. *X is hereditarily compact if and only if X is anti-(anti-compact).*

Proof. Firstly we observe that spec (anti-compact) $= \omega$, for the indiscrete topology on any infinite set is compact and hence not anti-compact.

Let X be hereditarily compact, and suppose X is not anti-(anti-compact). Then there is a subspace Y of X such that Y is anti-compact but $|Y| \notin$ spec (anti-compact). Hence Y is infinite and therefore not compact, contradicting the fact that X is hereditarily compact.

Conversely, let X be anti-(anti-compact) and suppose X is not hereditarily compact. Then there is a non-compact subspace Z of X. By the lemma, Z has an infinite anti-compact subspace E. But $|E| \notin$ spec (anti-compact), contradicting X is anti-(anti-compact). ▮

The following result can be proved in a manner analogous to the previous theorem.

THEOREM 9. *X is hereditarily Lindelof if and only if X is anti-(anti-Lindelof).*

We observe that spec (hereditarily compact) $= \omega$, and hence from Proposition 1 we have that anti-compact implies anti-hereditarily compact. The next result characterizes anti-hereditarily compact spaces.

THEOREM 10. *X is anti-hereditarily compact if and only if each infinite subset of X contains an infinite anti-compact subspace.*

Proof. Let X be anti-hereditarily compact and E be an infinite subset of X. Then E is not hereditarily compact, so there is a non-compact subspace F of E. By Lemma 1, F has an infinite anti-compact subspace.

Conversely, let E be an infinite subset of X. Then E has an infinite anti-compact subspace, and hence E is not hereditarily compact. Thus X is anti-hereditarily compact. ∎

COROLLARY 2. *Any Hausdorff space is anti-hereditarily compact.*

Proof. If the space is finite, we are done. Otherwise, any infinite Hausdorff space contains a discrete sequence of distinct points, and hence is anti-hereditarily compact by Theorem 10.

The observation of Stone [7, page 900] that a T_2 hereditarily compact space is necessarily finite follows immediately from the corollary. We also note that the real line with the usual topology is an anti-hereditarily compact space which is not anti-compact. The Hausdorff condition in this corollary cannot be weakened to T_1, as the cofinite topology on an infinite set shows. The following example shows that the converse of this corollary is false. Let N have the topology $\mathscr{T}$ with base $\{\{2n-1, 2n\}: n \in N\}$. Then $(N, \mathscr{T})$ is not T_0 but it is anti-hereditarily compact since it is anti-compact.

The results of Lemma 1 and Theorems 8, 9 and 10 can be generalized to higher cardinal numbers as follows. A space X is called *λ-compact* (λ is a cardinal) iff every open cover of X has a subcover of power $< \lambda$.

LEMMA 2. *Every non-λ-compact space has a subspace which is well orderable in type λ in such a way that initial segments are relatively open. Such a subspace is anti-λ-compact whenever λ is a regular cardinal.*

THEOREM 11. *Let λ be a regular cardinal. Then X is hereditarily λ-compact iff X is anti-anti-λ-compact.*

THEOREM 12. *X is anti-hereditarily-λ-compact iff each subset of X of power at least λ contains a λ-sequence all of whose initial segments are relatively open.*

LEMMA 3. *X is hereditarily-λ-compact iff X contains no scattered subsets of power λ.* (In one direction use Lemma 2. In the other, show that if Y is hereditarily-λ-compact of power λ then the set of points of Y with no nbds of power $< \lambda$ is nonempty dense-in-itself.)

THEOREM 13. *X is anti-hereditarily-λ-compact iff each subset of power at least λ contains a scattered subset of power λ.*

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THE UNIVERSITY OF AUCKLAND
AUCKLAND, NEW ZEALAND