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INTENSIONAL SETS

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In his proof of the consistency of the continuum hypothesis [4], Gödel introduced the technique of building a model of set theory in stages L_α, where each successor stage $L_{\alpha+1}$ contains exactly those subsets of L_α which are definable in a first order logic for set theory using parameters from L_α. In 1971, Chang [1], extended this procedure by taking the infinitary language $\mathscr{L}_{\kappa,\kappa}$ for the underlying logic, obtaining sequences C_α^κ where $C_{\alpha+1}^\kappa$ contains just those subsets of C_α^κ definable in $\mathscr{L}_{\kappa,\kappa}$ using parameters from C_α^κ. The general procedure has recently been investigated by Gloede [2] and [3] and by the author. Suppose we are at stage M_α and desire to construct $M_{\alpha+1}$. Whatever process has given us M_α is assumed to have also given us both a language $\mathscr{L}_\alpha \subseteq \mathscr{L}_{\infty,\infty}(\varepsilon)$ and a distinguished collection F_α of subsets of M_α. Then $M_{\alpha+1}$ is defined as the collection of subsets of M_α that are definable by a formula of $\mathscr{L}_\alpha$ using as the set of parameters one of the elements of F_α.

The following examples are from Gloede [2, p. 313]:

(i) For all α, let $\mathscr{L}_\alpha$ be $\mathscr{L}_{\omega\omega}$ and let F_α be the collection of finite subsets of M_α. Then for all α, $M_{\alpha+1}$ is simply $L_{\alpha+1}$.

(ii) For all α, let $\mathscr{L}_\alpha$ be $\mathscr{L}_{\kappa\kappa}$ and let F_α be the collection of subsets of M_α of cardinal less than κ. Then for all α, $M_{\alpha+1}$ is $C_{\alpha+1}^\kappa$.

(iii) For all α, let $\mathscr{L}_\alpha$ be $\mathscr{L}_{M_\alpha^+,M_\alpha^+}$ and let F_α be the collection of finite subsets of M_α. Then $\bigcup_{\gamma\in n} M_\gamma$ is the collection HOD. (M_α^+ is the least admissible set A such that $M_\alpha \in A$.)

The particular sequence M_α that forms the starting point for this paper is the one obtained by taking, for all α, $\mathscr{L}_\alpha$ to be the language $\mathscr{L}_{\omega\omega} \cup \mathscr{L}_{cf(\bar{\bar{\alpha}}),cf(\bar{\bar{\alpha}})}$ and F_α to be the collection of subsets of M_α of cardinal less than $cf(\bar{\bar{\alpha}})$. The motivation for singling out this particular sequence is that it allows our construction process to grow in a natural way along with the stages of our construction since we continually increase the definitional complexity of our language but only at a pace that keeps $\mathscr{L}_\alpha$ inside of M_α.

DEFINITION 1. (i) $M_0 = \emptyset$;
(ii) for limit ordinals λ, $M_\lambda = \bigcup_{\gamma<\lambda} M_\gamma$;
(iii) for all α, $M_{\alpha+1}$ is the collection of subsets of M_α definable in the

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language $\mathscr{L}_{cf(\bar{\bar{\alpha}}),cf(\bar{\bar{\alpha}})} \cup \mathscr{L}_{\omega\omega}$ using as parameters a subset of M_α of cardinal less than $cf(\bar{\bar{\alpha}})$.

A routine induction shows that every set is in some M_α; in fact, for all x, $x \in M_{\overline{\overline{TC(x)}}^+ + 1}$. This is so because our language eventually grows to the point where the set x can simply be enumerated as $\{y \mid \bigvee_{b \in x}[y = b]\}$. Nevertheless there is clearly a difference between those sets x that make their first appearance at stage $M_{\overline{\overline{TC(x)}}^+ + 1}$ and those that appear at an earlier level, for these latter must be defined by one of their properties and not simply enumerated.

Definition 2. (i) For all x, x is intensional iff there is an $\alpha < \overline{\overline{TC(x)}}^+$ such that $x \in M_{\alpha+1}$; (ii) For all x, x is hereditarily intensional (HI(x)) iff for all $y \in TC(\{x\})$, y is intensional.

It is immediate from the definitions that every constructible set is hereditarily intensional; that the collection of hereditarily intensional sets is transitive; that, for subsets x of ω, HI(x) iff $x \in L$; and, since $V = L$ implies $\forall x.\mathrm{HI}(x)$, Con (ZF) $\Rightarrow$ Con (ZF + $\forall x.\mathrm{HI}(x)$).

The most interesting property possessed by the collection of hereditarily intensional sets is that they satisfy the GCH; in fact, loosely speaking, they are the largest natural collection of sets that necessarily satisfies the GCH. The proof that the HI sets satisfy the GCH uses the following lemma.

Lemma. *For all regular cardinals κ and λ, if $\kappa \leq \lambda$ and if the* GCH *holds below λ, then $\lambda^{\underset{\smile}{\kappa}} = \lambda$.*

Proof. It suffices to establish the lemma in the case $\lambda = \kappa$. If λ is a successor cardinal μ^+, then $(\mu^+)^{\underset{\smile}{\mu^+}} = (\mu^+)^\mu = \mu^\mu \cdot \mu^+ \leq 2^\mu \cdot \mu^+ = \mu^+ \cdot \mu^+ = \mu^+ = \lambda$. (The identity $(\mu^+)^\mu = \mu^\mu \cdot \mu^+$ is the Hausdorff recursion formula, [5, p. 289].) If λ is a regular limit cardinal, then the conclusion follows from the hypothesis that the GCH holds below λ which guarantees that λ is strongly inaccessible. ▮

Theorem. ZFC + $\forall x.\mathrm{HI}(x) \vdash$ GCH.

Proof. Observe first that for all $\alpha \geq \omega$ and all $x\alpha$, if HI(x), then there is a $\beta < \bar{\bar{\alpha}}^+$ with $x \in M_\beta$. The proof that for all α, $2^{\aleph_\alpha} \subseteq \aleph_{\alpha+1}$ is by induction of α and is seen true for $\alpha = 0$ by the remark following Definition 2. Suppose then that for all $\beta < \alpha$, $2^{\aleph_\beta} = \aleph_{\beta+1}$. If $\aleph_\alpha$ is singular then a trivial counting argument on $\mathscr{L}_{\aleph_\alpha}$ and $F_{\aleph_\alpha}$ shows that at any stage $M_{\gamma+1}$ with $\bar{\bar{\gamma}} = \aleph_\alpha$, only $\aleph_\alpha$ new subsets of $\aleph_\alpha$ can be defined so that only $\aleph_{\alpha+1}$ subsets of $\aleph_\alpha$ can be hereditarily intensional. In the event that $\aleph_\alpha$ is regular, the associated language has cardinal $\aleph_\alpha$ and the cardinal of the collection of sets of parameters is $\aleph_\alpha^{\underset{\smile}{\aleph_\alpha}}$ which by the induction hypothesis and the lemma is just $\aleph_\alpha$, so that, here too, only $\aleph_\alpha$ new subsets of $\aleph_\alpha$ can be defined at any stage M_γ for which $\bar{\bar{\gamma}} = \aleph_\alpha$ and, therefore, there are only $\aleph_{\alpha+1}$ hereditarily intensional subsets of $\aleph_\alpha$. ▮

The next theorem shows that in any model of ZFC in which there is a set which is not hereditarily intensional, the collection of hereditarily intensional sets does not form an inner model.

THEOREM. *Suppose there is a set which is not hereditarily intensional. Then* $\langle \mathrm{HI}, \varepsilon \rangle$ *fails to satisfy the axiom of subsets.*

Proof. Choose a non-hereditarily intensional set a of minimal order, so that $\sim \mathrm{HI}(a)$ but for all $x \in a$, $\mathrm{HI}(x)$. Let $\kappa = \overline{\overline{TC(a)}}^{+}$ and let $b = a \cup \{\kappa^+\}$. Then $b \in M_{\kappa^+ + 2}$ (via the definition, over $M_{\kappa^+ + 1}$, $y \in b \leftrightarrow y = \kappa^+ \vee \bigvee_{u \in a} [y = u]$) and $\mathrm{HI}(b)$ since a was chosen minimal. However $a = \{x \mid x \in b \wedge x \neq \kappa^+\}$ is then a definable subset of b not hereditarily intensional. ∎

We conclude with the following open question suggested by K. Bowen: Is it true that for every regular κ, there is a structure $\mathscr{M}$ such that $\mathscr{M} \models (V = \mathrm{HI} \wedge V \neq C^{\kappa})$? The strongest possible alternative, that $V = HI$ already implies $V = L$, is also open.

REFERENCES

1. C. C. CHANG, "Sets constructible using $L_{\kappa\kappa}$" in *Axiomatic set theory*, D. Scott, editor, Amer. Math. Soc., Providence, R.I., 1971, pp. 1-8.
2. K. GLOEDE, "Set theory in infinitary languages" in *Logic Conference, Kiel 1974*, Müller et al, editors, Springer-Verlag LNM 499, Berlin, 1975, pp. 311-362.
3. ———, *Mengenlehre in infinitären Sprachen*, Universität Heidelberg, 1974.
4. K. GÖDEL, *Consistency proof for the generalized continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A., vol. 25 (1939), pp. 220-224.
5. K. KURATOWSKI and A. MOSTOWSKI, *Set theory*, North-Holland, Amsterdam, 1968.

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