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FREE CIRCLE ACTIONS ON HOMOTOPY NINE SPHERES

BY

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1. Introduction

This paper presents an explicit construction of a smooth free circle action on a nine-dimensional exotic homotopy sphere which, in an appropriate framing, represents the framed cobordism/stable homotopy class $\nu^3 \in \pi_9^S = \Omega_9$. The construction is done, in Section 2, by performing what is essentially equivariant framed surgery on a representative V_0 of ν^3 with a specified circle action, yielding a homotopy sphere $V^9 \in \nu^3$ with a smooth free circle action on it. The construction also yields a "conjugation" of the orbit space P of V^9 in this action, i.e., a self-diffeomorphism γ of P such that $\gamma_* = -1$ on $H_2(P; Z)$—analogous to the conjugation of CP^4 induced by standard complex conjugation in S^9.

Brumfiel has apparently obtained the result that ν^3 contains a sphere with an action by means of computations in stable homotopy: cf. Theorem I.10(ii) of [4] and the remarks preceding it. Unfortunately, these computations do not appear there and indeed seem never to have been published. However, Brumfiel does show in [4] that there are infinitely many differentiably—indeed, topologically—distinct free circle actions on the standard nine-sphere S^9 (this is a consequence of his result [4, p. 401, I.5(ii), I.8(ii)] that the nine-dimensional exotic Kervaire sphere does not admit any such action). Moreover, Bredon in [2] constructed a free circle action on S^9 which is differentiably distinct from but PL equivalent to the standard action. In Section 3 these results are adapted to produce actions on V^9 which are differentiably distinct from the one obtained in Section 2. Although we do not do so here, it is not difficult to show that these actions on S^9 and V^9 are all the smooth free circle actions on homotopy nine-spheres.

Regarding the terminology: all manifolds are smooth manifolds, all embeddings are smooth embeddings, and all actions are smooth circle actions. All actions are also free except on discs and where otherwise specified. Homology and cohomology are ordinary with integral coefficients.

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2. An action on V^9

Since the nine-dimensional simply-connected surgery obstruction groups are all zero, all of the classes in the nine-dimensional framed cobordism group $\Omega_9 = \pi_9^S = Z_2 \oplus Z_2 \oplus Z_2$ contain homotopy spheres—in fact, two each, since $bP_{10} = Z_2$. Any class which contains a framed standard S^9 is in the image of $J\colon \pi_9(SO) \to \Omega_9$; this image does not contain the class ν^3 (cf. [1]), which consequently does not contain S^9 in any framing (nor the nine-dimensional Kervaire sphere, which is framed cobordant to S^9).

The class $\nu \in \pi_3^s$ contains the standard 3-sphere S^3 in the framing given by the standard injection $q\colon S^3 = Sp(1) \to SO(4)$ which expresses each quaternion as a real linear transformation (i.e., q is the Lie group framing), so ν^3 contains $S^3 \times S^3 \times S^3$ in the framing given by the product

$$q \times q \times q\colon S^3 \times S^3 \times S^3 \to SO(4) \times SO(4) \times SO(4) \to SO(12)$$

(all framings are considered to be of the stable normal bundle of some suitable dimension). Denote this framed manifold by V_0.

Put an action on $S^3 \times S^3 \times S^3$ (and hence on V_0) as follows: consider S^3 as a subset of complex 2-space, so that each element may be written as a pair (α, β) of complex numbers, and each element of $V_0 = S^3 \times S^3 \times S^3$ may be expressed as a sextuplet $((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ of them; then let

$$z((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)) = ((z\alpha_1, z\beta_1), (z\alpha_2, \beta_2), (\alpha_3, \beta_3))$$

for $z \in S^1$. This action is clearly smooth, and is free, being free on the first factor. Consequently the orbit space, which we denote by M, is a smooth 8-manifold; it is a homology $S^2 \times S^3 \times S^3$.

If we consider $S^3 \times S^3 \times S^3$ to be the product of the principal S^1 fiber bundle S^3 over S^2 (the first factor) with an S^1-space $S^3 \times S^3$ (the latter two factors, in the restriction of the above action)—that is, as a fiber bundle over S^2 with fiber $S^1 \times S^3 \times S^3$—we see that M is also a fiber bundle over S^2, with fiber $S^3 \times S^3$. The point of this is that we can thus obtain M as the union of two "hemispheres" of the form $D^2 \times S^3 \times S^3$, over the hemispheres S^2; these are coordinate neighborhoods of the S^1 bundle $S^3 \times S^3 \times S^3$ over M, and we may arrange things in such a way that the projection map $S^3 \times S^3 \times S^3 \to M$ restricted to one of these hemispheres is the map of

$$S^1 \times D^2 \times S^3 \times S^3 = \{((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)) \mid \beta_1 \neq 0\}$$

in $S^3 \times S^3 \times S^3$ to $D^2 \times S^3 \times S^3$ in M which takes

$$((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)) \in S^3 \times S^3 \times S^3$$

to

$$(\alpha_1/\beta_1, (\alpha_2, \beta_2), (\alpha_3, \beta_3)) \in D^2 \times S^3 \times S^3.$$

This will occasionally be called the front hemisphere of M.

Define an embedding $f_1: S^3 \to S^3 \times S^3 \times S^3$ by

$$f_1(\alpha, \beta) = ((\alpha, \beta), (0, 1), (0, 1))$$

for $(\alpha, \beta) \in S^3$. This is clearly equivariant with respect to the standard action on S^3, so it covers a map of orbit spaces $f': S^2 \to M$ which is easily seen to be a smooth degree 1 embedding. There is a normal tube $S^3 \times D^6$ of $f_1(S^3)$ which has the form

$$S^3 \times D^2 \times D^4 = f_1(S^3) \times (D^2 \times D^1) \times (D^2 \times D^1) \subset S^3 \times S^3 \times S^3$$

(the correspondence of a 3-disc about $(0, 1)$ in S^3 takes (α, β) to $(\alpha, \text{im } \beta)$ where im β is the imaginary part of β); note that the action induced on this tube is the product of the standard actions on S^3 and D^2 with the trivial action on D^4. This tube covers a normal tube T_0 of $f'(S^2)$ in M, and we may arrange things so that in the front hemisphere

$$T_0 \cap D^2 \times S^3 \times S^3 \subset D^2 \times D^3 \times D^3$$

(where the latter two D^3's are simply a half-disc about $(0, 1)$ in S^3). Now choose two widely separated points a_2 and a_3 in the interior of D^2 which are real (that is, self-conjugate with respect to the conjugation on D^2 induced from that on S^3 which takes α_1/β_1 to $\bar{\alpha}_1/\bar{\beta}_1$) and define two embeddings f_2 and f_3 of S^3 into the interior of the front hemisphere of M by

$$f_2(\alpha, \beta) = (a_2, (\alpha, \beta), (1, 0)), \qquad f_3(\alpha, \beta) = (a_3, (1, 0), (\alpha, \beta))$$

for $(\alpha, \beta) \in S^3$. Clearly $f_2(S^3)$ and $f_3(S^3)$ are disjoint from each other and from T_0. The product of $f_2(S^3)$ with small discs about $a_2 \in D^2$ and $(1, 0) \in S^3$ gives a normal tube

$$f_2(S^3) \times D^5 = D^2 \times f_2(S^3) \times D^3 \subset M$$

for $f_2(S^3)$ which is disjoint from $D^2 \times D^3 \times D^3$, and a similar normal tube for $f_3(S^3)$ may be found which is disjoint from the normal tube about $f_2(S^3)$ and from $D^2 \times D^3 \times D^3$.

The framing on V_0 restricted to $D^2 \times S^1 \times S^3 \times S^3 \subset V_0$ over the front hemisphere of M is the product of a framing on $D^2 \times S^1$—hence simply on S^1—with another on $S^3 \times S^3$. We may put this latter framing on the stable normal bundle $\nu(M) | D^2 \times S^3 \times S^3$ of the front hemisphere, which is trivial, by means of a local cross-section. (Because of the nontrivial action of S^1 on the second factor of $S^3 \times S^3 \times S^3$, the pullback under the projection of the framing on the first S^3 factor in the front hemisphere will, strictly speaking, be an element of $[S^1 \times S^3, SO]$ rather than of $\pi_1(SO) \oplus \pi_3(SO)$. However, the fiber sequence $S^1 v S^3 \to S^1 \times S^3 \to S^4$ yields the exact sequence

$$0 = \pi_4(SO) \to [S^1 \times S^3, SO] \to \pi_1(SO) \oplus \pi_3(SO),$$

so there is no problem.) To simplify the notation, let an element of the normal tube of $f_2(S^3)$ be written simply as $(f_2(s), \lambda)$ where s and λ are the appropriate elements of S^3 and D^5 respectively, and similarly for the normal tube of $f_3(S^3)$. Let $p\colon S^3 \to SO(5)$, a homotopy inverse of the framing q, be the composition of the map taking $s \in S^3$ to $q(s^{-1})$ with the usual inclusion of $SO(4)$ in $SO(5)$. Define embeddings F_2 and F_3 of $S^3 \times D^5$ onto the normal tubes of $f_2(S^3)$ and $f_3(S^3)$, respectively, in M by

$$F_2(s, \lambda) = (f_2(s), p(s)\lambda), \qquad F_3(s, \lambda) = (f_3(s), p(s)\lambda),$$

for $(s, \lambda) \in S^3 \times D^5$.

F_2 induces a framing on the normal bundle of $f_2(S^3)$ in M which is given by p; the framing on $\nu(M)|\, f_2(S^3)$ is given by q (from the second factor of the framing map on V_0). Thus the total framing on $f_2(S^3)$ is given by the sum

$$p \oplus q\colon S^3 \to SO(5) \times SO(4) \subset SO(9),$$

which is nullhomotopic, so the total framing is concordant to the trivial framing—which is, of course, the total framing on

$$S^3 \times \{0\} = F_2^{-1}(f_2(S^3)) \subset S^3 \times D^5$$

as a submanifold of $D^4 \times D^5$. Thus attaching $D^4 \times D^5$ to $M \times I$ with F_2 yields a normal cobordism which is the trace of a framed surgery based on the embedding $f_2(S^3)$ in M (cf. [3, pp. 83–86]). Since $f_2(S^3)$ is in the interior of a coordinate neighborhood of M and the framing on a pullback in V_0 is the product of one induced from the front hemisphere and one on the fiber, this cobordism may be covered—that is, we may use the product of F_2 and the identity on S^1 to attach $S^1 \times D^4 \times D^5$ to $V_0 \times I$, thus getting a framed cobordism with V_0 as one boundary component. The same thing may be done with the embedding $f_3(S^3)$ and the map F_3, that is, another copy of $S^1 \times D^4 \times D^5$ may be attached to $V_0 \times \{1\}$ via

$$1 \times F_3\colon S^1 \times S^3 \times D^5 \to V_0,$$

yielding an equivariant framed covering of the trace of a surgery performed on $F_3(S^3) \subset M$.

We have now killed two of the three 3-cycles of V_0. We may deal with the remaining one, which is carried by the embedding f_1 of S^3, by means of a surgery performed directly on V_0, as follows. As remarked above just after the definition of f_1, the embedding $f_1(S^3)$ has a normal tube of the form $f_1(S^3) \times D^2 \times D^4$ the induced action on which is the product of the standard actions on the first two factors and the trivial action on D^4. Simplifying the notation again, let elements of this normal tube be represented as (s, a, b) where $s \in f(S^3)$, $a \in D^2$ and $b \in D^4$, so that the induced action is given as $z(s, a, b) = (zs, za, b)$ for $z \in S^1$.

Consider the standard product $S^3 \times D^2 \times D^4$ with the standard action on all three factors, i.e., $z(s, a, b) = (zs, za, zb)$ (this is, of course, the action induced on $S^3 \times D^2 \times D^4$ as a boundary subspace of $D^{10} = D^4 \times D^2 \times D^4$ with the standard action). We may define an equivariant embedding F_1 of this onto the normal tube of $f_1(S^3)$ in V_0, extending f_1, by

$$F_1(s, a, b) = (s, a, s^{-1}b) \in f(S^3) \times D^2 \times D^4 \subset V_0$$

for $(s, a, b) \in S^3 \times D^2 \times D^4$; the product $s^{-1}b$ in the last factor is ordinary inversion and multiplication of quaternions. To verify the equivariance, let $z \in S^1$ and $(s, a, b) \in S^3 \times D^2 \times D^4$; then

$$\begin{aligned} F_1(z(s, a, b)) &= F_1(zs, za, zb) \\ &= (zs, za, (zs)^{-1}zb) \\ &= (zs, za, s^{-1}z^{-1}zb) \\ &= (zs, za, s^{-1}b) \\ &= z(s, a, s^{-1}b) \\ &= zF_1(s, a, b) \end{aligned}$$

(where the first equality is from the action on $S^3 \times D^2 \times D^4 \subset D^{10}$ and the fifth is from the action on V_0). Furthermore, the quaternionic action of S^3 on D^4 in F_1 gives a framing on the normal tube of $f_1(S^3)$ which is a homotopy inverse of q, so that it cancels the framing on $v(V_0) | f_1(S^3)$, and consequently attaching D^{10} to $V_0 \times I$ by identification via F_1 of the boundary submanifold $S^3 \times D^2 \times D^4$ gives the trace of a surgery based on $f_1(S^3)$.

A framed cobordism U of V_0 to a homotopy sphere V^9 may be constructed by attaching D^{10} and two copies of $S^1 \times D^9$ to one boundary component of $V_0 \times I$ by the identifications described above. Since these identifications are equivariant with respect to the specified actions, U has a semifree action with exactly one fixed point—the zero of D^{10}. Hence V^9 has a smooth free circle action.

V^9 cannot be the standard sphere S^9, since it is in v^3 (in the framing induced from U above). However, it contains $D^4 \times S^5$ with the standard smooth structure and product circle action in a smooth equivariant embedding—the "outside" of the D^{10} attached to $V_0 \times I$ above. Let $V' = V^9 - (D^4 \times S^5)$ be the complement of this, and let T' be its orbit space. Thus V' is essentially $S^3 \times D^6$ with an exotic smooth structure and circle action on the interior but the standard smooth structure and product action on the boundary $S^3 \times S^5$ and similarly T' is essentially the nontrivial 6-disc bundle T over S^2 (the orbit space of $S^3 \times D^6$ in the standard action) with an exotic smooth structure on the interior and the same boundary $\partial T' = \partial T$.

The orbit space of $D^4 \times S^5$ is $CP^4 - T$, the complement in CP^4 of the normal tube of an embedded S^2 carrying a generator of $H_2(CP^4)$; thus the orbit space P of V^9 in this action decomposes as $(CP^4 - T) \cup T'$. Since $CP^4 - T$ is also a normal tube of an embedded CP^2 which is a cohomology 4-skeleton of

CP^4, substituting T' for T does not change the first Pontrjagin class and consequently P has the same tangent bundle as CP^4 (as an element of $KO(CP^4) = KO(P)$).

To obtain the conjugation of P promised in Section 1, let us return to $S^3 \times S^3 \times S^3$ and M. Define a self-diffeomorphism c of $S^3 \times S^3 \times S^3$ by ordinary complex conjugation on each S^3 factor. While c is not equivariant, it is conjugate linear, so that it carries orbits to orbits and consequently induces a self-diffeomorphism c' of M which preserves hemispheres; on the front hemisphere,

$$c'(a, (\alpha_2, \beta_2), (\alpha_3, \beta_3)) = (\bar{a}, (\bar{\alpha}_2, \bar{\beta}_2), (\bar{\alpha}_3, \bar{\beta}_3)).$$

Since a_2 and a_3 were chosen to be self-conjugate, c' preserves the embeddings $f_2(S^3)$ and $f_3(S^3)$ and their normal tubes, and is in fact given on each fiber of each of these tubes by a certain linear transformation $L \in O(5)$. Also, c' preserves the embedding $f'(S^2)$ in M since c preserves the embedding $f_1(S^3)$ in $S^3 \times S^3 \times S^3$, and c' is the antipodal map on $f'(S^2)$.

Let M_0 be the manifold obtained from M by performing framed surgery to remove the two 3-cycles as above; the trace of this surgery is a cobordism W of M to M_0 which consists of $M \times I$ with two 9-disks $D^4 \times D^5$ attached to one boundary by the embeddings F_2 and F_3 of $S^3 \times D^5$ into M. Now

$$F_2^{-1} c' F_2 \colon S^3 \times D^5 \to S^3 \times D^5$$

is given for $((\alpha, \beta), \lambda) \in S^3 \times D^5$ by

$$F_2^{-1} c' F_2((\alpha, \beta), \lambda) = ((\bar{\alpha}, \bar{\beta}), p(\bar{\alpha}, \bar{\beta})^{-1} L p(\alpha, \beta)\lambda).$$

Since complex conjugation of S^3 is a degree 1 map, the function of S^3 into $O(5)$ taking (α, β) to $p(\bar{\alpha}, \bar{\beta})^{-1} L p(\alpha, \beta)$ is homotopic to a constant map, so that $c' \times 1$ extends across this attached $D^4 \times D^5$. Similarly $c' \times 1$ extends across the other attached $D^4 \times D^5$, and we obtain a diffeomorphism d of M to itself which induces the antipodal map on the embedded 2-sphere $f'(S^2)$. Since this carries the 2-cycle in homology, d^* is -1 on $H^2(M_0)$.

Now T' is the complement of the interior of a normal tube of $f'(S^2)$; this normal tube may be chosen invariant under c' and hence under d. Thus the restriction of d to T' is a self-diffeomorphism of T' which restricts on the boundary $\partial T' = \partial T$ to a bundle map over the antipodal map of S^2. Since $\pi_2(SO(6)) = 0$, there is up to smooth isotopy only one such bundle map, so the restriction of d to T' may be altered by a differentiable isotopy on a collar of the boundary to a self-diffeomorphism χ' of T' which coincides on the boundary with the restriction to ∂T of the bundle map $\chi \colon T \to T$ induced by complex conjugation on $S^3 \times D^6$. The desired conjugation γ of P may now be put together as χ on $CP^4 - T$ and χ' on T'.

3. Other actions on V^9

Bredon and Brumfiel have demonstrated the existence of various actions on S^9 differentiably distinct from the standard one. We may use these to produce actions on V^9 differentiably distinct from the action of Section 2.

First, Bredon ([2, Theorem 4.5]) constructed an action on S^9 with orbit space the connected sum $CP^4 \# \Sigma^8$ of CP^4 with the exotic 8-sphere Σ^8. This construction readily generalizes to give an action on V^9 with orbit space $P \# \Sigma^8$. This is clearly PL equivalent to the action of Section 2; the conjugation γ on P and some smoothing theory show that these actions are nevertheless differentiably distinct, as follows (the obstruction theory of [6], which can be used to show CP^4 and $CP^4 \# \Sigma^8$ nondiffeomorphic, cannot be used here because—see, e.g. [4, p. 400, I.3]—P does not have a smoothly embedded homotopy CP^3 as cohomological 6-skeleton). By standard obstruction theory, there are two homotopy classes of homotopy equivalences of $P \# \Sigma^8$ to P. Since the 6-skeleton collapse $P \to S^8$ induces an isomorphism $\pi_8(PL/0) \cong [P, PL/0]$, the connected sum

$$1 \# f\colon P \# \Sigma^8 \to P \# S^8 = P,$$

$f\colon \Sigma^8 \to S^8$ an orientation preserving PL homeomorphism, corresponds to the nontrivial element of $[P, PL/0]$ and hence is not homotopic to a diffeomorphism (see [5]). If the other class of homotopy equivalences in $[P \# \Sigma^8, P]$ contained a diffeomorphism d, then the composition $\gamma \circ d$ would be a diffeomorphism homotopic to $1 \# f$, so there is no such d either.

Finally, Brumfiel's result [4, p. 400–401] that S^9 has infinitely many topologically distinct smooth free circle actions may be extended to V^9. If Q is the orbit space of an action on S^9, let $r\colon T \to Q$ be the inclusion of a normal tube of a smoothly embedded S^2 carrying a generator of $H_2(Q)$, and let $s\colon S^3 \times D^6 \to S^9$ be an equivariant embedding of $S^3 \times D^6$, in the standard product action, covering r. Remove the interiors of $r(T)$ and $s(S^3 \times D^6)$ from Q and S^9 and replace them with T' and V' of Section 2, attached along the boundaries by the restriction of r and s respectively. This yields a smooth free circle action on

$$(S^9 - (S^3 \times D^6)) \cup V' = V^9$$

the orbit space of which has the same tangent bundle as Q. The techniques of the previous paragraph can be adapted to show that $Q \# \Sigma^8$ and $((Q - T) \cup T') \# \Sigma^8$ are not diffeomorphic to Q and $(Q - T) \cup T'$ respectively, showing that all of the actions on S^9 and V^9 come in PL equivalent pairs.

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